§0. Original T-W method.

$\bar{\rho} : G_F \to GL_n(k)$

Mazur R

$\bar{\rho} \rightsquigarrow m, \mathbb{T}$

"$R = \mathbb{T}$"

1) understand $\mathbb{T}$. why $R \to \mathbb{T}$?

existence of Gal. rep.

2) understand R

3) why $R \to \mathbb{T}$ is Isom.

Minimal case, $\bar{\rho} = \bar{\rho}_E$ modular.

1) modular forms $\rightsquigarrow$ Gal rep.

2) R complete intersection / formally sm.

3) $\mathbb{T}$ acts on M

Q T-W set

$R_\infty \to R_Q \to R$

$\mathbb{T}_Q \to \mathbb{T}$

M_Q

$S_\infty \to R_\infty \to \mathbb{T}_\infty \curvearrowright M_\infty$

§1. Developments.

Full modularity for ell. curve $/\mathbb{Q}$.

$\Rightarrow$ understand def. ring attached to local

Gal. rep. at bad p.

Conrad: idea of Fontaine

equiv.

local Gal def. $\sim$ linear alg.

integral p-adic Hodge theory,

theory of f.f. gp schemes

p-div. gps

Breuil found new way understanding

the integral theory of f.f. gp schemes

over ramified bases

Breuil, Conrad, Diamond, Taylor [BCDT]

$\Rightarrow$ R formally smooth.

However the argument not applied to all 2-dim rep. (not from ell. curves)

Kisin 1) not only consider local def. rings attached to f.f. gp schemes but also moduli space, whose geom. relates to local models of SV

$\Rightarrow$ construct R whose generic fibre is often formally sm. and has expected dim.

2) modify T-W method so that R not formally sm.

global def ring R over $R^{loc} = \widehat{\bigotimes}_{v \in S} R_v$

M_∞ / R_∞, free over S_∞

R_∞ no longer power series over $W(k)$

but R^{loc}

$m[\frac{1}{p}]$ faithful $R[\frac{1}{p}]$-mod

$\Rightarrow$ $R[\frac{1}{p}] = \mathbb{T}[\frac{1}{p}]$.

§2. p-adic Langlands.

Motivated by [BCDT]

Breuil and Mézard study

a certain low weight potentially

semi-stable def ring

$\Rightarrow$ geom of these def. rings

?

mod p reduction of lattices inside

locally alg. p-adic rep.

of $GL_2(\mathbb{Z}_p)$

Breuil : mod-p / p-adic Banach rep. of $GL_2(\mathbb{Q}_p)$

$\updownarrow$

geom. 2-dim p-adic rep. of $G_{\mathbb{Q}_p}$

Colmez : studied various Banach space

completions of Breuil

showed non-zero using (φ, Γ)-theory

Since (φ, Γ)-theory applies to all Gal rep.

Colmez propose p-adic Langlands to all

2-dim rep. $G_{\mathbb{Q}_p} \to GL_2(E)$

and constructed functor

$\{\text{suitable } GL_2(\mathbb{Q}_p)\text{-rep.}\} \to \{\text{rep. of } G_{\mathbb{Q}_p}\}$

Kisin investigated Breuil-Mezard conj.

suggested Colmez that generalized

p-adic Langlands should take place over

local Gal def. spaces

Then Colmez constructed for general ρ

the rep. $\Pi(\rho)$ using (φ, Γ)-theory

Emerton : introduced the completed coh. gp

inspired by Breuil

formulated local-global compatibility

for $\hat{H}$ using p-adic Langlands

After Colmez & Kisin

Emerton was able to prove most of the conj. leading to a new proof of Fontaine-Mazur conj. (past).

Shortbacks: • technical issue with p-adic Langlands

e.g. $p = 2, 3$

• T-W hypothesis $\bar{\rho}$ irreducible

(Lue Pan ...)