
*1. What is $\sup \{\frac{n-1}{n} \mid n \in \mathbb{N} \setminus \{0\}\}$? What is $\inf \{\frac{n+1}{n} \mid n \in \mathbb{N} \setminus \{0\}\}$?
Prove your answers correct.

*2. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be even if $f(-x) = f(x)$ for all $x$, and odd if $f(-x) = -f(x)$ for all $x$.
   
   (a) Prove that if $f$ is both odd and even, then $f(x) = 0$ for all $x$.
   
   (b) Suppose $f$ is integrable on every closed interval $[a, b]$, and let $g(x) = \int_0^x f(t) \, dt$.
Prove that if $f$ is odd, then $g$ is even, and that if $f$ is even, then $g$ is odd.

*3. Prove that $f : [a, b] \rightarrow \mathbb{R}$ is integrable if and only if for all $\varepsilon > 0$, there exist step functions $s, t : [a, b] \rightarrow \mathbb{R}$ such that $s \leq f \leq t$ and $\int_a^b (t - s)(x) \, dx < \varepsilon$.
   
   (Hint: substitute $\varepsilon/2$ in the approximation property of the sup.)

*4. Prove that $\int_a^b x \, dx = (b^2 - a^2)/2$ in the following steps.
   
   (a) Use the properties of integration to show that the general case is implied by the case where $a = 0$ and $b = 1$.
   
   (b) Establish that $\int_0^1 x \, dx = 1/2$. (Hint: previous exercises may be useful.)

*5. If $a \leq c \leq d \leq b \in \mathbb{R}$, and $f : [a, b] \rightarrow \mathbb{R}$ is integrable on $[a, b]$, prove that it is integrable on $[c, d]$. (Hint: previous exercises may be useful.)

*6. Suppose that $f$ is integrable on $[a, b]$. Show that $|f|$ is integrable on $[a, b]$.
   
   (Hint: if $s$ and $t$ are step functions such that $s \leq f \leq t$ and $\int (t - s) \, dx < \varepsilon$, and if a partition for both $s$ and $t$ is chosen, what step functions with this partition best approximate $f$ above and below? Their definition will involve several cases.)

*7. For all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, define the $n$th power $x^n \in \mathbb{R}$ recursively (that is, inductively) by $x^0 = 1$ and $x^{n+1} = x \cdot x^n$.
   
   (a) Using this definition, prove that the function $f(x) = x^n$ is monotone on $(-\infty, 0]$, and also on $[0, \infty)$.
   
   (b) Prove that this function is integrable on any closed interval $[a, b]$.
   
   (c) Prove that any polynomial function $g(x) = \sum_{i=0}^n c_i x^i$, where the $c_i$ are constants, is integrable on $[a, b]$.
