What is a spectral sequence?

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Very short answer: A spectral sequence is a piece of terminology used by algebraists to intimidate other mathematicians.

Short answer / slogan: A spectral sequence is a machine which breaks difficult homological calculations apart into many easier (?) ones, and then glues their results together.

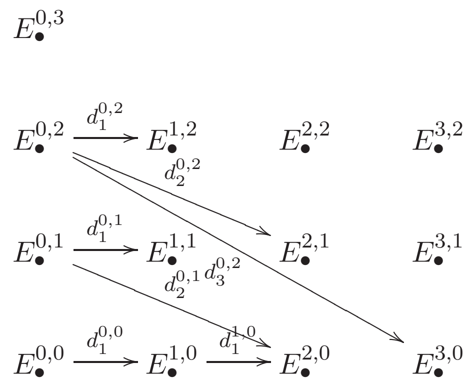
Long answer: A (first quadrant cohomology) E_r-page spectral sequence consists of the following data:

- A collection of abelian groups $E_s^{i,j}$ for integers $i \geq 0$, $j \geq 0$, $s \geq r$. The data of $E_s^{i,j}$ for s fixed and varying i, j is referred to as *the E_s-page.*

- Homomorphisms of abelian groups $d_s^{i,j} : E_s^{i,j} \to E_s^{i+s,j+1-s}$, the *differentials*, such that $\mathrm{im} d_s^{i-s,j-1+s} \subset \ker d_s^{i,j}$. The differentials into $E_s^{i,j}$ are trivial for $s > i$, and the differentials out of $E_s^{i,j}$ are trivial if $s > j+1$.

- Canonical isomorphisms $E_{s+1}^{i,j} \simeq \frac{\ker d_s^{i,j}}{\mathrm{im} d_s^{i-s,j-1+s}}$.

You can and should think of the $E_s^{i,j}$'s for fixed s as indexed by integer points in the first quadrant of the plane; the condition on the differentials says that the groups lying along each line of slope $-\frac{s-1}{s}$ form a complex, and the E_{s+1}-page formed by taking the cohomology of the E_s-page. Note also that each $E_{s+1}^{i,j}$ is a quotient of a subgroup of $E_s^{i,j}$, and that eventually every differential entering or leaving a fixed entry $E_s^{i,j}$ is identically zero, because either the i-coordinate of the source or the j-coordinate of the target is negative. In other words, the entries $E_s^{i,j}$ evolve as you "turn the pages", but eventually they stop changing: they *stabilize,* and we write $E_\infty^{i,j}$ for the stable value of $E_s^{i,j}$. The following picture must be regarded:



Note that no entry on the E_1-page has stabilized, since a differential leaves every entry on the page. However, on the E_2-page, the entries $E_2^{0,0}$ and $E_2^{1,0}$ have stabilized, so e.g. $E_\infty^{1,0} = E_2^{1,0}$.

This mass of data *abuts to an abelian group* $H^* = \oplus_{n\geq 0} H^n$ if there exists a descending chain of subgroups

$$H^n = \mathrm{Fil}^0 H^n \supset \mathrm{Fil}^1 H^n \supset \cdots \supset \mathrm{Fil}^i H^n \supset \cdots \supset \mathrm{Fil}^n H^n \supset \mathrm{Fil}^{n+1} H^n = 0$$

together with canonical isomorphisms $\mathrm{Fil}^{i+1} H^n \backslash \mathrm{Fil}^i H^n \simeq E_\infty^{i,n-i}$; more colloquially, H^n is determined *up to extension data* by the groups $E_\infty^{i,j}$ with $i+j=n$. Allllll this information is traditionally abbreviated into the notation

$$E_r^{i,j} \Rightarrow H^{i+j}.$$

The vast majority of spectral sequences are E_2-page spectral sequences, which is to say you begin with the data of the entries on the E_2-page.

Examples.

(Serre) Let $f : E \to B$ be a continuous map of topological spaces (say of CW complexes). This map f is a *Serre fibration* if the fibers $F = F_b = f^{-1}(b)$, $b \in B$ "satisfying the homotopy lifting property for CW complexes" - fiber bundles are a basic example. Given a Serre fibration, the cohomology groups $H^i(F_b, \mathbf{Q})$ naturally form the stalks of a locally constant sheaf $H^i(F)$ over B, and Serre constructed a spectral sequence

$$E_2^{i,j} = H^i(B, H^j(F)) \Rightarrow H^{i+j}(E, \mathbf{Q}).$$

The Serre spectral sequence has some very important extra structure, namely maps

$$E_r^{i,j} \times E_r^{k,l} \to E_r^{i+k,j+l}$$

which agree on the E_2-page with $(-1)^{jl}$ times the map induced by the cup product maps on the cohomologies of the base and fiber.

Exercise: If $F \to E \to B$ is a fiber bundle, with B path-connected and E orientable, show the equality

$$\chi(E) = \chi(B)\chi(F)$$

where χ denotes Euler characteristic and $\chi(F)$ is the Euler characteristic of "the" fiber.

Exercise: Using the family of fiber bundles $\mathrm{SO}(n-1) \to \mathrm{SO}(n) \to S^{n-1}$, inductively calculate the cohomology ring of $\mathrm{SO}(n)$.

(Grothendieck) Let X be a topological space, with $\mathcal{F} \to X$ a sheaf of abelian groups. Let $\mathcal{U}$ be a covering of X by open sets. Given any open set $U \subset X$, the rule

$$\begin{aligned} \mathcal{H}^j(-, \mathcal{F}) : \text{opens in } X &\to \text{abelian groups} \\ U &\mapsto H^j(U, \mathcal{F}|_U) \end{aligned}$$

defines a presheaf on X. Cech cohomology makes sense with coefficients in any presheaf, and we have the *Cech-to-sheaf spectral sequence*

$$E_2^{i,j} = \check{H}^i(\mathcal{U}, \mathcal{H}^j(-, \mathcal{F})) \Rightarrow H^{i+j}(X, \mathcal{F}).$$

Exercise: If $\mathcal{U} = \{U_1, U_2\}$ consists of two open sets, prove that the Cech-to-sheaf spectral sequence yields the Mayer-Vietoras long exact sequence

$$\cdots \to H^i(X, \mathcal{F}) \to H^i(U_1, \mathcal{F}) \oplus H^i(U_2, \mathcal{F}) \to H^i(U_1 \cap U_2, \mathcal{F}) \to H^{i+1}(X, \mathcal{F}) \to \dots.$$

Can you show anything when $\mathcal{U}$ consists of three open sets?

Exercise: The covering $\mathcal{U}$ is a *Leray covering relative to* $\mathcal{F}$ if $H^j(U, \mathcal{F}|_U) = 0$ for all $j > 0$ and all $U \in \mathcal{U}$. Supposing $\mathcal{U}$ is a Leray covering, show the spectral sequence degenerates to an isomorphism

$$H^i(X, \mathcal{F}) \simeq \check{H}^i(\mathcal{U}, \mathcal{F}).$$

(Grothendieck) Let X be a topological space, with $\mathcal{F} \to X$ a sheaf of abelian groups. Recall that $H^i(X, \mathcal{F})$ is defined as follows: one chooses a long exact sequence $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \dots$ with each $\mathcal{I}^i$ an injective sheaf, and $H^i(X, \mathcal{F})$ is the ith cohomology group of the complex

$$0 \to \Gamma(X, \mathcal{I}^0) \to \Gamma(X, \mathcal{I}^1) \to \Gamma(X, \mathcal{I}^2) \to \dots$$

Given a complex $\mathcal{G}^\bullet = 0 \to \mathcal{G}^0 \to \mathcal{G}^1 \to \dots$ of sheaves, the ith *cohomology sheaf of* $\mathcal{G}^\bullet$ is the sheaf $\mathcal{H}^i(\mathcal{G}^\bullet)$ defined as the sheafification of the presheaf

$$U \mapsto \frac{\ker \Gamma(U, \mathcal{G}^i) \to \Gamma(U, \mathcal{G}^{i+1})}{\mathrm{im}\Gamma(U, \mathcal{G}^{i-1}) \to \Gamma(U, \mathcal{G}^i)}.$$

In the setup above, note that the complexes $\mathcal{F}^\bullet : 0 \to \mathcal{F} \to 0$ and $\mathcal{I}^\bullet : 0 \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \dots$ have the same cohomology sheaves, namely $\mathcal{H}^0(\mathcal{F}^\bullet) \simeq \mathcal{H}^0(\mathcal{I}^\bullet) \simeq \mathcal{F}$ and $\mathcal{H}^i(\mathcal{F}^\bullet) \simeq \mathcal{H}^i(\mathcal{I}^\bullet) \simeq 0 \, \forall i \geq 1$ - in other words, resolving $\mathcal{F}$ by injective sheaves is equivalent to finding a complex $\mathcal{I}^\bullet$ of injective sheaves with $\mathcal{H}^0(\mathcal{I}^\bullet) \simeq \mathcal{F}$, $\mathcal{H}^i(\mathcal{I}^\bullet) \simeq 0 \, \forall i \geq 1$.

Definition / Theorem: *Given any complex of sheaves $\mathcal{F}^\bullet$, there exists a complex $\mathcal{I}^\bullet$ of injective sheaves with $\mathcal{H}^i(\mathcal{I}^\bullet) \simeq \mathcal{H}^i(\mathcal{F}^\bullet) \, \forall i \geq 0$; call any such complex an* injective resolution *of $\mathcal{F}^\bullet$. Choose an injective resolution $\mathcal{I}^\bullet$ of $\mathcal{F}^\bullet$ and define $\mathbf{H}^i(X, \mathcal{F}^\bullet)$ as the ith cohomology group of the complex*

$$0 \to \Gamma(X, \mathcal{I}^0) \to \Gamma(X, \mathcal{I}^1) \to \Gamma(X, \mathcal{I}^2) \to \dots ;$$

then $\mathbf{H}^i(X, \mathcal{F}^\bullet)$ a well-defined abelian group (i.e. it doesn't depend on the specific injective resolution $\mathcal{I}^\bullet$ we chose).

The group $\mathbf{H}^i(X, \mathcal{F}^\bullet)$ is the ith *hypercohomology* group of the complex $\mathcal{F}^\bullet$. There are two spectral sequences which abut to $\mathbf{H}^*(X, \mathcal{F}^\bullet)$, namely

$$E_2^{i,j} = H^i(X, \mathcal{H}^j(\mathcal{F}^\bullet)) \Rightarrow \mathbf{H}^{i+j}(X, \mathcal{F}^\bullet)$$

and

$$E_1^{i,j} = H^j(X, \mathcal{F}^i) \Rightarrow \mathbf{H}^{i+j}(X, \mathcal{F}^\bullet).$$

Example: Let X be a smooth complex manifold of complex dimension n. The *holomorphic de Rham complex of* X is the complex

$$\Omega_X^\bullet : 0 \to \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \cdots \to \Omega_X^n \to 0,$$

where Ω_X^i is the sheaf of holomorphic i-forms. The holomorphic functions with zero derivative are the locally constant functions, so $\mathcal{H}^0(\Omega_X^\bullet) \simeq \underline{\mathbf{C}}$; furthermore, a holomorphic analogue of the usual Poincare lemma yields $\mathcal{H}^i(\Omega_X^\bullet) = 0 \, \forall i \geq 1$. Hence the *first* hypercohomology spectral sequence degenerates to an isomorphism

$$H^i_{\text{sing}}(X, \mathbf{C}) \simeq H^i(X, \underline{\mathbf{C}}) \simeq \mathbf{H}^i(X, \Omega_X^\bullet),$$

so the *second* hypercohomology spectral sequence now reads

$$E_1^{i,j} = H^j(X, \Omega_X^i) \Rightarrow H^{i+j}_{\text{sing}}(X, \mathbf{C}).$$

In other words, sheaf cohomology of the holomorphic objects Ω_X^i suffices to compute the singular cohomology of X! This, to me at least, is much more surprising than the de Rham isomorphism - arbitrary smooth differential forms are very mushy objects, but holomorphic differential forms are extremely rigid.