Period maps, variations of p-adic Hodge structure, and Newton stratifications (research announcement)

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Let X be a connected complex manifold, and let $H = (H_{\mathbf{Z}}, \mathrm{Fil}^\bullet \subset H_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathcal{O}_X)$ be a rank n variation of Hodge structure over X. Let $\tilde{X}$ be the $\mathrm{GL}_n(\mathbf{Z})$-covering of X parametrizing trivializations $\underline{\mathbf{Z}}^n \overset{\sim}{\to} H_{\mathbf{Z}}$, and let $\mathrm{Gr}_{\mathbf{h}} = \mathrm{GL}_n(\mathbf{C})/P_{\mathbf{h}}(\mathbf{C})$ be the Grassmannian parametrizing flags in $\mathbf{C}^n$ with successive graded pieces having ranks $\mathbf{h} = \{h_i = \mathrm{rank}\,\mathrm{Fil}^i/\mathrm{Fil}^{i+1}\}$. Then we have a natural *period morphism*

$$\pi : \tilde{X} \to \mathrm{Gr}_{\mathbf{h}}$$

of complex manifolds, equivariant for the natural $\mathrm{GL}_n(\mathbf{Z})$-actions on the source and target.

In this note we sketch an analogous p-adic construction and some of its consequences for Shimura varieties. We came to these ideas while trying to understand Ana Caraiani and Peter Scholze's recent paper.

Precisely, if X is a rigid analytic space and $\mathbf{V}$ is a $\mathbf{Q}_p$-local system of rank n on X, we construct a space $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ parametrizing trivializations of $\mathbf{V}$. When X is smooth and $\mathbf{V}$ is *de Rham*, we construct a canonical p-adic period morphism from $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ to a *de Rham affine Grassmannian*. We'd like to say that $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ is a $\mathrm{GL}_n(\mathbf{Q}_p)$-torsor over X, but unfortunately it's not clear whether $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ is even a reasonable adic space. The correct setting here seems to be that of *diamonds*: we prove that $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ is naturally a diamond, and that its natural map to the diamond $X^\diamond$ associated with X is a pro-étale $\mathrm{GL}_n(\mathbf{Q}_p)$-torsor. The necessity of changing to a "looser" category of geometric objects here seems roughly analogous to the fact that, in the complex analytic situation above, $\tilde{X} \to X$ is defined only in the complex analytic category even when X begins its life as a projective variety. We also remark that the target of the period morphism out of $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ *only* exists as a diamond in general; analogously, the open subspace of $\mathrm{Gr}_{\mathbf{h}}$ parametrizing Hodge structures (and through which $\pi : \tilde{X} \to \mathrm{Gr}_{\mathbf{h}}$ factors) is a non-algebraic complex manifold.

Torsors for $\mathbf{Q}_p$-local systems, and period maps

We start with a very brief recollection on $\mathbf{Q}_p$-local systems.

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Proposition 1.1. *Let X be a locally Noetherian adic space, or a perfectoid space, or a diamond. Then we have a natural category $\mathbf{Q}_p\mathrm{Loc}(X)$ of $\mathbf{Q}_p$-local systems on X; these are (certain) sheaves of $\mathbf{Q}_p$-vector spaces on the pro-étale site X_{proet} which are locally free of finite rank. When X is locally Noetherian or perfectoid, with associated diamond $X^{\Diamond}$, there is a natural equivalence $\mathbf{Q}_p\mathrm{Loc}(X) \cong \mathbf{Q}_p\mathrm{Loc}(X^{\Diamond})$.*

These arise quite naturally in geometry: if $f : Y \to X$ is a proper smooth morphism of locally Noetherian adic spaces over $\mathbf{Q}_p$, then any $R^i f_{\mathrm{proet}*}\mathbf{Q}_p$ is naturally a $\mathbf{Q}_p$-local system on X (Gabber, Kedlaya-Liu, Scholze-Weinstein). We also remark that if X is a variety over $\mathbf{Q}_p$, then any lisse $\mathbf{Q}_p$-sheaf on X_{et} induces a $\mathbf{Q}_p$-local system on X^{ad}.

Our first result is the following (not very difficult) theorem.

Theorem 1.2. *Let $\mathcal{D}$ be any diamond, and let $\mathbf{V}$ be a $\mathbf{Q}_p$-local system on $\mathcal{D}$ of constant rank n. Then the functor*

$$\begin{aligned} \mathcal{T}\mathrm{riv}_{\mathbf{V}/\mathcal{D}} : \mathrm{Perf}_{/\mathcal{D}} &\to \mathrm{Sets} \\ \{f : T \to \mathcal{D}\} &\mapsto \mathrm{Isom}_{\mathbf{Q}_p\mathrm{Loc}(T)}(\underline{\mathbf{Q}_p}^n, f^*\mathbf{V}) \end{aligned}$$

is representable by a diamond pro-étale over $\mathcal{D}$, and the natural map $\mathcal{T}\mathrm{riv}_{\mathbf{V}/\mathcal{D}} \to \mathcal{D}$ is a pro-étale $\mathrm{GL}_n(\mathbf{Q}_p)$-torsor. The functor from rank n $\mathbf{Q}_p$-local systems on $\mathcal{D}$ to pro-étale $\mathrm{GL}_n(\mathbf{Q}_p)$-torsors over $\mathcal{D}$ given by $\mathbf{V} \mapsto \mathcal{T}\mathrm{riv}_{\mathbf{V}/\mathcal{D}}$ is an equivalence of categories, with essential inverse given by $\tilde{\mathcal{D}} \mapsto \tilde{\mathcal{D}} \times_{\underline{\mathrm{GL}_n(\mathbf{Q}_p)}} \underline{\mathbf{Q}_p}^n$.

Next we recall the definition of a *de Rham $\mathbf{Q}_p$-local system* on a smooth rigid analytic space X, following the notation and terminology in Scholze's p-adic Hodge theory paper. This seems to be the correct p-adic analogue of a variation of ($\mathbf{Q}-$ or $\mathbf{Z}-$)Hodge structures on a complex manifold.

Definition 1.3. Let $E/\mathbf{Q}_p$ be a discretely valued nonarchimedean field with perfect residue field of characteristic p, and let X be a smooth rigid analytic space over $\mathrm{Spa}\, E$. Let $\nu : X_{\mathrm{proet}} \to X_{\mathrm{et}}$ be the usual projection of sites, and let $\mathbb{B}_{\mathrm{dR}}^+, \mathbb{B}_{\mathrm{dR}}, \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+, \mathcal{O}\mathbb{B}_{\mathrm{dR}}$ be the usual period sheaves on X_{proet}. Given a $\mathbf{Q}_p$-local system $\mathbf{V}$ on X_{proet}, define

$$\mathbf{D}_{\mathrm{dR}}(\mathbf{V}) = \nu_*(\mathbf{V} \otimes_{\underline{\mathbf{Q}_p}} \mathcal{O}\mathbb{B}_{\mathrm{dR}}).$$

This is a locally free $\mathcal{O}_{X_{\mathrm{et}}}$-module of finite rank, equipped with a decreasing exhaustive separated filtration by $\mathcal{O}_{X_{\mathrm{et}}}$-local-direct summands and with an integrable connection satisfying Griffiths transversality. There is a natural injective map

$$\alpha_{\mathrm{dR}} : \nu^*\mathbf{D}_{\mathrm{dR}}(\mathbf{V}) \otimes_{\nu^*\mathcal{O}_{X_{\mathrm{et}}}} \mathcal{O}\mathbb{B}_{\mathrm{dR}} \to \mathbf{V} \otimes_{\underline{\mathbf{Q}_p}} \mathcal{O}\mathbb{B}_{\mathrm{dR}}$$

of $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$-modules compatible with all structures. We say $\mathbf{V}$ is *de Rham* if α_{dR} is an isomorphism.

Maintain the setup of the previous definition, and assume $\mathbf{V}$ is de Rham. Then we get two $\mathbb{B}_{\mathrm{dR}}^+$-lattices in the $\mathbb{B}_{\mathrm{dR}}$-local system $\mathbf{V} \otimes_{\underline{\mathbf{Q}_p}} \mathbb{B}_{\mathrm{dR}}$: one given by $\mathbf{M} := \mathbf{V} \otimes_{\underline{\mathbf{Q}_p}} \mathbb{B}_{\mathrm{dR}}^+$ and the other given by the image of

$$\left(\nu^*\mathbf{D}_{\mathrm{dR}}(\mathbf{V}) \otimes_{\nu^*\mathcal{O}_{X_{\mathrm{et}}}} \mathcal{O}\mathbb{B}_{\mathrm{dR}}^+\right)^{\nabla=0}$$

under the isomorphism

$$\alpha_{\mathrm{dR}}^{\nabla=0} : \left(\nu^*\mathbf{D}_{\mathrm{dR}}(\mathbf{V}) \otimes_{\nu^*\mathcal{O}_{X_{\mathrm{et}}}} \mathcal{O}\mathbb{B}_{\mathrm{dR}}\right)^{\nabla=0} \xrightarrow{\sim} \mathbf{V} \otimes_{\underline{\mathbf{Q}_p}} \mathbb{B}_{\mathrm{dR}}$$

induced by α_{dR}. Call this second one $\mathbf{M}_0$. When $\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(\mathbf{V}) = \mathbf{D}_{\mathrm{dR}}(\mathbf{V})$, we have $\mathbf{M}_0 \subseteq \mathbf{M}$. More generally, let

$$h_i = \mathrm{rank}_{\mathcal{O}_X} \mathrm{gr}^i \mathbf{D}_{\mathrm{dR}}(\mathbf{V}) = \mathrm{rank}_{\mathcal{O}_X}(\mathrm{Fil}^i \mathbf{D}_{\mathrm{dR}}/\mathrm{Fil}^{i+1}\mathbf{D}_{\mathrm{dR}})$$

be the Hodge numbers of $\mathbf{V}$, and let $\mu_{\mathbf{V}} : \mathbf{G}_m \to \mathrm{GL}_n$ be the cocharacter in which the weight i appears with multiplicity h_i. We remark that the h_i's are constant on connected components of X, and thus $\mu_{\mathbf{V}}$ is as well.

Proposition 1.4. *The relative positions of* $\mathbf{M}_0$ *and* $\mathbf{M}$ *inside* $\mathbf{V} \otimes_{\underline{\mathbf{Q}_p}} \mathbb{B}_{\mathrm{dR}}$ *are given by* $\mu_{\mathbf{V}}$.

With the setup as above, we now have the following theorem.

Theorem 1.5. *Let* X *be a smooth rigid analytic space over* $\mathrm{Spa}\, E$ *as above, and let* $\mathbf{V}$ *be a de Rham* $\mathbf{Q}_p$*-local system on* X *of rank* n*. Assume that* X *is connected, or more generally that the Hodge cocharacter* $\mu_{\mathbf{V}}$ *is constant. Let* $\mathrm{Gr}_{\mathrm{GL}_n,\mu_{\mathbf{V}}}$ *be the open Schubert cell in the (ind-)diamond* $\mathrm{Gr}_{\mathrm{GL}_n}$*. Then we have a natural* $\mathrm{GL}_n(\mathbf{Q}_p)$*-equivariant period morphism*

$$\pi_{\mathrm{dR}} : \mathcal{T}\mathrm{riv}_{\mathbf{V}/X^\diamond} \to \mathrm{Gr}_{\mathrm{GL}_n,\mu_{\mathbf{V}}}$$

of diamonds over $\mathrm{Spd}\, E$.

Corollary 1.6. *Let* X *be a connected rigid analytic space over* $\mathrm{Spa}\, E$ *as above, and let* $\mathbf{V}$ *be a de Rham* $\mathbf{Q}_p$*-local system on* X *of rank* n *with constant Hodge cocharacter* $\mu_{\mathbf{V}}$*. Then* $\mathbf{V}$ *induces a natural* Newton stratification *of* X *into locally closed subsets*

$$|X| = \coprod_{b \in B(\mathrm{GL}_n/\mathbf{Q}_p,\mu_{\mathbf{V}})} |X|^b$$

indexed by Newton polygons lying above the Hodge polygon of $\mathbf{V}$ *and with matching endpoints.*

Proof. Pulling back the Newton strata of $\mathrm{Gr}_{\mathrm{GL}_n,\mu_{\mathbf{V}}}$ defined by Caraiani-Scholze[1] under π_{dR}, we get a stratification of $|\mathcal{T}\mathrm{riv}_{\mathbf{V}/X^\diamond}|$ by $\mathrm{GL}_n(\mathbf{Q}_p)$-stable locally closed subsets, which descend under the identifications

$$|\mathcal{T}\mathrm{riv}_{\mathbf{V}/X^\diamond}|/\mathrm{GL}_n(\mathbf{Q}_p) \cong |X^\diamond| \cong |X|$$

to a stratification of $|X|$. □

Remark. The subsets $|X|^b$ are determined by their rank one points: $x \in |X|$ lies in a given $|X|^b$ if and only if its unique rank one generization $\tilde{x}$ lies in $|X|^b$. Also, it's not clear whether the closure of any $|X|^b$ is a union of $|X|^{b'}$'s, so the term "stratification" is being used in a loose sense here.

Next we compare this "generic fiber" Newton stratification with a suitable "special fiber" Newton stratification, in a situation where the latter exists in a reasonably canonical way. More precisely, let $\mathbf{f} : \mathfrak{Y} \to \mathfrak{X}$ be a smooth proper morphism of connected flat p-adic formal schemes over $\mathrm{Spf}\, \mathcal{O}_E$, with associated morphism of adic generic fibers $f : Y \to X$ over $\mathrm{Spa}\, E$. Assume X is smooth. Let $\mathcal{M} = R^i\mathbf{f}_{\mathrm{crys}*}(\mathcal{O}/W)[\frac{1}{p}]$ be the ith relative crystalline cohomology, and let $\mathbf{V} = R^i f_{\mathrm{proet}*}\mathbf{Q}_p$ be the ith relative p-adic étale cohomology. Then, on the one hand, the Frobenius action on the specializations of $\mathcal{M}$ at geometric points of $\mathfrak{X}$ gives a natural stratification

$$|\mathfrak{X}| = \coprod_{b \in B(\mathrm{GL}_n/\mathbf{Q}_p,\mu_{\mathbf{V}})} |\mathfrak{X}|^b.$$

[1]Warning: Here and in what follows, Caraiani-Scholze use $B(\mathbf{G},\mu^{-1})$ everywhere that we use $B(\mathbf{G},\mu)$, but everything matches after noting that the two are in bijection via $b \mapsto b^{-1}$.

In this context, the fact that the Newton polygon lies over the Hodge polygon is a famous theorem of Mazur. On the other hand, the previous corollary applied to $\mathbf{V}$ gives a decomposition $|X| = \coprod_{b\in B(\mathrm{GL}_n/\mathbf{Q}_p,\mu_{\mathbf{V}})} |X|^b$ with the same indexing set.

Theorem 1.7. *The stratification of $|X|$ associated with $\mathbf{V}$ agrees on rank one points with the pullback (under the specialization map $\mathbf{s} : |X| \to |\mathfrak{X}|$) of the stratification of $|\mathfrak{X}|$ associated with $\mathcal{M}$.*

We sketch the proof. Let $x = \mathrm{Spa}(K, \mathcal{O}_K) \to X$ be any rank one point, and choose a geometric point $\overline{x} = \mathrm{Spa}(C, \mathcal{O}_C)$ lying over it, with corresponding point $\overline{x} : \mathrm{Spf}\, \mathcal{O}_C \to \mathfrak{X}$. Let κ be the residue field of K, giving a point $s = \mathrm{Spec}\, \kappa \to \mathfrak{X}$. Let $k = \mathcal{O}_C/\mathfrak{m}_C = \mathcal{O}_{C^\flat}/\mathfrak{m}_{C^\flat}$ be the residue field of $\mathcal{O}_C$, so we get a natural geometric point $\overline{s} : \mathrm{Spec}\, k \to \mathfrak{X}$ over s. Note that $|s| = |\overline{s}| = |\overline{x}|$ as points in the topological space $|\mathfrak{X}|$, and that the specialization map $\mathbf{s}$ sends x to this point. Set $L = W(k)[\frac{1}{p}]$ and $\mathbb{A} = W(\mathcal{O}_{C^\flat})$, so the surjection $\mathcal{O}_{C^\flat} \twoheadrightarrow k$ induces a surjection $\mathbb{A}[\frac{1}{p}] \twoheadrightarrow L$.

Specializing the comparison map α_{dR} at $\overline{x}$ (and noting that $\mathbf{V}_{\overline{x}} = H^i_{\mathrm{et}}(Y_{\overline{x}}, \mathbf{Q}_p)$), we get a pair (T, Ξ) consisting of a finite free $\mathbf{Z}_p$-module $T = H^i_{\mathrm{et}}(Y_{\overline{x}}, \mathbf{Z}_p)_{/\mathrm{tors}} \subset \mathbf{V}_{\overline{x}}$ together with a $\mathbb{B}^+_{\mathrm{dR}}(C)$-lattice[2]

$$\Xi = \mathbf{M}_{0,\overline{x}} = \text{“}H^i_{\mathrm{dR}}(Y_x) \otimes_K \mathbb{B}^+_{\mathrm{dR}}(C)\text{”} \subset T \otimes_{\mathbf{Z}_p} \mathbb{B}_{\mathrm{dR}}(C).$$

We call such pairs *Fargues pairs,* in recognition of the following theorem:

Theorem 1.8 (Fargues)**.** *The following categories are naturally equivalent:*

i) *Pairs (T, Ξ) as above.*

ii) *Breuil-Kisin modules over $\mathbb{A} := W(\mathcal{O}_{C^\flat})$.*

iii) *Shtukas over $\mathrm{Spa}\, C^\flat$ with one paw at C.*

Applying this theorem to the particular Fargues pair associated with $\overline{x}$ above, we get a Breuil-Kisin module M over $\mathbb{A}$; specializing $M[\frac{1}{p}]$ along $\mathbb{A}[\frac{1}{p}] \twoheadrightarrow L$ gives a φ-isocrystal M_0 over L, whose Newton polygon turns out (after unwinding the definition of the Newton strata of $\mathrm{Gr}_{\mathrm{GL}_n,\mu_{\mathbf{V}}}$ and the equivalences in Fargues's theorem) to record exactly which stratum of $|X|$ contains x. On the other hand, Bhatt-Morrow-Scholze have recently defined a remarkable cohomology functor $H^i_{\mathbb{A}}(-)$ on smooth proper formal schemes over $\mathrm{Spf}\, \mathcal{O}_C$ taking values in Breuil-Kisin modules over $\mathbb{A}$; combining their work with Fargues's theorem, we get an identification $M[\frac{1}{p}] \cong H^i_{\mathbb{A}}(\mathfrak{Y}_{\overline{x}})[\frac{1}{p}]$. Since Bhatt-Morrow-Scholze's work also gives a canonical identification $H^i_{\mathbb{A}}(\mathfrak{Y}_{\overline{x}}) \otimes_{\mathbb{A}} L \cong H^i_{\mathrm{crys}}(\mathfrak{Y}_{\overline{s}}/W(k))[\frac{1}{p}]$ compatible with Frobenius, we deduce that the φ-isocrystal M_0 coincides with the ith rational crystalline cohomology of $\mathfrak{Y}_{\overline{s}}$. Since the Newton polygon of the latter determines the stratum of $|\mathfrak{X}|$ containing $|\overline{s}| = \mathbf{s}(x)$, we're done.

With applications in mind, we also consider the following situation. Let $\mathbf{G}$ be a reductive group over $\mathbf{Q}_p$, so $G = \mathbf{G}(\mathbf{Q}_p)$ is a locally profinite group. Then for X as in Proposition 1.1, we define G-local systems on X as exact additive tensor functors

$$\begin{aligned} \mathbf{V} : \mathrm{Rep}(\mathbf{G}) &\to \mathbf{Q}_p\mathrm{Loc}(X) \\ (W, \rho) &\mapsto \mathbf{V}_W \end{aligned}$$

in the obvious way.[3] This generality is not artificial; for example, polarized $\mathbf{Q}_p$-local systems correspond to GSp_{2n}- or GO_n-local systems, and we get even more general examples from Shimura

[2] One has to be a bit careful in the following expression since $- \otimes_K \mathbb{B}_{\mathrm{dR}}(C)$ doesn't quite make sense unless x is a classical rigid analytic point; hence the quotation marks.

[3] Here of course $\mathrm{Rep}(\mathbf{G})$ denotes the category of pairs (W, ρ) with W a finite-dimensional $\mathbf{Q}_p$-vector space and $\rho : \mathbf{G} \to \mathbf{GL}(W)$ a morphism of algebraic groups over $\mathbf{Q}_p$.

varieties (as we'll see below). When X is a diamond, the natural functor $\mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ is a pro-étale G-torsor over X, and we again prove that the association $\mathbf{V} \mapsto \mathcal{T}\mathrm{riv}_{\mathbf{V}/X}$ is an equivalence of categories, with essential inverse given by sending a pro-etale G-torsor $\tilde{X}$ over X to the G-local system

$$\begin{aligned} \mathbf{V}(\tilde{X}) : \mathrm{Rep}(\mathbf{G}) &\to \mathbf{Q}_p\mathrm{Loc}(X) \\ (W, \rho) &\mapsto \tilde{X} \times_{\underline{G},\rho} W. \end{aligned}$$

When X is a smooth rigid analytic space, we say a G-local system $\mathbf{V}$ is de Rham if the associated $\mathbf{Q}_p$-local systems $\mathbf{V}_W$ are de Rham for all $(W, \rho) \in \mathrm{Rep}(\mathbf{G})$. Using the fact that the comparison isomorphism α_{dR} is compatible with direct sums, tensor products, and subquotients, the Tannakian formalism (plus a little more) gives a Hodge cocharacter $\mu_{\mathbf{V}} : \mathbf{G}_{m,E} \to \mathbf{G}_E$ such that $\rho \circ \mu_{\mathbf{V}}$ measures the Hodge filtration on $\mathbf{D}_{\mathrm{dR}}(\mathbf{V}_W)$ for any (W, ρ). We then have the following generalization of Theorem 1.5.

Theorem 1.9. *Let $\mathbf{V}$ be a de Rham $G = \mathbf{G}(\mathbf{Q}_p)$-local system on a smooth rigid analytic space X, with constant Hodge cocharacter $\mu_{\mathbf{V}}$. Then we have a natural G-equivariant period morphism*

$$\pi_{\mathrm{dR}} : \mathcal{T}\mathrm{riv}_{\mathbf{V}/X^{\diamond}} \to \mathrm{Gr}_{\mathbf{G},\mu_{\mathbf{V}}}$$

of diamonds over $\mathrm{Spd}\, E$.

We again get a Newton stratification of $|X|$, indexed now by the Kottwitz set $B(\mathbf{G}, \mu_{\mathbf{V}})$ (which explains our use of this notation in the GL_n-setting earlier).

Some applications to Shimura varieties

Let $(\mathbf{G}, X)$ be a Shimura datum, with associated Hodge cocharacter μ and reflex field E. For any sufficiently small open subgroup $K \subset \mathbf{G}(\mathbf{A}_f)$, the associated Shimura variety Sh_K is a smooth quasiprojective variety over E. Choose a prime $\mathfrak{p}$ of E lying over p, and let $\mathcal{S}_K = (Sh_K \times_E E_{\mathfrak{p}})^{\mathrm{ad}}$ be the associated rigid analytic space over $\mathrm{Spa}\, E_{\mathfrak{p}}$. Let $\mathscr{S}_K = \mathcal{S}_K^{\diamondsuit}$ be the associated diamond over $\mathrm{Spd}\, E_{\mathfrak{p}}$. Finally, set $G = \mathbf{G}(\mathbf{Q}_p)$ as before.

Proposition 1.10. *For any sufficiently small $K^p \subset \mathbf{G}(\mathbf{A}_f^p)$, there is a diamond $\mathscr{S}_{K^p}$ over* $\mathrm{Spd}\, E_{\mathfrak{p}}$ *with an action of G such that*

$$\mathscr{S}_{K^p} \cong \varprojlim_{K_p} \mathscr{S}_{K^p K_p}$$

G-equivariantly and compatibly with changing K^p (or even with changing the Shimura data).

When $(\mathbf{G}, X)$ is of Hodge type, Scholze constructed a *perfectoid Shimura variety*, i.e. a perfectoid space $\mathcal{S}_{K^p}$ with continuous G-action such that

$$\mathcal{S}_{K^p} \sim \varprojlim_{K_p} \mathcal{S}_{K^p K_p}$$

as adic spaces. Whenever such an $\mathcal{S}_{K^p}$ exists, we necessarily have $\mathscr{S}_{K^p} \cong \mathcal{S}_{K^p}^{\diamondsuit}$ for formal reasons. However, unlike the difficult construction of $\mathcal{S}_{K^p}$, the proof of Proposition 1.10 is *essentially trivial* once the theory of diamonds is set up: the inverse limit of any projective system of diamonds with finite étale transition maps is a diamond. We also point out that knowledge of $\mathscr{S}_{K^p}$ is strictly weaker than knowledge of $\mathcal{S}_{K^p}$: the construction of the latter object also gives very rich information about the existence of certain affinoid coverings, formal models, compactifications, etc., and $\mathscr{S}_{K^p}$

doesn't a priori have the same applications to p-adic automorphic forms as $\mathcal{S}_{K^p}$. In any case, it seems likely that $\mathcal{S}_{K^p}$ always exists (and X. Shen has recently constructed such perfectoid Shimura varieties when $(\mathbf{G}, X)$ is of abelian type), but this is probably out of reach in general. Therefore, it might be surprising that the following result is within reach.

Theorem 1.11. *There is a natural G-equivariant period morphism*

$$\pi_{\mathrm{HT}} : \mathscr{S}_{K^p} \to \mathscr{F}\ell^{\diamondsuit}_{\mathbf{G},\mu}$$

of diamonds over $\operatorname{Spd} E_{\mathfrak{p}}$*, compatible with changing K^p and functorial in morphisms of* arbitrary *Shimura data. When $(\mathbf{G}, X)$ is of Hodge type (or more generally, of abelian type), this is the morphism of diamonds associated with the "refined Hodge-Tate period map"*

$$\pi_{\mathrm{HT}} : \mathcal{S}_{K^p} \to \mathscr{F}\ell_{\mathbf{G},\mu}$$

of Caraiani-Scholze.

We briefly sketch the construction of π_{HT}. For any fixed level K_p, the map $\mathscr{S}_{K^p} \to \mathscr{S}_{K^pK_p}$ is naturally a pro-étale K_p-torsor, so the pushout

$$\widetilde{\mathscr{S}_{K^pK_p}} := \mathscr{S}_{K^p} \times_{\underline{K_p}} \underline{G}$$

is naturally a pro-étale G-torsor over $\mathscr{S}_{K^pK_p}$, compatibly with varying K^pK_p. Note that via the G-action on $\mathscr{S}_{K^p}$, we get a canonical splitting

$$\widetilde{\mathscr{S}_{K^pK_p}} \cong \mathscr{S}_{K^p} \times K_p \backslash G$$

which is G-equivariant for the diagonal G-action on the right-hand side. By our previous equivalence of categories, $\widetilde{\mathscr{S}_{K^pK_p}}$ gives rise to a G-local system $\mathbf{V}$ over $\mathscr{S}_{K^pK_p}$, or equivalently over $\mathcal{S}_{K^pK_p}$, which we call the *tautological G-local system*. The crucial ingredient is then:

Theorem 1.12 (R. Liu-X. Zhu)**.** *The tautological G-local system $\mathbf{V}$ over $\mathcal{S}_{K^pK_p}$ is de Rham with Hodge cocharacter μ, for any Shimura variety.*

They prove, surprisingly, that if the stalk of a $\mathbf{Q}_p$-local system (or G-local system) $\mathbf{V}$ on a connected rigid analytic space is de Rham at one classical point (in the classical sense of Fontaine), then $\mathbf{V}$ is de Rham. For a Shimura variety, this allows one to check the de Rham property at "special points", where everything is explicit.

Thanks to this result, we can apply Theorem 1.9 to get a G-equivariant morphism of diamonds

$$\tilde{\pi}_{\mathrm{dR}} : \widetilde{\mathscr{S}_{K^pK_p}} \to \mathrm{Gr}_{\mathbf{G},\mu}$$

over $\operatorname{Spd} E_{\mathfrak{p}}$. One checks by hand, using the aforementioned splitting, that $\tilde{\pi}_{\mathrm{dR}}$ descends along the G-equivariant projection $\widetilde{\mathscr{S}_{K^pK_p}} \to \mathscr{S}_{K^p}$ to a G-equivariant morphism

$$\pi_{\mathrm{dR}} : \mathscr{S}_{K^p} \to \mathrm{Gr}_{\mathbf{G},\mu}$$

independent of the auxiliary choice of K_p. Finally, since μ is minuscule, a result of Caraiani-Scholze gives a canonical identification $\mathrm{Gr}_{\mathbf{G},\mu} \cong \mathscr{F}\ell^{\diamondsuit}_{\mathbf{G},\mu}$, and we define π_{HT} as the composite of π_{dR} with this isomorphism.

We remark that when $\mathscr{S}_{K^p}$ is the diamond of a perfectoid Shimura variety $\mathcal{S}_{K^p}$, any morphism of diamonds

$$f^{\Diamond} : \mathscr{S}_{K^p} \cong \mathcal{S}_{K^p}^{\Diamond} \to \mathscr{F}\!\ell_{\mathbf{G},\mu}^{\Diamond}$$

arises uniquely from a morphism

$$f : \mathcal{S}_{K^p} \to \mathscr{F}\!\ell_{\mathbf{G},\mu}$$

of adic spaces. In particular, the proof of Theorem 1.11 actually gives a new construction of the refined Hodge-Tate period map for perfectoid Shimura varieties of Hodge type (or abelian type); this proof is of course related to Caraiani-Scholze's proof, but the details are rather different.

Again $\mathrm{Gr}_{\mathbf{G},\mu} \cong \mathscr{F}\!\ell_{\mathbf{G},\mu}^{\Diamond}$ has a Newton stratification by G-invariant locally closed subsets, indexed by $B(\mathbf{G},\mu)$, so pulling back under π_{HT} and descending gives a "generic fiber" Newton stratification of $|\mathcal{S}_{K^pK_p}|$ as before. Suppose now that $(\mathbf{G},X)$ is a Shimura datum of Hodge type, with $p>2$ and $\mathbf{G}/\mathbf{Q}_p$ unramified. Let $K_p \subset G$ be hyperspecial, so by work of Kisin, $Sh_{K^pK_p} \times_E E_{\mathfrak{p}}$ has (among other things) a good integral model $S^{\circ}_{K^pK_p}$ over $\mathcal{O}_{E_{\mathfrak{p}}}$ and the special fiber of $S^{\circ}_{K^pK_p}$ has a natural Newton stratification. Let $\mathfrak{S}_{K^pK_p}$ be the formal scheme given as the p-adic completion of $S^{\circ}_{K^pK_p}$, so the associated rigid analytic space identifies with the locus of good reduction $\mathcal{S}^{gd}_{K^pK_p} \subset \mathcal{S}_{K^pK_p}$. Again we have a specialization map

$$\mathbf{s} : |\mathcal{S}^{gd}_{K^pK_p}| \to |\mathfrak{S}_{K^pK_p}| = |S^{\circ}_{K^pK_p} \times_{\mathcal{O}_{E_{\mathfrak{p}}}} \mathbf{F}_q|.$$

Theorem 1.13. *The generic fiber Newton strata of $|\mathcal{S}^{gd}_{K^pK_p}|$ coincide on rank one points with the pullback under $\mathbf{s}$ of the special fiber Newton strata.*

The proof naturally combines (the ideas in the proof of) Theorem 1.7 with the Tannakian formalism and some unwinding of Kisin's construction. When the Sh_K's are compact PEL Shimura varieties of type A or C, this is a result of Caraiani-Scholze (cf. Section 4.3 of their paper, especially Lemma 4.3.20 and the diagram immediately following).