Pairings on modules of analytic distributions

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This is an attempt to understand some very mysterious (and rather muddled) assertions in Section 3.5 of Walter Kim's Berkeley Ph.D. thesis.

The naive pairings

Let

$$\Sigma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}_p),\ a \in \mathbf{Z}_p^\times, c \in p\mathbf{Z}_p,\ ad - bc \neq 0 \right\}$$

denote the usual monoid. For an integer $k \geq 0$ and a ring R we write $V_k(R) = R[Z]^{\deg \leq k}$, with $V_k = V_k(\mathbf{Q}_p)$. Give V_k the right action $(p \cdot g)(Z) = (d + cZ)^k p\left(\frac{b+aZ}{d+cZ}\right)$. Let $\mathbf{D}_k$ denote the $\mathbf{Q}_p$-Banach dual of the Tate algebra $\mathbf{A} = \mathbf{Q}_p \langle z \rangle$, and equip $\mathbf{D}_k$ with the right action

$$(\mu|g)(f) = \mu\left((a + cz)^k f\left(\frac{b + dz}{a + cz}\right)\right),\ g \in \Sigma_0(p).$$

The map $\rho_k : \mathbf{D}_k \to V_k$ defined by

$$\rho_k : \mu \mapsto \int (Z + z)^k \mu(z)$$

is $\Sigma_0(p)$-equivariant.

We define a bilinear pairing $(\ ,\ )_k : \mathbf{D}_k \times \mathbf{D}_k \to \mathbf{Q}_p$ by the formula

$$\begin{aligned} (\mu_1, \mu_2)_k &\mapsto \int (z_1 - z_2)^k \mu_1(z_1)\mu_2(z_2) \\ &= \sum_{i=0}^{k} (-1)^i \begin{pmatrix} k \\ i \end{pmatrix} \mu_1(z_1^{k-i})\mu_2(z_2^i). \end{aligned}$$

Here we regard $(z_1 - z_2)^k$ as an element of $\mathbf{A} \otimes \mathbf{A}$ in the obvious way. Note that $(\ ,\ )_k$ is symmetric or skew-symmetric according to whether k is even or odd.

Proposition 1. *The pairing* $(\,,\,)_k$ *satisfies the equivariance property*

$$(\mu_1|g, \mu_2|g)_k = (\det g)^k\,(\mu_1, \mu_2)_k$$

for all $\mu_1, \mu_2 \in \mathbf{D}_k$ *and all* $g \in \Sigma_0(p)$.

Proof. A simple calculation verifies the identity

$$(b + dz_1)(a + cz_2) - (b + dz_2)(a + cz_1) = (\det g)(z_1 - z_2).$$

With this in mind, we simply unwind the actions:

$$\begin{aligned}
(\mu_1|g, \mu_2|g)_k &= \int \left(\frac{b + dz_1}{a + cz_1} - \frac{b + dz_2}{a + cz_2}\right)^k (a + cz_1)^k (a + cz_2)^k \mu_1(z_1)\mu_2(z_2) \\
&= \int \left((b + dz_1)(a + cz_2) - (b + dz_2)(a + cz_1)\right)^k \mu_1(z_1)\mu_2(z_2) \\
&= \int \left((\det g)(z_1 - z_2)\right)^k \mu_1(z_1)\mu_2(z_2) \\
&= (\det g)^k\,(\mu_1, \mu_2)_k\,,
\end{aligned}$$

as desired. □

Now, the module V_k admits a well-known bilinear pairing $\langle\,,\,\rangle_k : V_k \times V_k \to \mathbf{Q}_p$ satisfying the same equivariance property and unique up to scaling, defined on the obvious monomial basis of $V_k \otimes V_k$ by

$$Z_1^i \otimes Z_2^j \mapsto \begin{cases} (-1)^i \begin{pmatrix} k \\ i \end{pmatrix}^{-1} & \text{if } i + j = k \\ 0 & \text{if } i + j \neq k. \end{cases}$$

Proposition 2. *The diagram*

$$\begin{array}{ccc}
\mathbf{D}_k \times \mathbf{D}_k & \xrightarrow{\rho_k \otimes \rho_k} & V_k \times V_k \\
& {\scriptstyle (\,,\,)_k} \searrow & \downarrow {\scriptstyle \langle\,,\,\rangle_k} \\
& & \mathbf{Q}_p
\end{array}$$

commutes.

Proof. We calculate

$$\begin{aligned}
\langle \rho_k(\mu_1), \rho_k(\mu_2) \rangle_k &= \left\langle \sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} Z_1^i \mu_1(z_1^{k-i}), \sum_{j=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} Z_2^j \mu_2(z_1^{k-j}) \right\rangle_k \\
&= \sum_{i=0}^{k} (-1)^i \begin{pmatrix} k \\ i \end{pmatrix} \mu_1(z_1^{k-i})\mu_2(z_2^i) \\
&= (\mu_1, \mu_2)_k\,.
\end{aligned}$$

The enlightened pairings

Set $\mathcal{W} = \mathrm{Hom}_{\mathrm{cts}}(\mathbf{Z}_p^\times, \mathbf{G}_m)^{\mathrm{an}}$, the $\mathbf{Q}_p$-rigid analytic space of weights over $\mathbf{Q}_p$. Given an admissible affinoid open $\Omega \subset \mathcal{W}$, there is a universal character $\chi_\Omega : \mathbf{Z}_p^\times \to A(\Omega)^\times$ and a minimal integer $s[\Omega]$ such that $\chi_\Omega(1 + p^{s+1}z) : \mathbf{Z}_p \to A(\Omega)$ is analytic, i.e. is given by an element of the relative Tate algebra $A(\Omega)\langle z \rangle$. Let $\mathbf{A}^s$ denote the module of $\mathbf{Q}_p$-valued continuous functions on $\mathbf{Z}_p$ which are analytic on each coset of $p^s\mathbf{Z}_p$. For any $s \geq s[\Omega]$, set

$$\mathbf{A}^s_\Omega = \mathbf{A}^s \widehat{\otimes}_{\mathbf{Q}_p} A(\Omega),$$

equipped with the left action $(g \cdot f)(z) = \chi_\Omega(a + cz) f\left(\frac{b+dz}{a+cz}\right)$. Set

$$\begin{aligned} \mathbf{D}^s_\Omega &= \mathrm{Hom}^{\mathrm{cts}}_{A(\Omega)}(\mathbf{A}^s_\Omega, A(\Omega)) \\ &\simeq \mathrm{Hom}^{\mathrm{cts}}_{\mathbf{Q}_p}(\mathbf{A}^s, A(\Omega)). \end{aligned}$$

Suppose Ω contains a character of the form $x \mapsto x^k$ for some integer k; we write w_k for the corresponding point of Ω. By the basic properties of affinoid opens, we will have $w_{k'} \in \Omega$ for an infinitude of integers k', in fact for all integers with $(p-1)p^e | (k'-k)$ and e sufficiently large. Evaluation of an element of $A(\Omega)$ at a point $w_k \in \Omega$ is well-defined, and induces a well-defined $\Sigma_0(p)$-equivariant specialization map $\mathbf{D}^s_\Omega \to \mathbf{D}^s_k$ and therefore a map

$$\sigma_k : \mathbf{D}^s_\Omega \to V_k$$

obtained by composing with the natural morphisms $\mathbf{D}^s_k \to \mathbf{D}_k$ and $\rho_k : \mathbf{D}_k \to V_k$.

We define the enlightened pairings as follows. Set $W_p = \begin{pmatrix} & -1 \\ p & \end{pmatrix}$. The enlightened pairing on V_k is $\langle p_1, p_2 \rangle^\circ_k = \langle p_1, p_2 \cdot W_p \rangle$. The enlightened pairing on $\mathbf{D}_k$ is

$$(\mu_1, \mu_2)^\circ_k = \int_{\mathbf{Z}_p^2} (1 + p z_1 z_2)^k \, \mu_1(z_1)\mu_2(z_2).$$

Finally, we define an $A(\Omega)$-bilinear pairing

$$(\mu_1, \mu_2)^\circ_\Omega : \mathbf{D}^s_\Omega \times \mathbf{D}^s_\Omega \to A(\Omega)$$

by

$$(\mu_1, \mu_2)^\circ_\Omega = \int \chi_\Omega(1 + p z_1 z_2)\mu_1(z_1)\mu_2(z_2).$$

Proposition 3. *Each of the enlightened pairings satisfies the equivariance property* $\{\phi_1 \cdot g, \phi_2\} = \left\{\phi_1, \phi_2 \cdot W_p g^\iota W_p^{-1}\right\}$, *and the diagram*

$$\begin{array}{ccc} \mathbf{D}^s_\Omega \times \mathbf{D}^s_\Omega & \xrightarrow{\langle\, , \rangle^\circ_\Omega} & A(\Omega) \\ \Big\downarrow{\scriptstyle \sigma_k \otimes \sigma_k} & & \Big\downarrow{\scriptstyle f \mapsto f(w_k)} \\ V_k \times V_k & \xrightarrow[\langle\, , \rangle^\circ_k]{} & \mathbf{Q}_p \end{array}$$

commutes for all $w_k \in \Omega$.

Here ι is Shimura's main involution; note that

$$W_p \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\iota} W_p^{-1} = \begin{pmatrix} a & p^{-1}c \\ pb & d \end{pmatrix},$$

so $x \mapsto W_p x^{\iota} W_p^{-1}$ is an involution of $\Sigma_0(p)$. Presumably, $(\ ,\ )^{\circ}_{\Omega}$ is the *unique* $A(\Omega)$-bilinear pairing on $\mathbf{D}^s_{\Omega} \times \mathbf{D}^s_{\Omega}$ satisfying the claim of Proposition 3, and presumably this is an easy consequence of the Zariski-density of the points $w_k \in \Omega$ and the uniqueness of the pairings $\langle\ ,\ \rangle^{\circ}_k$, but I have made no attempt to verify this.