Geometry and cohomology of local Shimura varieties

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Introduction

Recall: A **Shimura datum** is (roughly) a pair (G, X) where $G/\mathbf{Q}$ is a connected reductive group and $X \simeq G(\mathbf{R})/K_\infty$ is a Hermitian symmetric domain.

Theorem (Bailey-Borel, Shimura, Deligne, Borovoi, Milne)

Given (G, X) as above, and $K \subset G(\mathbf{A}_f)$ a sufficiently small open compact subgroup, the locally symmetric manifold

$$G(\mathbf{Q})\backslash(X \times G(\mathbf{A}_f))/K$$

is $\cong \mathrm{Sh}(G, X)_K(\mathbf{C})$ for a certain smooth quasiprojective algebraic variety $\mathrm{Sh}(G, X)_K$ defined over a number field $E = E(G, X)$.

$\rightsquigarrow$ Get a tower $\{\mathrm{Sh}(G, X)_K\}_K$ with $G(\mathbf{A}_f)$-action.
When $(G, X) = (\mathrm{GL}_2, \mathfrak{H}^{\pm})$, this is the usual tower of modular curves.

Introduction (cont.)

Shimura varieties give a wonderful tool/playground for working out ideas in arithmetic geometry and the Langlands program. Some particular instances of this:

- $\mathrm{Sh}(G, X)_K$ can often (but not always!) be interpreted as a moduli space of abelian varieties w. extra structures. ($\sim$"Shimura varieties of Hodge type.") $\rightsquigarrow$ Canonical integral models / $\mathcal{O}_E[1/N]$ (Kottwitz, Kisin, ...) w. concrete description of their special fibers (Langlands-Rapoport, Kisin), canonical compactifications (Faltings-Chai, Lan, Madapusi Pera, ...), etc.
- $\lim_{K \to 1} H^*_{\text{ét}}(\mathrm{Sh}(G, X)_{K, \overline{E}}, \overline{\mathbf{Q}_\ell})$ has commuting $G(\mathbf{A}_f)$- and $\mathrm{Gal}(\overline{E}/E)$-actions. $\rightsquigarrow$ Construction of Galois representations associated with automorphic forms in many instances. (Huge industry; cf. recent work of Kret-Shin for an interesting example involving a Shimura variety *not* of Hodge type.)

Question: Is there a local analogue of Shimura varieties?

Local Shimura varieties

Fix a prime p, and set $\breve{\mathbf{Q}}_p = \widehat{\mathbf{Q}_p^{\mathrm{unr}}} \circlearrowleft \sigma =$ lift of $x \mapsto x^p$.

Definition (Rapoport-Viehmann)

A **local Shimura datum** *is a triple $(G, \{\mu\}, b)$ where $G/\mathbf{Q}_p$ is a connected reductive group, $\{\mu\}$ is a conjugacy class of minuscule cocharacters $\mathbf{G}_{m,\overline{\mathbf{Q}_p}} \to G_{\overline{\mathbf{Q}_p}}$, and $b \in G(\breve{\mathbf{Q}}_p)$ is an element such that [redacted].*

(Can and do fix $\mu \in \{\mu\}$ defined over a minimal fin. extension $E = E(G, \{\mu\})/\mathbf{Q}_p$; ignore difference between μ and $\{\mu\}$. Set $\breve{E} = E.\breve{\mathbf{Q}}_p$.)

Conjecture (Rapoport-Viehmann)

Given $(G, \{\mu\}, b)$ as above, there should be a naturally defined tower of smooth rigid analytic spaces $\{\mathcal{M}_K = \mathcal{M}(G, \{\mu\}, b)_K\}_K$ over $\breve{E}$, indexed by $K \subset G(\mathbf{Q}_p)$, equipped with commuting actions of $G(\mathbf{Q}_p)$ (on the whole tower) and $J_b(\mathbf{Q}_p) := \{j \in G(\breve{\mathbf{Q}}_p) \mid \sigma(j) = bjb^{-1}\}$ (on each $\mathcal{M}_K$). Moreover,...

Local Shimura varieties (cont'd)

Conjecture (Rapoport-Viehmann, cont'd)

Moreover,...

- $\mathcal{M}(G, \{\mu\}, b)_K$ *should often (but not always!) coincide with the generic fiber of a formal scheme (a "Rapoport-Zink space") parametrizing p-divisible groups w./w.o. extra structures . Works e.g. for* $G = \mathrm{GL}_n$, GSp_{2n}, GSpin_n *... Doesn't work if e.g.* $G = \mathrm{PGL}_n$, E_7, *...*
- *Should have a canonical (non-effective) descent datum to* $E \rightsquigarrow \lim_{K \to 1} H^*_{\text{ét}}(\mathcal{M}_{K,\overline{E}}, \overline{\mathbf{Q}_\ell})$ *has commuting* $G(\mathbf{Q}_p)$*-,* $J_b(\mathbf{Q}_p)$*, and* W_E*-actions, which should realize instances of the local Langlands and local Jacquet-Langlands correspondences (Carayol, Kottwitz, Harris, ...).*
- *Should have compatible étale and* $J_b(\mathbf{Q}_p)$*-equivariant period morphisms* $\pi_{\mathrm{GM}} : \mathcal{M}_K \to \mathcal{F}\ell_{G,\mu}$ *with target a rigid analytic flag variety.*
- *...*

Local Shimura varieties (cont'd)

Theorem (Scholze ∼late 2014, via Caraiani-Scholze, Fargues & Fargues-Fontaine, Kedlaya-Liu, Scholze-Weinstein)

Local Shimura varieties exist, as the solution to a canonical moduli problem. In fact, they exist in the larger category of diamonds for any μ, and even for more general input data $(G, \{\mu_i\}_{1 \leq i \leq n}, b)$, in which case the resulting spaces live over "n copies of $\mathrm{Spa}(\breve{E})$" in a precise sense.

This last construction is in parallel with the moduli spaces of local/global shtukas defined in equal characteristic p settings (Drinfeld, Varshavsky, Lafforgue, ...), which fiber over some finite self-product of copies of the base (which is either $\mathrm{Spec}\, \mathbf{F}_q((t))$ or a projective curve over $\mathbf{F}_q$).
$\rightsquigarrow$ Scholze (work in progress): Combining this with ideas of V. Lafforgue, get a construction associating a Weil parameter with any irreducible smooth representation of any $G(\mathbf{Q}_p)$, giving a candidate for the local Langlands correspondence.

Rough idea of construction

Fix (G, μ, b) as before, with E the field of def. of μ. Let $C/\breve{E}$ be any complete algebraically closed extension. Let's unwind $\mathcal{M}_\infty(C) \stackrel{def}{=} \lim_{\underset{K}{\leftarrow}} \mathcal{M}_K(C)$.

Fargues-Fontaine: Any such $C \rightsquigarrow$ a certain "curve" $X = X_{C^\flat}$ (a connected reg. Noeth. 1-dim'l scheme over $\operatorname{Spec} \mathbf{Q}_p$) together with a distinguished closed point $\infty \in X$ such that $\kappa(\infty) \cong C$.

Fargues: Any $b \in G(\breve{\mathbf{Q}}_p) \rightsquigarrow$ a G-bundle $\mathcal{E}_b$ over X, with $\operatorname{Aut}(\mathcal{E}_b) \supset J_b(\mathbf{Q}_p)$ (and sometimes with equality; in particular, $\operatorname{Aut}(\mathcal{E}_1) \cong G(\mathbf{Q}_p)$).

Idea

$\mathcal{M}_\infty(C) = \{$*modifications* $\mathcal{E}_1 \to \mathcal{E}_b$ *supported at* ∞, *of type* $\mu\}$, *with the natural group actions on the r.h.s.*

Can define a functor $\mathcal{M}_\infty$ representing essentially the same moduli problem on more general perfectoid spaces over $\breve{E}$; this turns out to be a diamond, and $\mathcal{M}_K \stackrel{def}{=} \mathcal{M}_\infty/K$ is a diamond too.

Some misc. remarks

- In the case of RZ spaces, Scholze-Weinstein had already reinterpreted their generic fibers in this language of modified vector bundles in 2012.
- For minuscule μ, not yet known in general whether $\mathcal{M}_\infty$ is representable by a perfectoid space. (True in situations related to RZ spaces, by Scholze-Weinstein, W. Kim,...)
- The map π_{GM} does indeed exist, and it plays a key role in the proof that $\mathcal{M}_\infty$ is a diamond. However, for general μ its target is a diamond $\mathrm{Gr}_{G,\mu}^{B_{\mathrm{dR}}}$ of non-classical provenance (a Schubert cell in a "B_{dR} affine Grassmannian").

Expectations for cohomology

Fix (G, μ, b). Two key conjectures (stated very roughly):

- Kottwitz: Suppose b is "basic" ($\rightsquigarrow J_b$ is an inner form of G). Let π (resp. ρ) be an irreducible smooth rep. of $G(\mathbf{Q}_p)$ (resp. $J_b(\mathbf{Q}_p)$). Then the $\rho \boxtimes \pi$-part of

$$\mathcal{H}(G, \mu, b) \stackrel{def}{=} \sum_{i \geq 0} (-1)^i H^i_{\text{ét},c}(\mathcal{M}_{\infty,\overline{E}}, \overline{\mathbf{Q}_\ell})$$

  encodes the refined local Langlands correspondences for disc. series L-packets of $G(\mathbf{Q}_p)$ and $J_b(\mathbf{Q}_p)$. In particular, every supercuspidal π should appear.

- Harris, Harris-Viehmann: Suppose b is *not* basic. Then

$$\mathcal{H}(G, \mu, b) \simeq \oplus_{i \in I} \mathrm{Ind}_{P_i(\mathbf{Q}_p)}^{G(\mathbf{Q}_p)} \mathcal{H}(M_i, \mu_i, b)$$

  for some explicit finite list of local Shimura data (M_i, μ_i, b) assoc. with Levi subgroups of G. In particular, no supercuspidal π should appear.

Expectations for cohomology (cont'd)

Status of these conjectures:

- Kottwitz's conjecture: Proved for Lubin-Tate and Drinfeld towers (in which case formulated by Carayol) by Harris, Harris-Taylor, Boyer, Dat via global arguments. (Also, LT tower $\cong$ Drinfeld tower via Faltings and Fargues.) Not known in full for any other (interesting) local Shimura datum! If one forgets W_E-action, purely local methods are possible, cf. work of Strauch and Mieda. (Also cf. ongoing work of Kaletha-Weinstein.)
- Harris's conjecture: Proved in many cases related to RZ spaces by Mantovan, assuming that the datum (G, μ, b) is "Hodge-Newton reducible" (roughly, it comes from a local Shimura datum for some Levi subgroup). (Cf. also forthcoming work of Scholze.)

We got interested in trying to understand Mantovan's method from a perfectoid POV. The outcome:

A result towards Harris's conjecture

Theorem (H.)

Fix (G, μ, b) as before (allowing arbitrary μ) which is HN-reducible. Assume G is split with simply connected derived group. Then:

- *$\exists!$ minimal standard Levi subgroup $M \subsetneq G$ such that (M, μ, b) is a local shtuka datum which is moreover HN-irreducible. Let P be the standard parabolic with Levi M.*
- *There is a natural isomorphism*

$$H^i_{\text{ét},c}(\mathcal{M}(G, \mu, b)_{\infty,\overline{E}}, \overline{\mathbf{Q}_\ell}) \cong \mathrm{Ind}_{P(\mathbf{Q}_p)}^{G(\mathbf{Q}_p)} H^{i-2d}_{\text{ét},c}(\mathcal{M}(M, \mu, b)_{\infty,\overline{E}}, \overline{\mathbf{Q}_\ell}(-d))$$

compatible with J_b and W_E-actions. (Here $d = \langle 2\rho_G - 2\rho_M, \mu \rangle$.)

Sketch of the argument

Key idea (d'après Mantovan): When (G, μ, b) is HN-reducible, the *spaces themselves* should be parabolically induced from a local Shimura variety for a Levi, up to a "fattening" by some piece which doesn't mess up cohomology. Tricky to make this work at finite level (which Mantovan did). At infinite level, though, everything becomes very clean:

Theorem (H.)

Assumptions as on the previous slide, we have a canonical isomorphism

$$\mathcal{M}(G, \mu, b)_\infty \cong \left(\mathcal{M}(M, \mu, b)_\infty \times_{\breve{E}} \mathcal{J}_b^U\right) \times^{\underline{P(\mathbf{Q}_p)}} \underline{G(\mathbf{Q}_p)}$$

of diamonds over $\breve{E}$. Here $\mathcal{J}_b^U$ is a certain d-dimensional smooth unipotent group diamond with trivial (= same as $\mathbf{A}^d_{\breve{E}}$) ℓ-adic cohomology.

This plus some cohomological formalism for diamonds (work in progress of Scholze; we only need a small extract) implies the previous theorem.

Sketch of the argument (cont'd)

Key technical input in Mantovan's work, proved by Mantovan-Viehmann: A p-divisible group w./w.o. extra structures over $\mathcal{O}_K$ (for $K/\mathbf{Q}_p$ any complete dvf) admits a canonical *Hodge-Newton filtration* refining the connected-étale decomposition.
The key technical ingredient in our argument is an analogue of this for bundles on the curve:

Theorem (H.)

Any G-bundle modification $\mathcal{E}_1 \to \mathcal{E}_b$ as before admits a canonical Hodge-Newton filtration, and in particular a canonical reduction to a P-bundle modification. Moreover, this spreads out in families.

The admissible locus

Fix (G, μ, b) with μ minuscule. As mentioned before, there is a natural étale and $J_b(\mathbf{Q}_p)$-equivariant period morphism

$$\pi_{\mathrm{GM}} : \mathcal{M}(G, \mu, b)_K \to \mathcal{F}\ell_{G,\mu}.$$

When the local Shimura variety comes from an RZ space parametrizing deformations of some p-divisible group $H_b/\overline{F}_p$, this map records the Hodge filtration on $\mathbf{D}(H_b)[1/p]$ coming from Grothendieck-Messing deformation theory (hence the name).

Question: What can we say about the image of π_{GM}? This is the so-called *(b-)admissible locus* $\mathcal{F}\ell^{\mathrm{a}}_{G,\mu}$.

Properties of the admissible locus

Basic properties:

- $\mathcal{F}\ell^{\mathrm{a}}_{G,\mu}$ is always open and stable under specialization (= "partially proper" in the rigid world = "without boundary" in the Berkovich world).
- There is an intrinsic description of $\mathcal{F}\ell^{\mathrm{a}}_{G,\mu}$ (Faltings, Hartl, Scholze-Weinstein). Not very explicit, though.
- Have an explicitly defined open subset $\mathcal{F}\ell^{\mathrm{wa}}_{G,\mu} \subseteq \mathcal{F}\ell_{G,\mu}$, the *weakly admissible locus*, containing the adm. locus and with the same classical points. Sometimes admissible = weakly admissible, e.g. in Lubin-Tate or Drinfeld situations, but this is not typical. In fact, can have strange situations where

$$\mathcal{F}\ell^{\mathrm{a}}_{G,\mu} \subsetneq \mathcal{F}\ell^{\mathrm{wa}}_{G,\mu} = \mathcal{F}\ell_{G,\mu}.$$

Question: Can we describe $\mathcal{F}\ell^{\mathrm{a}}_{G,\mu}$ explicitly in these strange situations?

Properties of the non-admissible locus

Idea: It turns out to be easier to describe the *complement* of $\mathcal{F}\ell^{\mathrm{a}}_{G,\mu}$.
Think of Drinfeld space: $\Omega^n = \mathbf{P}^n \smallsetminus (\mathrm{all}\, \mathbf{Q}_p - \mathrm{rational\, hyperplanes})$.
Note: The complement of the admissible locus here is not a rigid space!

Theorem (H.-Weinstein, work in progress)

The complement $\mathcal{F}\ell^{\mathrm{na}}_{G,\mu}$ of the admissible locus is naturally a diamond, and admits a canonical stratification into finitely many locally closed $J_b(\mathbf{Q}_p)$-stable strata $\mathcal{F}\ell^{b'-\mathrm{na}}_{G,\mu}$ indexed by some $b' \in G(\breve{\mathbf{Q}}_p)$. When b is basic, these strata are equidimensional with explicitly calculable dimensions, and their closure relations are pulled back from the closure relations in Bun_G.

This stratification = a cousin of the Newton stratification defined on any $\mathcal{F}\ell_{G,\mu}$ by Caraiani-Scholze.

Moreover, these strata often admit strange and beautiful explicit descriptions.

The first strange example

Example 1. Take $G = \mathrm{GL}_5$, $\mu = \mathrm{diag}(x, x, 1, 1, 1)$, $b =$ an elt. of $G(\mathbf{Q}_p)$ with char. poly $X^5 - p^2$. This example was singled out by Hartl as the first strange GL_n situation.

Claim: In this example, there is only one stratum, and

$$\mathcal{F}\ell^{\mathrm{na}}_{G,\mu} \simeq \left(\mathrm{Spf}\, \breve{\mathbf{Z}}_p[[T^{1/p^\infty}]] \smallsetminus V(pT)\right)^{\diamond} / D^\times$$

for a certain free action of $D^\times = D^\times_{1/3}$; here $D_{1/3}$ is the division alg. over $\mathbf{Q}_p$ of invt. $1/3$.

Another strange example

Example 2. Take $G = \mathrm{GL}_7$, $\mu = \mathrm{diag}(x, x, 1, 1, 1, 1, 1)$, $b =$ an elt. of $G(\mathbf{Q}_p)$ with char. poly $X^7 - p^2$. There are now three strata in $\mathcal{F}\ell^{\mathrm{na}}_{G,\mu}$, say S_1, S_2, and S_3. S_1 and S_2 are open and 3-dimensional, and S_3 is closed and 2-dimensional, with $\overline{S_i} = S_i \cup S_3$ for $i = 1, 2$. Moreover,

- $S_1 \cup S_3$ is $\simeq$ a $(\mathbf{P}^2)^\diamond$-bundle over some $(\mathrm{punctured}\, 1 - \mathrm{variable\, perfectoid\, disk})^\diamond / D^\times_{1/4}$.
- S_3 is $\simeq$ a $(\mathrm{punctured}\, 1 - \mathrm{variable\, perfectoid\, disk})^\diamond / \mathbf{Q}_p^\times$-bundle over the same $(\mathrm{punctured}\, 1 - \mathrm{variable\, perfectoid\, disk})^\diamond / D^\times_{1/4}$.
- S_2 is $\simeq (\mathrm{some\, open\, subset\, of\, a}\, 3 - \mathrm{variable\, perfectoid\, disk})^\diamond / D^\times_{1/5}$.

Scholze has proposed a definition of *smoothness* for (morphisms of) diamonds, satisfied by all the displayed examples so far.

Further computations suggest a conjecture:

Conjecture (H.-Weinstein)

When b is basic, any connected component of any stratum $\mathcal{F}\ell_{G,\mu}^{b'-\mathrm{na}} \subset \mathcal{F}\ell_{G,\mu}^{\mathrm{na}}$ is smooth.

Here is some evidence for this:

Theorem (H.-Weinstein)

When b is basic, each stack quotient $[\mathcal{F}\ell_{G,\mu}^{b'-\mathrm{na}}/J_b(\mathbf{Q}_p)]$ is a smooth diamond stack.

Believe it or not, this actually implies something down-to-earth:

Corollary (H.-Weinstein)

When b is basic, the b-admissible locus $\mathcal{F}\ell_{G,\mu}^{\mathrm{a}}$ is connected.

Previously proved by Hartl in some cases, via Berkovich space techniques.