1 Smoothness of Bun_n for dinguses[1]

Dear Jared,

Let $\mathrm{Bun}_n \to \mathrm{Perf}$ denote the stack of rank n vector bundles on "the" Fargues-Fontaine curve. Yesterday I figured out a fairly cheap argument for checking that Bun_n is a smooth diamond stack, using charts made out of de Rham affine Grassmannians. Of course Peter's charts made from those spaces X_b give more information, but it seems harder to check that they have the right properties.

A word on terminology: if S is any absolute diamond, we say S is *smooth* if for any diamond X, the projection map $X \times S \to X$ is smooth. Note that if S_1 and S_2 are smooth, then so is $S_1 \times S_2$. One can also check that if K is any finite extension of $\mathbf{Q}_p$ and S is a diamond with a smooth morphism $S \to \mathrm{Spd}\, K$, then S is smooth in this sense.

Let $\mathrm{Bun}_n^d \subset \mathrm{Bun}_n$ denote the open-closed substack of bundles of constant degree d. Let $\mathrm{Gr}_{n,k}/\mathrm{Spd}\,\mathbf{Q}_p$ denote the de Rham affine Grassmannian sending $T \in \mathrm{Perf}$ with specified untilt $T^\sharp$ to the set of subsheaves

$$\mathcal{E} \subset \mathcal{O}_{\mathcal{X}_T}^n$$

such that $\mathcal{E} \to \mathcal{O}_{\mathcal{X}_T}^n$ is a modification supported along $T^\sharp \subset \mathcal{X}_T$ of (constant) meromorphy type $(k, 0, \ldots, 0)$. Note that $\mathcal{E}$ has constant degree $-k$. In particular, for any $m \geq d/n$ there is a natural morphism

$$\mathrm{Gr}_{n,mn-d} \to \mathrm{Bun}_{n,d}$$

given by sending $\mathcal{E} \subset \mathcal{O}_{\mathcal{X}_T}^n$ as above to the degree d bundle $\mathcal{E}(m) := \mathcal{E} \otimes_{\mathcal{O}} \mathcal{O}(m)$. This clearly factors through a morphism

$$f_m : [\mathrm{Gr}_{n,mn-d}/\underline{\mathrm{GL}_n(\mathbf{Q}_p)}] \to \mathrm{Bun}_{n,d},$$

where $\underline{\mathrm{GL}_n(\mathbf{Q}_p)}$ acts on any $\mathrm{Gr}_{n,k}$ in the usual way.

Proposition 1.1. *The morphism f_m is smooth.*

Proof. We need to check that for any $S \in \mathrm{Perf}$ and any morphism $a : S \to \mathrm{Bun}_{n,d}$, the fiber product

$$S \times_{a, \mathrm{Bun}_{n,d}, f_m} [\mathrm{Gr}_{n,mn-d}/\underline{\mathrm{GL}_n(\mathbf{Q}_p)}]$$

is a diamond smooth over S. What functor does this fiber product represent? Well, giving a is equivalent to giving a degree d rank n bundle $\mathcal{E}/\mathcal{X}_S$. Unwinding definitions then shows that this fiber product represents the set of isomorphism classes of pairs $(S^\sharp, \mathcal{E} \hookrightarrow \mathcal{F})$ where $S^\sharp$ is an untilt of S and $\mathcal{E} \hookrightarrow \mathcal{F}$ is a modification supported along $S^\sharp \subset \mathcal{X}_S$ and of meromorphy type $(mn - d, 0, \ldots, 0)$, such that moreover $\mathcal{F}$ is pointwise-semistable.[2] Ignoring the last condition, this functor is representable by a "twisted de Rham affine Grassmannian" $\mathrm{Gr}_{n,d-mn}^{\mathcal{E}}/S$, which locally on S is isomorphic to $\mathrm{Gr}_{n,d-mn} \times S$ and therefore is smooth over S. Enforcing the semistability of $\mathcal{F}$ then cuts out (by Kedlaya-Liu) an open subspace

$$\mathrm{Gr}_{n,d-mn}^{\mathcal{E},ss} \subset \mathrm{Gr}_{n,d-mn}^{\mathcal{E}},$$

so $\mathrm{Gr}_{n,d-mn}^{\mathcal{E},ss} \to S$ is still smooth, and

$$\mathrm{Gr}_{n,d-mn}^{\mathcal{E},ss} \cong S \times_{\mathrm{Bun}_{n,d}} [\mathrm{Gr}_{n,mn-d}/\underline{\mathrm{GL}_n(\mathbf{Q}_p)}]$$

so we win. □

[1] Version of 12/25/2016

[2] More precisely, this fiber product should be regarded as a functor on $\mathrm{Perf}_{/S}$, but whatever.

Next we describe the image of f_m on geometric points.

Proposition 1.2. *Let $C/\mathbf{F}_p$ be an algebraically closed perfectoid field, and let $a : \mathrm{Spd}\, C \to \mathrm{Bun}_{n,d}$ be any point, with associated bundle $\mathcal{E}/\mathcal{X}_C$. Then a lifts along f_m to a C-point of $[\mathrm{Gr}_{n,mn-d}/\underline{\mathrm{GL}_n(\mathbf{Q}_p)}]$ if and only if the maximal Harder-Narasimhan slope of $\mathcal{E}$ is $\leq m$.*

Proof. "Only if" is an easy exercise: if a lifts, then by definition there is some inclusion $\mathcal{E}(-m) \subset \mathcal{O}^n_{\mathcal{X}_C}$, so $\mathcal{E}(-m)$ has maximal HN slope ≤ 0. "If" can be deduced from various results of the form "weakly admissible filtrations of specified Hodge type on specified φ-modules exist when they should". □

The condition on HN slopes in the previous proposition cuts out an open substack $\mathrm{Bun}^{\leq m}_{n,d}$ such that f_m factors through the inclusion of this substack. Clearly $\mathrm{Bun}^{\leq m}_{n,d} \subset \mathrm{Bun}^{\leq m+1}_{n,d}$ and

$$\mathrm{Bun}_{n,d} = \bigcup_{m \gg 0} \mathrm{Bun}^{\leq m}_{n,d}.$$

It is true, but not *a priori* obvious, that $f_m : [\mathrm{Gr}_{n,mn-d}/\underline{\mathrm{GL}_n(\mathbf{Q}_p)}] \to \mathrm{Bun}^{\leq m}_{n,d}$ is surjective in the pro-étale topology, i.e. that given any $S \in \mathrm{Perf}$ and any $x \in \mathrm{Bun}^{\leq m}_{n,d}(S)$ we can lift x along f_m after passing to some pro-étale cover of S. This can be deduced as follows: Using the previous two proposition, one first checks that the morphism of diamonds

$$S \times_{s,\mathrm{Bun}^{\leq m}_{n,d},f_m} [\mathrm{Gr}_{n,mn-d}/\underline{\mathrm{GL}_n(\mathbf{Q}_p)}] \to S$$

is smooth, and moreover surjective on topological spaces, with locally spatial source. One then applies the following result (whose straightforward proof is omitted; the key point in the proof is that smooth maps of diamonds are universally open).

Proposition 1.3. *Let $f : Y \to X$ be any map of locally spatial diamonds. If f is smooth and $|Y| \to |X|$ is surjective, then f is surjective as a map of pro-étale sheaves.*

OK, so we have a family of smooth maps

$$f_m : [\mathrm{Gr}_{n,mn-d}/\underline{\mathrm{GL}_n(\mathbf{Q}_p)}] \to \mathrm{Bun}_{n,d}$$

which together cover the target. Now comes the fun part.

Proposition 1.4. *The stack $[\mathrm{Gr}_{n,mn-d}/\underline{\mathrm{GL}_n(\mathbf{Q}_p)}]$ is a smooth diamond stack.*

With this in hand, we're done: after choosing some smooth diamonds X_m with some smooth surjective maps

$$g_m : X_m \to [\mathrm{Gr}_{n,mn-d}/\underline{\mathrm{GL}_n(\mathbf{Q}_p)}],$$

the composite maps $f_m \circ g_m : X_m \to \mathrm{Bun}_{n,d}$ are smooth and give a collection of charts which verify that $\mathrm{Bun}_{n,d}$ is a smooth diamond stack.

So now we need to show that $[\mathrm{Gr}_{n,mn-d}/\underline{\mathrm{GL}_n(\mathbf{Q}_p)}]$ is smooth. We'd like to deduce this from the smoothness of $\mathrm{Gr}_{n,k}$. It turns out there's a really cute general argument for this sort of thing (which is what I missed until yesterday).

Proposition 1.5. *Fix a locally profinite group G, and let X be any absolute diamond with $\underline{G}$-action. If there exists some smooth diamond W with a* free *$\underline{G}$-action, then $[X/\underline{G}]$ is a diamond stack. If moreover W can be chosen such that $W/\underline{G}$ is smooth, then $[X/\underline{G}]$ is smooth whenever X is smooth.*

Proof. Give $X \times W$ the diagonal $\underline{G}$-action; this action is free, since the action on W is free, so $(X \times W)/\underline{G}$ is a diamond. The projection map $X \times W \to X$ is smooth, surjective and $\underline{G}$-equivariant, so we get a smooth surjective map

$$(X \times W)/\underline{G} \to [X/\underline{G}]$$

whose source is a diamond.[3] Hence the target is a diamond stack.[4]

Suppose now that X is smooth. The natural projection map $(X \times W)/\underline{G} \to W/\underline{G}$ is then smooth. Indeed, we get a pullback diagram

$$\begin{array}{ccc} X \times W & \longrightarrow & W \\ \downarrow & & \downarrow \\ (X \times W)/\underline{G} & \longrightarrow & W/\underline{G} \end{array}$$

with surjective pro-étale vertical maps, and smoothness of X implies that the upper horizontal map is smooth; since smoothness can be checked (quasi-)pro-étale-locally on the target, we get that the lower horizontal map is smooth as desired. But now, if $W/\underline{G}$ is smooth as well, we're looking at a smooth map $(X \times W)/\underline{G} \to W/\underline{G}$ with smooth target, which implies that $(X \times W)/\underline{G}$ is smooth. But then $(X \times W)/\underline{G} \to [X/\underline{G}]$ is a smooth surjective map whose source is a smooth diamond, so we win. □

Returning to our specific situation, we just need to find *some* smooth diamond W with a free $\underline{\mathrm{GL}_n(\mathbf{Q}_p)}$-action, such that $W/\underline{\mathrm{GL}_n(\mathbf{Q}_p)}$ is also smooth. To do this, suppose we can find smooth diamonds W_1 and W_2, where W_1 has a free $\underline{\mathrm{SL}_n(\mathbf{Q}_p)}$-action and W_2 has a free $\underline{\mathbf{Q}_p^\times}$-action, such that $W_1/\underline{\mathrm{SL}_n(\mathbf{Q}_p)}$ and $W_2/\underline{\mathbf{Q}_p^\times}$ are both smooth. Letting $m : \mathrm{SL}_n(\mathbf{Q}_p) \times \mathbf{Q}_p^\times \to \mathrm{GL}_n(\mathbf{Q}_p)$ be the group homomorphism which is inclusion on the first factor and which sends $(1, a)$ to $\mathrm{diag}(a, \ldots, a)$, the diamond

$$W = (W_1 \times W_2) \times^{\underline{\mathrm{SL}_n(\mathbf{Q}_p) \times \mathbf{Q}_p^\times}} \underline{\mathrm{GL}_n(\mathbf{Q}_p)}$$

then does what we want: since $\ker m$ is finite and $\mathrm{im}\, m \subset \mathrm{GL}_n(\mathbf{Q}_p)$ is a finite-index normal subgroup, W is étale over the smooth diamond $W_1 \times W_2$, hence smooth itself, and

$$W/\underline{\mathrm{GL}_n(\mathbf{Q}_p)} \cong W_1/\underline{\mathrm{SL}_n(\mathbf{Q}_p)} \times W_2/\underline{\mathbf{Q}_p^\times}$$

is smooth.

For W_2, we just take $\mathrm{Spd}\, \mathbf{Q}_p^{\mathrm{cyc}} \cong \mathrm{Spd}\, \mathbf{F}_p((t^{1/p^\infty}))$ with the usual $\mathbf{Q}_p^\times$-action. For W_1, it turns out that the following thing works. Let W_1 be the functor on Perf sending S to the set of pointwise-injective bundle maps $i : \mathcal{O}^n \hookrightarrow \mathcal{O}(\frac{1}{n+1})$ over the relative curve $\mathcal{X}_S$. There is an obvious $\underline{\mathrm{GL}_n(\mathbf{Q}_p)}$-action given by precomposition with i. I claim that W_1 and $W_1/\underline{\mathrm{SL}_n(\mathbf{Q}_p)}$ are smooth.[5]

[3] This follows from a general lemma: If P is some property of morphisms of diamonds which is stable under base change and quasi-pro-étale-local on the target, and $Y \to X$ is a $\underline{G}$-equivariant morphism of absolute diamonds which has P, then $[Y/\underline{G}] \to [X/\underline{G}]$ has P, in the sense that for any diamond W with a map $W \to [X/\underline{G}]$, $[Y/\underline{G}] \times_{[X/\underline{G}]} W$ is a diamond and $[Y/\underline{G}] \times_{[X/\underline{G}]} W \to W$ has P.

[4] One also checks that $[X/\underline{G}]$ always has diagonal representable in diamonds, for any absolute diamond with $\underline{G}$-action, cf. the "Notes on diamonds".

[5] It seems very likely that $W_1/\underline{\mathrm{GL}_n(\mathbf{Q}_p)}$ is actually smooth, in which case one could avoid the silly circumlocutions of the previous paragraph, but I wasn't able to see this smoothness immediately.

For the smoothness of W_1, consider the functor W' on Perf sending S to the set of sections $s \in H^0(\mathcal{X}_S, \mathcal{O}(\frac{1}{n+1}))$ such that s does not vanish identically on any fiber of the map $|\mathcal{X}_S| \to |S|$. This functor is representable by a spatial diamond, which turns out by some games with Lubin-Tate formal modules to be of the shape $\mathrm{Spd}\, \mathbf{F}_q((t^{1/p^\infty}))/\underline{\mathbf{Z}_{p^{n+1}}^\times}$ for some free action of $\mathbf{Z}_{p^{n+1}}^\times$ on some $\mathrm{Spd}\, \mathbf{F}_q((t^{1/p^\infty}))$; in particular, this thing is smooth. (Here $\mathbf{Z}_{p^h}$ = ring of integers in the degree h unramified extension of $\mathbf{Q}_p$.) Then W_1 is an open subfunctor of $\underbrace{W' \times \cdots \times W'}_{n}$, so W_1 is smooth.

For the smoothness of $W_1/\underline{\mathrm{SL}_n(\mathbf{Q}_p)}$, we first observe that this thing has a moduli interpretation: it is the functor on Perf sending S to the set of pairs $(\mathcal{E}, i)$ where $\mathcal{E} \subset \mathcal{O}(\frac{1}{n+1})/\mathcal{X}_S$ is a rank n subbundle which is pointwise-semistable of degree zero and i is a trivialization $i : \mathcal{O} \xrightarrow{\sim} \wedge^n \mathcal{E}$. By some easy games with the classification, one can check that given any such $\mathcal{E}$, $\mathcal{O}(\frac{1}{n+1})/\mathcal{E}$ is a line bundle on $\mathcal{X}_S$ of constant degree 1, and that i together with the trivialization $\mathcal{O}(1) \cong \wedge^{n+1}\mathcal{O}(\frac{1}{n+1})$ induce a canonical trivialization $\mathcal{O}(\frac{1}{n+1})/\mathcal{E} \cong \mathcal{O}(1)$. Pushing this further, $W_1/\underline{\mathrm{SL}_n(\mathbf{Q}_p)}$ identifies with the functor sending S to the set of surjections $\mathcal{O}(\frac{1}{n+1}) \twoheadrightarrow \mathcal{O}(1)$ of bundles over $\mathcal{X}_S$; indeed, any such surjection has kernel $\mathcal{E}$ which is pointwise-semistable of degree zero and which comes with a canonical trivialization of its determinant, and then W_1 is the $\underline{\mathrm{SL}_n(\mathbf{Q}_p)}$-torsor over this guy parametrizing trivializations $\mathcal{O}^n \xrightarrow{\sim} \mathcal{E}$ compatible with the trivialization of $\wedge^n\mathcal{E}$. Applying $\mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathcal{X}_S}}(-, \mathcal{O}(1))$ to such a surjection gives an inclusion $\mathcal{O} \hookrightarrow \mathcal{O}(\frac{n}{n+1})/\mathcal{X}_S$, nonzero on each fiber of the map $|\mathcal{X}_S| \to |S|$, with cokernel $\simeq \mathcal{O}(1)^n$ at all geometric points of S. In particular, we get a natural transformation

$$f : W_1/\underline{\mathrm{SL}_n(\mathbf{Q}_p)} \to X$$

where X is the functor sending S to the set of sections $s \in H^0(\mathcal{X}_S, \mathcal{O}(\frac{n}{n+1}))$ which are not identically zero on any fiber of $|\mathcal{X}_S| \to |S|$. I claim that f is an open immersion and that X is smooth. For openness, one easily checks that f is an injection. We then observe that f identifies its source with the subfunctor of its target cut out by the requirement that the vector bundle $\mathcal{O}(\frac{n}{n+1})/\mathcal{O} \cdot s$ be pointwise-semistable, and the habitual openness of the latter condition gives what we want. Smoothness of X, finally, is analogous to the smoothness of W' and is left as an exercise.

Cheers,
Dave