

On the GL_n -eigenvariety and a conjecture of Venkatesh

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Abstract

Let π be a cuspidal, cohomological automorphic representation of $GL_n(\mathbb{A})$. Venkatesh has suggested that there should exist a natural action of the exterior algebra of a certain motivic cohomology group on the π -part of the Betti cohomology (with rational coefficients) of the $GL_n(\mathbb{Q})$ -arithmetic locally symmetric space.

Venkatesh has given evidence for this conjecture by showing that its ‘ l -adic realization’ is a consequence of the Taylor–Wiles formalism. We show that its ‘ p -adic realization’ is related to the properties of eigenvarieties.

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1 Introduction

Automorphic representations. Let $n \geq 2$ be an integer, and consider a regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$. This paper is about the arithmetic structures which are (in some cases conjecturally) associated to π .

When $n = 2$, π can be generated by vectors which are naturally interpreted as classical cuspidal modular forms of weight $k \geq 2$. The eigenvalues of Hecke operators acting on such vectors are algebraic numbers, and in fact all lie in a common number field E_π . Deligne constructed a family of p -adic Galois representations associated to π , indexed by finite places of the number field E_π , which can be characterized by a relation between the characteristic polynomials of Frobenius elements and these Hecke eigenvalues. The existence of these Galois representations suggests the existence of a common motive $M(\pi)$ giving rise to them, and this motive was constructed by Scholl [Sch90], using the geometry of modular curves.

When $n > 2$, the picture is currently incomplete. Clozel [Clo90] showed, using the theory of cuspidal cohomology, that the Hecke eigenvalues of π are still algebraic numbers, belonging to a common number field E_π . The corresponding Galois representations were constructed more recently in [HLTT]. It remains an open problem to show that they are geometric (i.e. potentially semi-stable, in the sense of p -adic Hodge theory). Clozel conjectures again [Clo90] the existence of a motive $M(\pi)$ whose Frobenius elements are related to the Satake parameters of π .

Cohomology. When $n = 2$, the arithmetic locally symmetric spaces Y_K associated to the group GL_n are algebraic curves. When $n > 2$, these spaces have no direct link to algebraic geometry. Venkatesh [Vena, Venb] has suggested a more subtle relation, that we now describe. For any value of $n \geq 2$, the representation π contributes to cuspidal cohomology in degrees in the interval $[q_0, q_0 + l_0]$, and (ignoring multiplicities coming from the finite places) the contribution in degree $q_0 + i$ has dimension $\binom{l_0}{i}$. The integer l_0 is given by the formula $l_0 = \lfloor (n - 1)/2 \rfloor$.

Venkatesh suggests that there should exist a natural action of the exterior algebra

$$\wedge^* \mathrm{Ext}_{\mathcal{MM}_{\mathbb{Z}}}^1(M(\pi), M(\pi)(1))$$

on the π -part of the cohomology of Y_K , making this π -part free over this commutative graded ring; here $\mathcal{MM}_{\mathbb{Z}}$ denotes a suitable category of mixed motives over \mathbb{Z} [Fon92, Nek94]. The Bloch–Kato conjecture gives a prediction for the dimension of the \mathbb{Q} -vector space $\mathrm{Ext}_{\mathcal{MM}_{\mathbb{Z}}}^1(M(\pi), M(\pi)(1))$ depending only on the zeroes and poles of Euler factors at infinity of the adjoint motive $\mathrm{ad} M(\pi)$, and an explicit calculation then gives the value l_0 above. The conjecture of Venkatesh would therefore give an arithmetic explanation for the above formula for the dimension of the π -part of cohomology.

This conjecture is naturally hard to attack directly. However, by applying various realization functors from motivic cohomology to more concrete cohomology theories, one can attempt to draw more concrete consequences. Venkatesh and his collaborators [PV, GV] have carried this out already for the Hodge and l -adic realizations, using the theory of cuspidal cohomology and (assuming the existence of certain Galois representations) the Taylor–Wiles method. Additional evidence is provided by recent work of Balasubramanyam–Raghuram [BR].

In this paper, we try to understand Venkatesh’s conjecture from a p -adic or ‘crystalline’ perspective. Our point of departure is the observation that the integer l_0 also plays an essential role in the theory of eigenvarieties.

Eigenvarieties. Hida [Hid86] and Coleman [Col97] observed that classical modular forms of weight $k \geq 2$ can naturally be put into p -adic families depending continuously on the weight variable, whenever they have finite slope (= p -adic valuation of T_p - or U_p -eigenvalue). Coleman–Mazur [CM98] then systematized these families, constructing a geometric object called the eigencurve, whose points are in bijection with finite slope p -adic eigenforms and which locally are modelled on the families constructed by Coleman.

Eigenvarieties for reductive groups other than GL_2 have since been constructed by a number of different authors; let us mention in particular here the work of Buzzard, Chenevier, Emerton, and Urban [Buz07, Che04, Eme06, Urb11]. The eigenvarieties studied in this paper are those constructed by the first named author [Hanb], following an earlier construction of Ash–Stevens [AS]. Let \mathbf{G} be a split reductive group over \mathbb{Z} , and let $\mathbf{T} \subset \mathbf{G}$ be a split maximal torus, \mathbf{B} a Borel subgroup containing \mathbf{T} . Fix a tame level subgroup $K^p \subset \mathbf{G}(\mathbb{A}^{\infty,p})$, and let \mathscr{W} be the rigid space over \mathbb{Q}_p representing the functor

$$X \mapsto \mathrm{Hom}_{\mathrm{cts}}(\mathbf{T}(\mathbb{Z}_p), \mathcal{O}(X)^\times).$$

We call this the ‘weight space’, as there is a natural injection $X^*(\mathbf{T}) \hookrightarrow \mathscr{W}$. The eigenvariety of tame level K^p is a rigid space $w : \mathscr{X} \rightarrow \mathscr{W}$ whose closed points are in bijection with systems of Hecke eigenvalues which appear in the cohomology groups

$$H^*(K^p I, \mathscr{D}_\lambda)$$

of p -adic coefficient systems \mathscr{D}_λ which are indexed by points $\lambda \in \mathscr{W}$. (These groups will be defined in the case $\mathbf{G} = \mathrm{GL}_n$ in §4 below.)

The morphism w is finite locally in the domain, with discrete fibers. We expect that \mathscr{X} is generically of dimension equal to $\dim \mathscr{W} - l_0$, where¹

$$l_0 = \mathrm{rank} \mathbf{G}(\mathbb{R}) - \mathrm{rank} A_\infty K_\infty$$

¹Here A_∞ denotes the real points of the center of \mathbf{G} , and K_∞ is a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$. Note again that \mathbf{G} is assumed to be split.

is again the length of the range of degrees in which tempered cusp forms contribute to cohomology. When $l_0 = 0$, one expects that the map $w : \mathcal{X} \rightarrow \mathcal{W}$ is generically finite, and that points of \mathcal{X} corresponding to classical automorphic representations are dense. However, when $l_0 > 0$, the image of w should have positive codimension in \mathcal{W} , and the classical points should no longer be Zariski dense in the eigenvariety.

The goals of this paper. We can now discuss what we aim to achieve in this paper. We take $\mathbf{G} = \mathrm{GL}_n/\mathbb{Q}$ with $n \geq 2$, and study the geometry of the eigenvariety in a neighbourhood of a point corresponding to a regular algebraic, cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$ together with a chosen refinement t (i.e. t is a choice of ordering of the eigenvalues of the Satake parameter of π_p). We assume that the pair (π, t) satisfies the following conditions:

- A1.** The representation π is unramified at p , and the Satake parameter of π_p is regular semi-simple.
- A2.** The refinement t has ‘small slope’, in a sense which generalizes the sufficient condition ‘ $\mathrm{val}(a_p) < k - 1$ ’ of Coleman’s classicality criterion [Col96]. (See Theorem 4.7 for the precise condition we impose; this condition depends on a choice of isomorphism $\iota : \overline{\mathbb{Q}_p} \cong \mathbb{C}$.)
- A3.** The infinite component π_∞ satisfies the parity condition (4.3). (This condition is less essential and could probably be removed at the cost of more notation.)

We then study the eigenvariety \mathcal{X} of tame level $K_1(N)$, where N is the conductor of π . Under the above conditions, we construct a point $x \in \mathcal{X}(L)$ corresponding to (π, t) , for some finite extension L/\mathbb{Q}_p . Let $\mathbf{T}_x = \widehat{\mathcal{O}}_{\mathcal{X},x}$ denote the completed local ring at the point x , and let $\Lambda = \widehat{\mathcal{O}}_{\mathcal{W},\lambda}$ denote the completed local ring at the point $\lambda = w(x)$; these are both complete local Noetherian L -algebras, and \mathbf{T}_x is naturally a finite Λ -algebra. Our first main result is as follows.

Theorem 1.1. [Theorem 4.9] *Under the assumptions above, we have $\dim \mathbf{T}_x \geq \dim \mathcal{W} - l_0$, and if equality holds then the following are true:*

1. *The natural map $\Lambda \rightarrow \mathbf{T}_x$ is surjective, and \mathbf{T}_x is a complete intersection ring.*
2. *Let $V_x = \ker(\Lambda \rightarrow \mathbf{T}_x) \otimes_\Lambda L$, so $\dim_L V_x = l_0$. Let $\mathcal{L}_{\lambda,L}$ be the algebraic coefficient system of weight λ , and let \mathfrak{M} be the maximal ideal of the classical Hecke algebra corresponding to the system of Hecke eigenvalues of π and the refinement t . Then the graded vector space $H^*(K_1(N; p), \mathcal{L}_{\lambda,L})[\mathfrak{M}]$ has a canonical structure of free module of rank 1 over the commutative graded ring $\wedge_L^* V_x$.²*

²This structure of $\wedge_L^* V_x$ -module respects the grading up to sign, in the sense that multiplication by elements of $\wedge_L^i V_x$ takes H^j to H^{j-i} . See Theorem 4.9 for a precise statement.

Our second main result builds upon this using the conjectural relation to Galois representations. To this end, we state below Conjecture 4.11. This is an $R = \mathbb{T}$ type conjecture which (roughly speaking) asserts that \mathbf{T}_x carries a family of trianguline Galois representations deforming the representation $\rho_\pi : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_n(L)$, and that this family is universal in a precise sense. Our theorem is then as follows.

Theorem 1.2. *[Theorem 4.13] Let x be a closed point of \mathcal{X} associated to a regular algebraic, cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A})$ equipped with a small slope refinement t , as above. Assume Conjecture 4.11 and the equality $\dim \mathbf{T}_x = \dim \Lambda - l_0$. Then:*

1. *The Bloch–Kato Selmer groups of $\mathrm{ad} \rho_\pi$ and $\mathrm{ad} \rho_\pi(1)$ have dimensions*

$$\dim_L H_f^1(\mathbb{Q}, \mathrm{ad} \rho_\pi) = 0, \quad \dim_L H_f^1(\mathbb{Q}, \mathrm{ad} \rho_\pi(1)) = l_0,$$

and there is a canonical exact sequence

$$H_f^1(\mathbb{Q}, \mathrm{ad} \rho_\pi(1)) \rightarrow \mathfrak{m}_\Lambda / \mathfrak{m}_\Lambda^2 \rightarrow \mathfrak{m}_{\mathbf{T}_x} / \mathfrak{m}_{\mathbf{T}_x}^2 \rightarrow 0.$$

2. *Suppose moreover that \mathcal{X} is smooth at the point x . Then the above sequence is also left exact, and induces an isomorphism $H_f^1(\mathbb{Q}, \mathrm{ad} \rho_\pi(1)) \cong V_x$ (notation as in Theorem 1.1). Consequently, the graded vector space $H^*(K_1(N; p), \mathcal{L}_{\lambda, L})[\mathfrak{M}]$ has a canonical structure of free module of rank 1 over the commutative graded ring*

$$\wedge_L^* H_f^1(\mathbb{Q}, \mathrm{ad} \rho_\pi(1)).$$

Since $H_f^1(\mathbb{Q}, \mathrm{ad} \rho_\pi(1))$ should be exactly the p -adic realization of the group

$$\mathrm{Ext}_{\mathcal{MM}_{\mathbb{Z}}}^1(M(\pi), M(\pi)(1))$$

(see [Jan89, §6]), this is the desired result.

Remarks.

1. As will be clear to the reader, everything we do here could be done with \mathbb{Q} replaced by a general number field F .³ We restrict our attention to the case $F = \mathbb{Q}$ for clarity, since all of the main new ideas already appear here.
2. According to a conjecture of the first author [Hanb, Conj. 1.2.5/6.2.3], our main results should hold under a much weaker assumption than the condition **A2** above: it should suffice to assume that the refinement t corresponds to a “noncritical” refinement of $D_{\mathrm{crys}}(\rho_\pi|G_{\mathbb{Q}_p})$.

³Strictly speaking, the deformation-theoretic results proved in [BC09, Che11] which we crucially use in §3 are only stated in the literature for $G_{\mathbb{Q}_p}$ -representations; however, it is entirely straightforward to generalize these results to G_K -representations for any finite K/\mathbb{Q}_p , taking into account the tools developed in [KPX14].

3. The assumption $\dim \mathbf{T}_x = \dim \Lambda - l_0$, conjectured by Hida in the ordinary case [Hid98] and Urban in the general finite-slope case [Urb11], is a non-abelian analogue of the Leopoldt conjecture, and seems to be of equivalent difficulty [KT]. However, when $n = 3, 4$, we have $l_0 = 1$, in which case this equality follows from [Hanb, Theorem 4.5.1].
4. Conjecture 4.11 represents a serious assumption. However, it does not seem completely out of reach. In the ordinary (i.e. slope zero) case, the second author, together with C. Khare, has studied the problem of proving $R = \mathbb{T}$ theorems assuming only the existence of suitable families of Galois representations [KT]. When the base field is \mathbb{Q} these families have been essentially constructed by Scholze [Sch] (see also [NT]), so the only remaining barrier to proving some cases of Conjecture 4.11 in this situation is establishing local-global compatibility at p .
5. In light of Theorem 1.2, it seems natural to ask the following question which is independent of Venkatesh's conjecture: let $x \in \mathcal{X}$ be the point of the eigenvariety associated to a regular algebraic cuspidal automorphic representation π , which is equipped with a small slope refinement t . Is it always true that \mathcal{X} is smooth at the point x ? It is true if $n = 2$ (in which case the small slope condition implies that the weight map w is even étale at x , see [Bel12, Theorem 2.16]).

The structure of this paper. We now describe the organization of this paper. In §2, we recall the canonical graded ring and module structures on Tor . In §3, we define some crystalline and trianguline deformation rings, following [BC09]. Finally in §4 we make our study of eigenvarieties, prove Theorems 1.1 and 1.2, and discuss some numerical examples using data from [AGG84].

1.1 Notation

We will fix throughout this paper a prime p and an algebraic closure $\overline{\mathbb{Q}}_p$. A finite extension L/\mathbb{Q}_p inside $\overline{\mathbb{Q}}_p$ will be called a coefficient field. If H^i is a cohomology group which is naturally an L -vector space, we will write $h^i = \dim_L H^i$.

All undecorated tensor products are taken over \mathbb{Z} . We write \mathbb{A} for the adèle ring of \mathbb{Q} , and \mathbb{A}^∞ for the ring of finite adeles. All rings without grading will be assumed commutative in the usual sense. If G is a locally profinite group (such as the group $\mathbf{G}(\mathbb{A}^\infty)$, where \mathbf{G} is a reductive group over \mathbb{Q}) and U is an open compact subgroup, then we write $\mathcal{H}(G, U)$ for the algebra of compactly supported functions $U \backslash G / U \rightarrow \mathbb{Z}$. It is a free \mathbb{Z} -module with basis the characteristic functions of double cosets $[UgU]$, and multiplication $[UgU][UhU] = \sum_{i,j} [\alpha_i \beta_j U]$, where $UgU = \coprod_i \alpha_i U$ and $UhU = \coprod_j \beta_j U$.

If F is a perfect field, then we write $G_F = \text{Gal}(\overline{F}/F)$ for the absolute Galois group of F with respect to a fixed algebraic closure. If l is a prime, then we write $\text{Art}_l : \mathbb{Q}_l^\times \rightarrow G_{\mathbb{Q}_l}^{\text{ab}}$ for

the Artin map, normalized to send l to a lift of Frob_l (= geometric Frobenius). If Ω is an algebraically closed field of characteristic 0 and $n \geq 1$ is an integer, then we write rec_l^T for the Tate-normalized local Langlands correspondence for $\text{GL}_n(\mathbb{Q}_p)$, as described in [CT14, §2.1].

If $\rho : G_{\mathbb{Q}_p} \rightarrow \text{GL}_n(\overline{\mathbb{Q}_p})$ is a continuous representation, then we define $D_{\text{crys}}(\rho) = (B_{\text{crys}} \otimes_{\mathbb{Q}_p} \rho)^{G_{\mathbb{Q}_p}}$, as in [Fon94, Exposé III]. If $\rho : G_{\mathbb{Q}_l} \rightarrow \text{GL}_n(\overline{\mathbb{Q}_p})$ is a continuous representation, which is de Rham if $l = p$, then we define a Weil–Deligne representation $\text{WD}(\rho)$ over $\overline{\mathbb{Q}_p}$ as in [TY07, §1]. We note in particular that if $l = p$ and ρ is crystalline, then the eigenvalues of $\text{WD}(\rho)(\text{Frob}_p)$ are the same as the eigenvalues of the crystalline Frobenius on $D_{\text{crys}}(\rho)$. We write $\text{val} : \overline{\mathbb{Q}_p} \rightarrow \mathbb{Q} \cup \{\infty\}$ for the unique p -adic valuation such that $\text{val}(p) = 1$.

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2 Ring structure of Tor

If $A = \bigoplus_{n \geq 0} A_n$ is a graded ring, we say A is *commutative* if the product on A satisfies $a_m \cdot a_n = (-1)^{mn} a_n \cdot a_m$ for any $a_m \in A_m, a_n \in A_n$.

The following proposition plays a key role in our main construction.

Proposition 2.1. *Let R be a commutative ring, and let $I, J \subset R$ be ideals.*

1. *The graded R -module*

$$\text{Tor}_*^R(R/I, R/J) = \bigoplus_{i=0}^{\infty} \text{Tor}_i^R(R/I, R/J)$$

has a canonical structure of commutative graded R -algebra.

2. *If M (resp. N) is an R/I - (resp. R/J -) module, then $\text{Tor}_*^R(M, N)$ has a canonical structure of graded $\text{Tor}_*^R(R/I, R/J)$ -module.*

Proof. We recall that for any pair of commutative rings R, S and any R -modules A, B and S -modules P, Q , there is a natural product map

$$\text{Tor}_*^R(A, B) \otimes \text{Tor}_*^S(P, Q) \rightarrow \text{Tor}_*^{R \otimes S}(A \otimes P, B \otimes Q). \quad (2.1)$$

Taking $R = S$ and $A = P = R/I, B = Q = R/J$, the product map gives morphisms of rings $R \otimes R \rightarrow R, R/I \otimes R/I \rightarrow R/I$ and $R/J \otimes R/J \rightarrow R/J$ (because R is commutative), so combining the map (2.1) with the usual change of ring functoriality for Tor gives a map

$$\text{Tor}_*^R(R/I, R/J) \otimes \text{Tor}_*^R(R/I, R/J) \rightarrow \text{Tor}_*^R(R/I, R/J). \quad (2.2)$$

By [CE99, Ch. XI, §§1–2], this map induces the desired structure of commutative graded R -algebra. An analogous argument with the map (2.1) in the case $R = S$ and $A = R/I$, $B = R/J$, $P = M$, $Q = N$ gives the claimed module structure on $\mathrm{Tor}_*^R(M, N)$. \square

We now want to discuss this structure in a special case. Let R be a Noetherian local ring with residue field L , and let $I \subset R$ be an ideal generated by a regular sequence.

Lemma 2.2. *The isomorphism $\mathrm{Tor}_1^R(R/I, L) \cong I \otimes_R L$ induces a canonical isomorphism*

$$\mathrm{Tor}_*^R(R/I, L) \cong \wedge^*(I \otimes_R L)$$

of commutative graded L -algebras.

Proof. Let $\mathbf{x} = (x_1, \dots, x_r)$ be an R -sequence generating I , and let $K_\bullet(\mathbf{x})$ be the associated Koszul complex with its natural structure as a commutative differential graded R -algebra. We recall that in any fixed degree n ,

$$K_n(\mathbf{x}) = \bigoplus_{1 \leq j_1 < \dots < j_n \leq r} R \cdot e_{j_1 \dots j_n}$$

is a free R -module of rank $\binom{r}{n}$, and the differential on K_\bullet is defined by $d(e_{j_1 \dots j_n}) = \sum_{i=1}^n (-1)^{i+1} x_{j_i} e_{j_1 \dots \hat{j}_i \dots j_n}$. Since $K_\bullet(\mathbf{x})$ gives a minimal free resolution of R/I , the induced differential on $K_\bullet(\mathbf{x}) \otimes_R L$ is zero, and there is a canonical isomorphism

$$f_{\mathbf{x}} : \mathrm{Tor}_*^R(R/I, L) \cong K_\bullet(\mathbf{x}) \otimes_R L \cong \wedge^* I \otimes_R L.$$

It follows from the discussion in [CE99, Ch. XI, §5] that this is in fact an isomorphism of commutative graded algebras (and not just of R -modules). We must check that this isomorphism is independent of the choice of regular sequence \mathbf{x} . However, if $\mathbf{y} = (y_1, \dots, y_r)$ is another choice of regular sequence, then we can write $y_i = \sum_j a_{ij} x_j$ for some matrix $A = (a_{ij})$ with R -coefficients and $\det A \in R^\times$. It is then clear that A induces an isomorphism $K_n(\mathbf{x}) \cong K_n(\mathbf{y})$ of commutative differential graded R -algebras such that the automorphism of $\wedge^* I \otimes_R L$ given by $f_{\mathbf{y}} f_{\mathbf{x}}^{-1}$ is the identity. This completes the proof. \square

3 Galois theory

In this section we recall, following [BC09], some simple cases of the deformation theory of crystalline and trianguline Galois representations in characteristic 0. We fix throughout this section an integer $n \geq 1$, a prime p , and a coefficient field L .

3.1 Recollections on (ϕ, Γ) -modules

We begin by recalling some of the theory of (ϕ, Γ) -modules. We take as given the theory of $\mathcal{D}_{\text{crys}}$. We define the Robba ring \mathcal{R} over \mathbb{Q}_p to be the ring of power series $f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$ with $a_n \in \mathbb{Q}_p$ which converge in some annulus of the form $r(f) < |z - 1| < 1$. This ring has natural commuting actions of the group $\Gamma = \mathbb{Z}_p^*$ and the endomorphism ϕ , determined by the following formulae ($\gamma \in \Gamma$):

$$\gamma(f)(z) = f(z^\gamma), \quad \phi(f)(z) = f(z^p).$$

Let \mathcal{C}_L denote the category of Artinian local L -algebras with residue field L ; objects of \mathcal{C}_L will always be considered with their natural p -adic (L -vector space) topology. If $A \in \mathcal{C}_L$, then we define $\mathcal{R}_A = \mathcal{R} \otimes_{\mathbb{Q}_p} A$; then Γ and ϕ act on \mathcal{R}_A by scalar extension of their actions on \mathcal{R} .

Definition 3.1. *Let $A \in \mathcal{C}_L$. A (ϕ, Γ) -module over A is a finite \mathcal{R}_A -module D , free over \mathcal{R} , equipped with commuting, continuous, \mathcal{R}_A -semilinear actions of ϕ and Γ , and satisfying the condition $\mathcal{R}\phi(D) = D$. A morphism of (ϕ, Γ) -modules over A is a morphism of underlying \mathcal{R}_A -modules which respects the actions of ϕ and Γ .*

Let us write $\text{Mod}_{\phi, \Gamma, A}$ for the category of (ϕ, Γ) -modules over A . We write $\text{Mod}_{\mathbb{Q}_p, A}$ for the category of finitely generated A -modules equipped with a continuous action of the group $G_{\mathbb{Q}_p}$.

Theorem 3.2. *There is an exact, fully faithful functor $D_{\text{rig}} : \text{Mod}_{\mathbb{Q}_p, A} \rightarrow \text{Mod}_{\phi, \Gamma, A}$. If $V \in \text{Mod}_{\mathbb{Q}_p, A}$ is free over A , then $D_{\text{rig}}(V)$ is free over \mathcal{R}_A , and conversely.*

Proof. This follows from works of Fontaine, Chebonnier–Colmez, and Kedlaya, which also allow us to characterize the essential image of the functor D_{rig} as the full subcategory of étale (ϕ, Γ) -modules. For the final statement, see [BC09, Lemma 2.2.7]. \square

If $\delta : \mathbb{Q}_p^\times \rightarrow A^\times$ is a continuous character, then we can define an object $\mathcal{R}_A(\delta) \in \text{Mod}_{\phi, \Gamma, A}$ by taking the underlying module $\mathcal{R}_A(\delta) = \mathcal{R}_A$ and setting $\phi(1) = \delta(p)$, $\gamma(1) = \delta(\gamma)$. It is known ([BC09, Proposition 2.3.1]) that any element of $\text{Mod}_{\phi, \Gamma, A}$ which is free of rank 1 over \mathcal{R}_A is isomorphic to $\mathcal{R}_A(\delta)$ for a uniquely determined continuous homomorphism $\delta : \mathbb{Q}_p^\times \rightarrow A^\times$.

Definition 3.3. *Let $D \in \text{Mod}_{\phi, \Gamma, A}$ be free of rank n over \mathcal{R}_A . A triangulation of D is an increasing filtration $\Delta = (\Delta_i)_{i=0}^n$ by direct summand \mathcal{R}_A -submodules, stable under ϕ, Γ and such that Δ_i is free over \mathcal{R}_A of rank i .*

A representation $V \in \text{Mod}_{\mathbb{Q}_p, A}$ is trianguline if $D_{\text{rig}}(V)$ admits a triangulation.

If $D \in \text{Mod}_{\phi, \Gamma, A}$ is free of rank n over \mathcal{R}_A , then a choice of triangulation determines a tuple $\delta = (\delta_1, \dots, \delta_n)$ of homomorphisms $\delta_i : \mathbb{Q}_p^\times \rightarrow A^\times$, by the formula $\text{gr}_i \Delta = \Delta_i / \Delta_{i-1} \cong \mathcal{R}_A(\delta_i)$. This is called the parameter of the triangulation (cf. [BC09, §2.3.2]).

We now describe the relation between crystalline representations and trianguline representations.

Theorem 3.4. *Let $V \in \text{Mod}_{\mathbb{Q}_p, L}$ be crystalline with $\dim_L V = n$.*

1. *There is an isomorphism $D_{\text{rig}}(V)[1/t]^\Gamma \cong D_{\text{crys}}(V)$ of finite free L -modules equipped with ϕ -action.*
2. *The assignment $\Delta_i \mapsto \Delta_i[1/t]^\Gamma$ defines a bijection between the set of triangulations of $D_{\text{rig}}(V)$ and the set of filtrations $\mathcal{F} = (\mathcal{F}_i)_{i=0}^n$ of $D_{\text{crys}}(V)$ such that \mathcal{F}_i is ϕ -stable and $\dim_L \mathcal{F}_i = i$.*
3. *The representation V is trianguline.*

Proof. The first part is a theorem of Berger [Ber02, Théorème 0.2], who also shows how to recover the filtration on $D_{\text{crys}}(V)$. The second part is deduced from this by Bellaïche–Chenevier [BC09, Proposition 2.4.1]. The third part is an immediate consequence of the second. \square

Following [BC09], we refer to a ϕ -stable filtration $\mathcal{F} = (\mathcal{F}_i)_{i=0}^n$ of $D_{\text{crys}}(V)$ as a refinement. The above theorem shows that giving a refinement of V is the same as giving a triangulation of $D_{\text{rig}}(V)$. It is shown in [BC09, §2.4] that the parameter $\delta = (\delta_1, \dots, \delta_n)$ of the triangulation corresponding to a refinement \mathcal{F} can be described explicitly as follows: we can associate to \mathcal{F} a sequence $\alpha = (\alpha_1, \dots, \alpha_n)$ of elements of L , where α_i is the eigenvalue of ϕ on $\text{gr}_i \mathcal{F}$. We can also associate a sequence $s = (s_1, \dots, s_n)$ of integers, where s_1, \dots, s_i are the jumps in the induced filtration (Hodge) filtration of $\mathcal{F}_i \subset D_{\text{crys}}(V)$. The character $\delta_i : \mathbb{Q}_p^\times \rightarrow L^\times$ is then determined by the formulae

$$\delta_i(p) = \alpha_i p^{-s_i}, \quad \delta_i|_{\mathbb{Z}_p^\times} = \chi^{-s_i},$$

where $\chi : \mathbb{Z}_p^\times \rightarrow L^\times$ is the identity character (which corresponds to the restriction of the cyclotomic character to inertia under our normalization of local class field theory).

Finally, we wish to single out a few particularly pleasant classes of refinements.

Definition 3.5. *Let $V \in \text{Mod}_{\mathbb{Q}_p, L}$ be crystalline with pairwise distinct Hodge–Tate weights $k_1 < k_2 < \dots < k_n$. Let $\mathcal{F} = (\mathcal{F}_i)_{i=0}^n$ be a refinement of V , and let $\alpha = (\alpha_1, \dots, \alpha_n)$, $s = (s_1, \dots, s_n)$ be the tuples defined above.*

1. *We say that the refinement \mathcal{F} is non-critical if we have $s_1 < s_2 < \dots < s_n$.*

2. We say that \mathcal{F} is numerically non-critical if for each $i = 1, \dots, n-1$, we have

$$\text{val}(\alpha_1) + \dots + \text{val}(\alpha_i) < k_1 + \dots + k_{i-1} + k_{i+1}.$$

(In the case $i = 1$, we interpret $k_0 = 0$.)

A numerically non-critical V is non-critical (as follows from the fact that $D_{\text{crys}}(V)$ is weakly admissible), but the converse does not hold.

Definition 3.6. Let $V \in \text{Mod}_{\mathbb{Q}_p, L}$ be crystalline with pairwise distinct Hodge–Tate weights $k_1 < k_2 < \dots < k_n$. Let $\mathcal{F} = (\mathcal{F}_i)_{i=0}^n$ be a refinement of V , with $\alpha = (\alpha_1, \dots, \alpha_n)$ the associated ordering on the eigenvalues of crystalline Frobenius. We say \mathcal{F} is very generic if the following conditions hold:

1. The refinement \mathcal{F} is non-critical.
2. For each $1 \leq i < j \leq n$, we have $\alpha_i \alpha_j^{-1} \notin \{1, p^{-1}\}$.
3. We have $H^0(\mathbb{Q}_p, \text{ad } V(-1)) = 0$.

3.2 Galois deformations

Let S be a finite set of primes containing p , let \mathbb{Q}_S/\mathbb{Q} denote the maximal extension in a fixed algebraic closure unramified outside S , and let $G_{\mathbb{Q}, S} = \text{Gal}(\mathbb{Q}_S/\mathbb{Q})$. Consider a continuous representation $\rho : G_{\mathbb{Q}, S} \rightarrow \text{GL}_n(L)$ satisfying the following conditions:

- The representation ρ is absolutely irreducible.
- The restricted representation $\rho|_{G_{\mathbb{Q}_p}}$ is crystalline, with pairwise distinct Hodge–Tate weights and pairwise distinct crystalline Frobenius eigenvalues which all lie in L .
- For each complex conjugation $c \in G_{\mathbb{Q}, S}$, we have $\text{tr } \rho(c) \in \{-1, 0, 1\}$ (in other words, ρ is odd).

We write $\text{Def}_{\rho, \text{crys}} : \mathcal{C}_L \rightarrow \text{Sets}$ for the functor which assigns to $A \in \mathcal{C}_L$ the set of strict equivalence classes of homomorphisms $\rho_A : G_{\mathbb{Q}, S} \rightarrow \text{GL}_n(A)$ satisfying the following condition:

- C1.** The representation ρ_A lifts ρ , in the sense that $\rho_A \bmod \mathfrak{m}_A = \rho$.
- C2.** For each prime $l \in S, l \neq p$, there is an (unspecified) isomorphism of $I_{\mathbb{Q}_l}$ -representations $\rho_A|_{I_{\mathbb{Q}_l}} \simeq \rho \otimes_L A|_{I_{\mathbb{Q}_l}}$.
- C3.** The restricted representation $\rho_A|_{G_{\mathbb{Q}_p}}$ is crystalline (as an L -representation, forgetting the A -structure).

(Two such ρ_A, ρ'_A are said to be strictly equivalent if they are conjugate under the action of the group $\ker \mathrm{GL}_n(A) \rightarrow \mathrm{GL}_n(L)$.)

Proposition 3.7. *1. The functor $\mathrm{Def}_{\rho, \mathrm{crys}}$ is pro-represented by a complete Noetherian local L -algebra $R_{\rho, \mathrm{crys}}$.*

2. There is a canonical isomorphism

$$\mathrm{Def}_{\rho, \mathrm{crys}}(L[\epsilon]) \cong H_f^1(\mathbb{Q}, \mathrm{ad} \rho),$$

where

$$H_f^1(\mathbb{Q}, \mathrm{ad} \rho) = \ker \left[H^1(\mathbb{Q}_S/\mathbb{Q}, \mathrm{ad} \rho) \rightarrow \frac{H^1(\mathbb{Q}_p, \mathrm{ad} \rho)}{H_f^1(\mathbb{Q}_p, \mathrm{ad} \rho)} \oplus \bigoplus_{\substack{l \in S \\ l \neq p}} H^1(\mathbb{Q}_l^{\mathrm{nr}}, \mathrm{ad} \rho) \right]$$

is the usual Bloch–Kato Selmer group.

Proof. This is standard Galois deformation theory; we briefly sketch the proof. Let $\mathrm{Def}_\rho : \mathcal{C}_L \rightarrow \mathrm{Sets}$ denote the functor of strict equivalence classes of liftings $\rho_A : G_{\mathbb{Q}, S} \rightarrow \mathrm{GL}_n(A)$ of ρ (with no local conditions). The representability of Def_ρ by a complete Noetherian local L -algebra R_ρ is a consequence of Mazur’s theory and standard finiteness properties of Galois cohomology [Maz89].

For each prime $l \in S$, let $\mathrm{Def}_{\rho, l} : \mathcal{C}_L \rightarrow \mathrm{Sets}$ denote the functor of strict equivalence classes of liftings $\rho_A : G_{\mathbb{Q}_l} \rightarrow \mathrm{GL}_n(A)$ of $\rho|_{G_{\mathbb{Q}_l}}$. For any prime $l \in S$ with $l \neq p$ (resp. for $l = p$), condition C2. (resp. C3.) defines a subfunctor $\mathrm{Def}_{\rho, l}^{\mathrm{nr}} \subset \mathrm{Def}_{\rho, l}$ (resp. $\mathrm{Def}_{\rho, p}^{\mathrm{crys}} \subset \mathrm{Def}_{\rho, p}$). We claim these subfunctors in fact define local deformation problems [Maz97, §23], and hence are relatively representable. For $\mathrm{Def}_{\rho, l}^{\mathrm{nr}}$, this is proved in e.g. [BC09, Propositions 7.6.3 and 7.8.5]. For $\mathrm{Def}_{\rho, p}^{\mathrm{crys}}$, we recall that by a classical theorem of Fontaine, the category of crystalline representations of $G_{\mathbb{Q}_p}$ on finite-dimensional L -vector spaces is closed under passage to subobjects, quotients, and direct sums; in particular, $\mathrm{Def}_{\rho, p}^{\mathrm{crys}}$ satisfies the conditions of Ramakrishna’s criterion [Maz97, §25, Proposition 1], and therefore defines a local deformation problem. Since

$$\mathrm{Def}_{\rho, \mathrm{crys}} = \mathrm{Def}_\rho \times \prod_{l \in S} \mathrm{Def}_{\rho, l} \left(\mathrm{Def}_{\rho, p}^{\mathrm{crys}} \times \prod_{\substack{l \in S \\ l \neq p}} \mathrm{Def}_{\rho, l}^{\mathrm{nr}} \right),$$

the first part of the theorem follows as usual.

The second part follows from [Maz97, §26, Proposition 2] together with the identifications

$$\mathrm{Def}_{\rho, l}^{\mathrm{nr}}(L[\epsilon]) = H_{\mathrm{nr}}^1(\mathbb{Q}_l, \mathrm{ad} \rho) := \ker (H^1(\mathbb{Q}_l, \mathrm{ad} \rho) \rightarrow H^1(\mathbb{Q}_l^{\mathrm{nr}}, \mathrm{ad} \rho))$$

and

$$\mathrm{Def}_{\rho,p}^{\mathrm{crys}}(L[\epsilon]) = H_f^1(\mathbb{Q}_p, \mathrm{ad} \rho) := \ker \left(H^1(\mathbb{Q}_p, \mathrm{ad} \rho) \rightarrow H^1(\mathbb{Q}_p, \mathrm{ad} \rho \otimes B_{\mathrm{crys}}) \right),$$

since these are exactly the local conditions defining the global H_f^1 . \square

By our assumption that the crystalline Frobenius φ has no repeated eigenvalues on $D_{\mathrm{crys}}(\rho|_{G_{\mathbb{Q}_p}})$, choosing a refinement $\mathcal{F} = (\mathcal{F}_i)_{i=0}^n$ of $\rho|_{G_{\mathbb{Q}_p}}$ is equivalent to choosing an ordering $\alpha = (\alpha_1, \dots, \alpha_n)$ of these eigenvalues. Let us now fix an ordering α such that the associated refinement $\mathcal{F}(\alpha)$ is very generic, and let $\delta = (\delta_1, \dots, \delta_n)$ be the parameter of the triangulation corresponding to $\mathcal{F}(\alpha)$. We write $\mathrm{Def}_{\rho,\alpha} : \mathcal{C}_L \rightarrow \mathrm{Sets}$ for the functor which assigns to $A \in \mathcal{C}_L$ the set of strict equivalence classes of homomorphisms $\rho_A : G_{\mathbb{Q},S} \rightarrow \mathrm{GL}_n(A)$ satisfying the following conditions:

T1. The representation ρ_A lifts ρ , in the sense that $\rho_A \bmod \mathfrak{m}_A = \rho$.

T2. For each prime $l \in S$, $l \neq p$, there is an (unspecified) isomorphism of $I_{\mathbb{Q}_l}$ -representations $\rho_A|_{I_{\mathbb{Q}_l}} \simeq \rho \otimes_L A|_{I_{\mathbb{Q}_l}}$.

T3. The (ϕ, Γ) -module $D_{\mathrm{rig}}(\rho_A|_{G_{\mathbb{Q}_p}})$ admits a triangulation with parameter δ_A lifting δ .

We write $\mathrm{Def}_{\delta} : \mathcal{C}_L \rightarrow \mathrm{Sets}$ for the functor of continuous lifts $\lambda_A : (\mathbb{Z}_p^\times)^n \rightarrow A^\times$ of the character $\lambda = \delta|_{(\mathbb{Z}_p^\times)^n}$. Our assumptions imply that if $[\rho_A] \in \mathrm{Def}_{\rho,\alpha}(A)$, then there is exactly one triangulation of $D_{\mathrm{rig}}(\rho_A|_{G_{\mathbb{Q}_p}})$ lifting the fixed triangulation of $D_{\mathrm{rig}}(\rho|_{G_{\mathbb{Q}_p}})$, and hence the parameter δ_A of $[\rho_A]$ is well-defined. (It is clearly independent of the choice of representative in the strict equivalence class.) In this way, we obtain a natural transformation $\mathrm{Def}_{\rho,\alpha} \rightarrow \mathrm{Def}_{\delta}$, $[\rho_A] \mapsto \delta_A|_{(\mathbb{Z}_p^\times)^n}$.

Proposition 3.8. *1. The functor $\mathrm{Def}_{\rho,\alpha}$ is pro-represented by a complete Noetherian local L -algebra $R_{\rho,\alpha}$.*

2. There exists a subspace $H_\alpha^1(\mathbb{Q}_p, \mathrm{ad} \rho) \subset H^1(\mathbb{Q}_p, \mathrm{ad} \rho)$ and an isomorphism

$$\mathrm{Def}_{\rho,\alpha}(L[\epsilon]) \cong H_\alpha^1(\mathbb{Q}, \mathrm{ad} \rho),$$

where we define

$$H_\alpha^1(\mathbb{Q}, \mathrm{ad} \rho) = \ker \left[H^1(\mathbb{Q}_S/\mathbb{Q}, \mathrm{ad} \rho) \rightarrow \frac{H^1(\mathbb{Q}_p, \mathrm{ad} \rho)}{H_\alpha^1(\mathbb{Q}_p, \mathrm{ad} \rho)} \oplus \bigoplus_{\substack{l \in S \\ l \neq p}} H^1(\mathbb{Q}_l^{\mathrm{nr}}, \mathrm{ad} \rho) \right].$$

Proof. For the first part, it again suffices to show that the natural transformation $\text{Def}_{\rho,\alpha} \rightarrow \text{Def}_\rho$ is relatively representable. This is exactly analogous to the proof of Proposition 3.7, except we need to know that a suitable local trianguline deformation functor $\text{Def}_{\rho,p}^\alpha \subset \text{Def}_{\rho,p}$ is relatively representable; this follows from [BC09, Proposition 2.5.8].

For the second part, we simply define $H_\alpha^1(\mathbb{Q}_p, \text{ad } \rho)$ as the image of

$$\text{Def}_{\rho,p}^\alpha(L[\epsilon]) \subset \text{Def}_{\rho,p}(L[\epsilon]) \cong H^1(\mathbb{Q}_p, \text{ad } \rho).$$

Having made this definition,⁴ the result again follows as in the proof of Proposition 3.7. \square

We now discuss the relation between the functors $\text{Def}_{\rho,\text{crys}}$ and $\text{Def}_{\rho,\alpha}$. Let us write Λ for the representing object of Def_δ ; it is a formally smooth L -algebra, and there is a canonical augmentation $\Lambda \rightarrow L$, corresponding to the constant homomorphism $\lambda \otimes_L A$. This choice of base point determines a natural isomorphism $\text{Def}_\delta(A) \cong \text{Hom}_{\text{cts}}((\mathbb{Z}_p^\times)^n, 1 + \mathfrak{m}_A)$, $\lambda_A \mapsto \lambda_A(\lambda \otimes_L A)^{-1}$, hence a canonical isomorphism

$$\text{Def}_\delta(L[\epsilon]) \cong \text{Hom}_{\text{cts}}((\mathbb{Z}_p^\times)^n, L) \cong L^n,$$

which sends a character $\lambda_{L[\epsilon]} = \lambda \cdot (1 + \epsilon\mu)$ to the point $-\log(\mu(1+p))/\log(1+p)$.

Proposition 3.9. *We have $\text{Def}_{\rho,\text{crys}} \subset \text{Def}_{\rho,\alpha}$ as subfunctors of Def_ρ , and this inclusion leads to an identification*

$$\text{Def}_{\rho,\text{crys}} = \text{Def}_{\rho,\alpha} \times_{\text{Def}_\delta} \{\delta\}. \quad (3.1)$$

Dually, there is a canonical surjection $R_{\rho,\alpha} \rightarrow R_{\rho,\text{crys}}$, which factors through an isomorphism $R_{\rho,\alpha} \otimes_\Lambda L \cong R_{\rho,\text{crys}}$, and (taking $L[\epsilon]$ -points) an exact sequence of L -vector spaces:

$$0 \longrightarrow H_f^1(\mathbb{Q}, \text{ad } \rho) \longrightarrow H_\alpha^1(\mathbb{Q}, \text{ad } \rho) \longrightarrow L^n. \quad (3.2)$$

Proof. The inclusion $\text{Def}_{\rho,\text{crys}} \subset \text{Def}_{\rho,\alpha}$ is a consequence of [BC09, Proposition 2.5.8]. The identification (3.1) then follows by [BC09, Proposition 2.3.4]. (We invite the reader to compare the exact sequence (3.2) with the exact sequence appearing in the statement of [BC09, Theorem 2.5.10].) \square

Finally, we discuss the information about these deformation functors given by the Euler characteristic formula and Poitou–Tate duality. Let us write $H_f^1(\mathbb{Q}, \text{ad } \rho(1))$ and $H_\alpha^1(\mathbb{Q}, \text{ad } \rho(1))$ for the dual Selmer groups of $H_f^1(\mathbb{Q}, \text{ad } \rho)$ and $H_\alpha^1(\mathbb{Q}, \text{ad } \rho)$, respectively (defined by local conditions for $\text{ad } \rho(1)$ which are the annihilators under Tate local duality of the conditions for $\text{ad } \rho$).

⁴There is a natural description of $H_\alpha^1(\mathbb{Q}_p, \text{ad } \rho)$ in terms of Fontaine–Herr cohomology [Che11, Proposition 3.6.ii], but we don’t need this description in the present paper.

Proposition 3.10. 1. We have

$$h_f^1(\mathbb{Q}, \text{ad } \rho) = h_f^1(\mathbb{Q}, \text{ad } \rho(1)) - \lfloor (n-1)/2 \rfloor$$

and

$$h_\alpha^1(\mathbb{Q}, \text{ad } \rho) = h_\alpha^1(\mathbb{Q}, \text{ad } \rho(1)) + n - \lfloor (n-1)/2 \rfloor.$$

2. The exact sequence of Proposition 3.9 can be extended to an exact sequence

$$0 \rightarrow H_f^1(\mathbb{Q}, \text{ad } \rho) \rightarrow H_\alpha^1(\mathbb{Q}, \text{ad } \rho) \rightarrow L^n \rightarrow H_f^1(\mathbb{Q}, \text{ad } \rho(1))^\vee \rightarrow H_\alpha^1(\mathbb{Q}, \text{ad } \rho(1))^\vee \rightarrow 0.$$

3. There exists a presentation $R_{\rho, \alpha} \cong L[x_1, \dots, x_g]/(f_1, \dots, f_r)$, where $g - r = n - \lfloor (n-1)/2 \rfloor$.

Proof. By the (p -adic version of the) Greenberg–Wiles Euler characteristic formula, we have for any collection of subspaces $\mathcal{L} = \{\mathcal{L}_l\}_{l \in S}$, $\mathcal{L}_l \subset H^1(\mathbb{Q}_l, \text{ad } \rho)$, a formula

$$h_{\mathcal{L}}^1(\mathbb{Q}, \text{ad } \rho) - h_{\mathcal{L}^\perp}^1(\mathbb{Q}, \text{ad } \rho(1)) = 1 - \sum_{l \in S} [\dim_L \mathcal{L}_l - h^0(\mathbb{Q}_l, \text{ad } \rho)] - h^0(\mathbb{R}, \text{ad } \rho).$$

In our case this formula simplifies to become

$$h_{\mathcal{L}}^1(\mathbb{Q}, \text{ad } \rho) - h_{\mathcal{L}^\perp}^1(\mathbb{Q}, \text{ad } \rho(1)) = 1 - [\dim_L \mathcal{L}_p - h^0(\mathbb{Q}_p, \text{ad } \rho)] - \begin{cases} \frac{n^2}{2} & n \text{ even;} \\ \frac{n^2+1}{2} & n \text{ odd.} \end{cases}$$

(This calculation is the only part in this section where we actually use the assumption that ρ is odd.) The first part now follows from the calculation of $\dim_L \mathcal{L}_p$ in each case, as in [BC09, Theorem 2.5.10]. The second part is a consequence of Poitou–Tate duality. (These theorems are most often stated for torsion coefficients, but it is easy to extend them to the case of L -coefficients by an argument of passage to the limit; compare the argument of [Kis03, Lemma 9.7].) For the third part, a standard argument shows the existence of a surjection

$$L[x_1, \dots, x_g] \rightarrow R_{\rho, \alpha}$$

inducing an isomorphism on Zariski tangent spaces, where

$$g = h_\alpha^1(\mathbb{Q}, \text{ad } \rho) = \dim_L \text{Def}_{\rho, \alpha}(L[\epsilon]).$$

To bound the number of relations, we use the fact (see [BC09, Theorem 2.5.10] again) that the functor of trianguline lifts of $\rho|_{G_{\mathbb{Q}_p}}$ is formally smooth over L . This implies that $H_\alpha^1(\mathbb{Q}, \text{ad } \rho(1))^\vee$ forms an obstruction space, so the existence of the desired presentation follows from the first part of the proposition. (For a similar argument in the torsion context, see [CHT08, Corollary 2.2.12] and [CHT08, Lemma 2.3.4].) \square

4 Eigenvarieties and Venkatesh's conjecture

In this section we discuss arithmetic locally symmetric spaces and their cohomology. We then introduce the eigenvariety for GL_n , following [Hanb]. Having done this, we will be able to state and prove our main result.

4.1 Cohomology of GL_n

Fix an integer $n \geq 1$, a prime p , and a coefficient field L . Let $\mathbf{G} = \mathrm{GL}_n$, an algebraic group over \mathbb{Z} , and fix a maximal compact subgroup $K_\infty \subset \mathbf{G}(\mathbb{R})$. If $K \subset \mathbf{G}(\widehat{\mathbb{Z}})$ is an open compact subgroup, we have an associated topological space defined as a double quotient

$$Y_K = \mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}) / K \cdot \mathbb{R}_{>0} K_\infty.$$

If K is neat (in the sense that the associated arithmetic groups $gKg^{-1} \cap \mathbf{G}(\mathbb{Q})$ ($g \in \mathbf{G}(\mathbb{A}^\infty)$) are all neat), then Y_K is a topological manifold of dimension $d = \frac{1}{2}(n-1)(n+2)$. If $N \geq 1$ is an integer prime to p , then we write $K_1(N)$ for the usual mirabolic congruence subgroup consisting of matrices in $\mathrm{GL}_n(\widehat{\mathbb{Z}})$ whose last row is congruent to $(0, \dots, 0, 1)$ modulo N .

Let $\mathbf{T} \subset \mathbf{B} \subset \mathbf{G}$ denote the standard diagonal torus and upper-triangular Borel subgroup, respectively, and let $\mathbf{N} \subset \mathbf{B}$ denote the unipotent radical. Let $\overline{\mathbf{B}}$ denote the opposite Borel subgroup. We write $I \subset \mathbf{G}(\mathbb{Z}_p)$ for the standard Iwahori subgroup (pre-image in $\mathbf{G}(\mathbb{Z}_p)$ of $\mathbf{B}(\mathbb{F}_p)$ under reduction modulo p), and also define $K_1(N; p) = K_1(N)^p I$.

Let \mathfrak{S}_n denote the Weyl group of \mathbf{T} , which we identify with the symmetric group on $\{1, \dots, n\}$. For any $w \in \mathfrak{S}_n$ and any $\lambda \in X^*(\mathbf{T})$ we set $w*\lambda = (\lambda + \rho)^w - \rho$, with $\rho \in \frac{1}{2}X^*(\mathbf{T})$ the usual half-sum of \mathbf{B} -positive roots.

We define abstract Hecke algebras

$$\mathbf{T}^{(N)} = \mathcal{H}(\mathbf{G}(\mathbb{A}^{\infty, Np}), \mathbf{G}(\widehat{\mathbb{Z}}^{Np}))$$

and

$$\mathbf{T}^{(N), p} = \mathbf{T}^{(N)} \otimes \mathbb{Z}[X_*(\mathbf{T})^-],$$

where $X_*(\mathbf{T})^- \subset X_*(\mathbf{T})$ is the subset of anti-dominant cocharacters. These are commutative rings. If A is any ring then we define $\mathbf{T}_A^{(N)} = \mathbf{T}^{(N)} \otimes A$ and $\mathbf{T}_A^{(N), p} = \mathbf{T}^{(N), p} \otimes A$. There is an injective algebra homomorphism (see [Hid98, §4])

$$\mathbb{Z}[X_*(\mathbf{T})^-] \rightarrow \mathcal{H}(\mathbf{G}(\mathbb{Q}_p), I), \quad \mu \mapsto U_\mu := [I\mu(p)I]. \quad (4.1)$$

We write $U_{p,i}$ for the operator associated with the cocharacter

$$\mu_i : t \mapsto \mathrm{diag} \left(1, \dots, 1, \underbrace{t, \dots, t}_i \right),$$

and we set $U_p = U_{p,1} \cdots U_{p,n-1}$.

Let us now fix an integer $N \geq 1$ prime to p . Let $\Delta_p \subset \mathbf{G}(\mathbb{Q}_p)$ denote the monoid

$$\Delta_p = \prod_{\mu \in X_*(\mathbf{T})^-} I\mu(p)I.$$

If M is any $\mathbb{Q}[\Delta_p]$ -module, and $K = \prod_l K_l$ is a neat open compact subgroup of $\mathbf{G}(\widehat{\mathbb{Z}})$ such that $K_p \subset I$, then we can define an associated locally constant sheaf \underline{M} on Y_K as the sheaf of sections of the map

$$\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}) \times M/K \cdot \mathbb{R}_{>0} K_\infty \rightarrow Y_K,$$

where $\mathbf{G}(\mathbb{Q})$ acts trivially on M and K acts via projection to K_p . Similarly, if M is a $\mathbb{Q}[\mathbf{G}(\mathbb{Q})]$ -module, then we can define an associated locally constant sheaf \underline{M} on Y_K as the sheaf of sections of the map

$$\mathbf{G}(\mathbb{Q}) \backslash \mathbf{G}(\mathbb{A}) \times M/K \cdot \mathbb{R}_{>0} K_\infty \rightarrow Y_K,$$

where K acts trivially on M . If M is a $\mathbb{Q}[\mathbf{G}(\mathbb{Q}_p)]$ -module then it has natural structures both of $\mathbb{Q}[\Delta_p]$ - and of $\mathbb{Q}[\mathbf{G}(\mathbb{Q})]$ -module, and the associated sheaves \underline{M} are canonically isomorphic.

We will be interested only in cohomology at level $K_1(N;p)$. If M is a $\mathbb{Q}[\Delta_p]$ - or $\mathbb{Q}[\mathbf{G}(\mathbb{Q})]$ -module, then we define the groups

$$H^*(K_1(N;p), M) = H^*(Y_K, \underline{M})^{K_1(N;p)},$$

where $K \subset K_1(N;p)$ is any neat, normal open compact subgroup. This is canonically independent of the choice of subgroup K , and the cohomology groups have a canonical structure of $\mathbf{T}^{(N),p}$ -module. If M is a $\mathbb{Q}[\mathbf{G}(\mathbb{Q}_p)]$ -module, then the two different ways of defining the Hecke action (via Δ_p and $\mathbf{G}(\mathbb{Q})$) are the same.

If X is any $\mathbf{T}_A^{(N)}$ -module then we write $\mathbf{T}_A^{(N)}(X)$ for the image

$$\mathbf{T}_A^{(N)}(X) = \text{im}(\mathbf{T}_A^{(N)} \rightarrow \text{End}_A(X)),$$

and similarly for $\mathbf{T}_A^{(N),p}$. If X is finitely generated as an A -module, then $\mathbf{T}_A^{(N)}(X)$ is a finite A -algebra.

We now introduce algebraic coefficient systems. If $\lambda \in X^*(\mathbf{T})^+$ is a dominant weight, and E is a field of characteristic 0, then we define

$$\mathcal{L}_{\lambda,E} = \text{Ind}_{\mathbf{B}}^{\mathbf{G}} \lambda = \{f \in E[\mathbf{G}] \mid \forall g \in \mathbf{G}, b \in \overline{\mathbf{B}}, f(bg) = \lambda(b)f(g)\}.$$

This is the abelian group of E -points of the algebraic representation of \mathbf{G} of highest weight λ ; it is an absolutely irreducible $E[\mathbf{G}(E)]$ -module. Any character $\lambda \in X^*(\mathbf{T})$ determines a character $\Delta_p \rightarrow \mathbb{Q}_p^\times$, by the formula $I\mu(p)I \mapsto p^{(\lambda,\mu)}$, and we will also use the twisted

$L[\Delta_p]$ -modules $\mathcal{L}_{\lambda,L} \otimes_L L(\lambda)^{-1}$. It is helpful to note at this point that if $\mu \in X_*(\mathbf{T})^-$, then the element $\mu(p) \in \Delta_p$ has eigenvalues on $\mathcal{L}_{\lambda,L} \otimes_L L(\lambda)^{-1}$ of non-negative valuation, and acts trivially on the highest weight space.

For any field E of characteristic 0, the Hecke action is defined over \mathbb{Q} in the sense that there is a canonical isomorphism

$$H^*(K_1(N; p), \mathcal{L}_{\lambda,E}) \cong H^*(K_1(N; p), \mathcal{L}_{\lambda,\mathbb{Q}}) \otimes_{\mathbb{Q}} E \quad (4.2)$$

which respects the isomorphism $\mathbf{T}_E^{(N),p} \cong \mathbf{T}_{\mathbb{Q}}^{(N),p} \otimes_{\mathbb{Q}} E$.

Definition 4.1. Let $\pi = \otimes'_l \pi_l$ be an irreducible admissible $\mathbb{C}[\mathrm{GL}_n(\mathbb{A}^\infty)]$ -module such that π_l is unramified if $l \nmid Np$, and let

$$\mathfrak{N} = \ker(\mathbf{T}_{\mathbb{C}}^{(N)} \rightarrow \mathrm{End}_{\mathbb{C}}(\otimes'_{l|Np} (\pi_l^{\mathrm{GL}_n(\mathbb{Z}_l)}))),$$

a maximal ideal of $\mathbf{T}_{\mathbb{C}}^{(N)}$ with residue field \mathbb{C} . If M is a $\mathbf{T}_{\mathbb{C}}^{(N)}$ -module, we say that π contributes to M if the localization $M_{\mathfrak{N}}$ is non-zero.

We set $l_0 = \lfloor \frac{n-1}{2} \rfloor$ and $q_0 = (d - l_0)/2$.

Proposition 4.2. Let π be a regular algebraic, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ of conductor dividing N satisfying the following parity condition: if $n = 2m + 1$ is odd, then the central character of π_∞ has the form $x \in \mathbb{R}^\times \mapsto |x|^a \mathrm{sgn}(x)^\epsilon$, and we require

$$\epsilon \equiv a \pmod{2}. \quad (4.3)$$

(Note that this condition can always be achieved after perhaps replacing π with a twist by a quadratic character.)

1. There exists a unique $\lambda \in X^*(\mathbf{T})^+$ such that π contributes to $H^*(K_1(N; p), \mathcal{L}_{\lambda,\mathbb{C}})$. Moreover, these groups are non-zero exactly in the range $[q_0, q_0 + l_0]$.
2. Let $\mathfrak{N} \subset \mathbf{T}_{\mathbb{C}}^{(N)}$ be the maximal ideal associated to π^∞ by Definition 4.1. Then we have

$$H^*(K_1(N; p), \mathcal{L}_{\lambda,\mathbb{C}})_{\mathfrak{N}} = H^*(K_1(N; p), \mathcal{L}_{\lambda,\mathbb{C}})[\mathfrak{N}]$$

and hence

$$\mathbf{T}_{\mathbb{C}}^{(N)}(H^*(K_1(N; p), \mathcal{L}_{\lambda,\mathbb{C}}))_{\mathfrak{N}} = \mathbb{C}.$$

3. Suppose that N equals the conductor of π . Then for each $i = 0, \dots, l_0$ we have

$$\dim_{\mathbb{C}} H^{q_0+i}(K_1(N; p), \mathcal{L}_{\lambda,\mathbb{C}})_{\mathfrak{N}} = n! \binom{l_0}{i}.$$

4. Suppose that the Satake parameter of π_p is regular semi-simple. Then there are exactly $n!$ maximal ideals $\tilde{\mathfrak{N}}$ of $\mathbf{T}_{\mathbb{C}}^{(N),p}$ lying above \mathfrak{N} , which are in natural bijection with the set of orderings of the eigenvalues of the Satake parameters of π_p , and for each one we have

$$\dim_{\mathbb{C}} H^{q_0+i}(K_1(N;p), \mathcal{L}_{\lambda, \mathbb{C}})_{\tilde{\mathfrak{N}}} = \binom{l_0}{i}$$

for each $i = 0, \dots, l_0$.

Proof. Let $\mathfrak{m} = \text{Lie SL}_n(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. According to the theory of Eisenstein cohomology, there is for any $\mu \in X^*(\mathbf{T})^+$ a canonical decomposition

$$H^*(K_1(N;p), \mathcal{L}_{\mu, \mathbb{C}}) \cong \bigoplus_{\{\mathbf{P}\} \in \mathcal{C}} \bigoplus_{\varphi \in \Phi_{\mu, \{\mathbf{P}\}}} H^*(\mathfrak{m}, K_{\infty}; A_{\mu, \{\mathbf{P}\}, \varphi}^{K_1(N;p)} \otimes_{\mathbb{C}} \mathcal{L}_{\mu, \mathbb{C}})(\chi_{\mu}), \quad (4.4)$$

where \mathcal{C} is the set of associate classes of parabolic subgroups, $\Phi_{\mu, \{\mathbf{P}\}}$ is a set of finite sets $\varphi = \{\varphi_{\mathbf{P}}\}_{\mathbf{P} \in \{\mathbf{P}\}}$ of cuspidal automorphic representations of the Levi factors of elements of the class $\{\mathbf{P}\}$ satisfying certain conditions, and $A_{\mu, \{\mathbf{P}\}, \varphi}$ is a space of automorphic forms on \mathbf{G} defined in terms of Eisenstein series. The cohomology on the right-hand side is relative Lie algebra cohomology. See [FS98, §1] for a precise statement. The final (χ_{μ}) represents a character twist in the Hecke action which depends only on the central character of $\mathcal{L}_{\mu, \mathbb{C}}$ [FS98, §2.1].

In particular, the summand corresponding to \mathbf{G} is the ‘cuspidal cohomology’

$$H_{\text{cusp}}^*(K_1(N;p), \mathcal{L}_{\mu, \mathbb{C}}) = \bigoplus_{\pi \in \Phi_{\mu, \{\mathbf{G}\}}} (\pi^{\infty})^{K_1(N;p)} \otimes_{\mathbb{C}} H^*(\mathfrak{m}, K_{\infty}; \pi_{\infty} \otimes_{\mathbb{C}} \mathcal{L}_{\mu, \mathbb{C}})(\chi_{\mu}),$$

where $\Phi_{\mu, \{\mathbf{G}\}}$ is now just the set of cuspidal automorphic representations of $\text{GL}_n(\mathbb{A})$ with the same infinitesimal character as $\mathcal{L}_{\mu, \mathbb{C}}^{\vee}$. We first claim that for any $\mu \in X^*(\mathbf{T})^+$, we have

$$H^*(K_1(N;p), \mathcal{L}_{\mu, \mathbb{C}})_{\mathfrak{N}} = H_{\text{cusp}}^*(K_1(N;p), \mathcal{L}_{\mu, \mathbb{C}})_{\mathfrak{N}}.$$

In other words, the other summands in the decomposition (4.4) all vanish after localization at \mathfrak{N} . This follows immediately from [FS98, Proposition 4.1], which relies on the Jacquet–Shalika classification of automorphic representations of $\text{GL}_n(\mathbb{A})$. The first part of the proposition now follows. Indeed, there is exactly one $\lambda \in X^*(\mathbf{T})^+$ such that $\mathcal{L}_{\lambda, \mathbb{C}}^{\vee}$ has the same infinitesimal character as π_{∞} , which is a necessary condition for the relative Lie algebra cohomology to be non-zero [BW00, Ch. II, Proposition 3.1]. On the other hand, one knows ([Clo90, Lemme 4.9]) that π_{∞} is essentially tempered, and [Clo90, Lemme 3.14] then shows that when the parity condition (4.3) holds, the groups $H^i(\pi_{\infty} \otimes_{\mathbb{C}} \mathcal{L}_{\lambda, \mathbb{C}})$ are non-zero if and only if $i \in [q_0, q_0 + l_0]$, in which case they have dimension $\binom{l_0}{i - q_0}$. (If the parity condition does not hold, then the groups are 0.) This also shows the third part of the proposition. (Note that π_p is generic, since π is cuspidal, and so π_p is equal to a full parabolic induction and therefore has $\dim_{\mathbb{C}} \pi_p^I = n!$.) The second part simply asserts that

$$H_{\text{cusp}}^*(K_1(N;p), \mathcal{L}_{\mu, \mathbb{C}})_{\mathfrak{N}} \cong (\pi^{\infty})^{K_1(N;p)} \otimes_{\mathbb{C}} H^*(\mathfrak{m}, K_{\infty}; \pi_{\infty} \otimes_{\mathbb{C}} \mathcal{L}_{\mu, \mathbb{C}})(\chi_{\mu})$$

is a semi-simple $\mathbf{T}_{\mathbb{C}}^{(N)}$ -module, which follows from the existence of this isomorphism. The fourth part of the proposition is equivalent to the local statement that π_p^I splits as a direct sum of $n!$ 1-dimensional $\mathbb{Z}[X_*(\mathbf{T})^-]$ -modules. Let $\alpha : \mathbb{Z}[X_*(\mathbf{T})^-] \rightarrow \mathbb{Z}[\mathbf{T}(\mathbb{Q}_p)]$ be the homomorphism given on elements $\mu \in X_*(\mathbf{T})^-$ by the formula $\mu \mapsto \mu(p)$. If we write $\pi_p = \mathfrak{n}\text{-Ind}_{\mathbf{B}(\mathbb{Q}_p)}^{\mathbf{G}(\mathbb{Q}_p)} \chi$, for some unramified character $\chi : \mathbf{T}(\mathbb{Q}_p) \rightarrow \mathbb{C}^\times$, then a standard calculation (see [Che04, Lemma 4.8.4]) shows that π_p^I is isomorphic as $\mathbb{Z}[X_*(\mathbf{T})^-]$ -module to the direct sum

$$\pi_p^I \cong \bigoplus_{w \in \mathfrak{S}_n} (\chi^w \delta_B^{1/2}) \circ \alpha,$$

where $\delta_B : \mathbf{T}(\mathbb{Q}_p) \rightarrow \mathbb{C}^\times$, $t \mapsto |\det(\text{Ad}_{\mathbf{G}}(t)|_{\text{Lie } \mathbf{N}})|_p$, is the standard modulus character. This completes the proof. \square

Definition 4.3. *If π is an irreducible admissible $\mathbb{C}[\text{GL}_n(\mathbb{A}^\infty)]$ -module, then for any $\sigma \in \text{Aut}(\mathbb{C})$ we can define the conjugated representation $\pi^\sigma = \pi \otimes_{\mathbb{C}, \sigma} \mathbb{C}$. We write $\mathbb{Q}(\pi)$ for the fixed field of the stabilizer in $\text{Aut}(\mathbb{C})$ of the isomorphism class of π , and call it the field of definition of π .*

It is known ([Clo90, Proposition 3.1]) that π can be defined over its field of definition. In particular, if π is unramified outside N then the homomorphism $\mathbb{T}^{(N)} \rightarrow \mathbb{C}$ describing the Hecke eigenvalues of π takes values in $\mathbb{Q}(\pi)$. If π is the finite part of a regular algebraic, cuspidal automorphic representation of $\text{GL}_n(\mathbb{A})$, then $\mathbb{Q}(\pi)$ is a number field [Clo90, Théorème 3.13].

Corollary 4.4. *Let π be a regular algebraic, cuspidal automorphic representation of $\text{GL}_n(\mathbb{A})$. Suppose that π_p is unramified, that π has conductor N , and that the Satake parameter of π_p is regular semi-simple. Fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \cong \mathbb{C}$ and an ordering $t = (t_1, \dots, t_n)$ of the eigenvalues of the Satake parameter of π_p . Let $\tilde{\mathfrak{N}} \subset \mathbf{T}_{\mathbb{C}}^{(N),p}$ be the maximal ideal associated to the pair (π, t) by Proposition 4.2, and suppose that $\iota(L)$ contains the image of the natural homomorphism $\mathbf{T}^{(N),p} \rightarrow \mathbf{T}_{\mathbb{C}}^{(N),p}/\tilde{\mathfrak{N}}$. (This condition can always be achieved, after possibly enlarging the coefficient field L .) Let $\mathfrak{M} = \iota^{-1}\tilde{\mathfrak{N}} \cap \mathbf{T}_L^{(N),p}$.*

1. We have

$$H^*(K_1(N; p), \mathcal{L}_{\lambda, L})_{\mathfrak{M}} = H^*(K_1(N; p), \mathcal{L}_{\lambda, L})[\mathfrak{M}]$$

and

$$\mathbf{T}_L^{(N),p}(H^*(K_1(N; p), \mathcal{L}_{\lambda, L})_{\mathfrak{M}}) = L.$$

We have $\iota(U_{p,i} \bmod \mathfrak{M}) = t_n \dots t_{n-i+1} p^{\sum_{j=1}^i (n-1)/2 - (j-1)}$.

2. For each $i = 0, \dots, l_0$ we have

$$\dim_L H^{q_0+i}(K_1(N; p), \mathcal{L}_{\lambda, L})_{\mathfrak{M}} = \binom{l_0}{i},$$

and $H^i(K_1(N; p), \mathcal{L}_{\lambda, L})_{\mathfrak{M}} = 0$ if $i \notin [q_0, q_0 + l_0]$.

Proof. The corollary follows from Proposition 4.2, the isomorphism (4.2), and the fact that everything is defined over a common algebraic subfield of L and \mathbb{C} . \square

By twisting, we deduce a trivial variant of Corollary 4.4 for the coefficient system $\mathcal{L}_{\lambda, L} \otimes_L L(\lambda)^{-1}$.

4.2 Recollections on the eigenvariety

We have described the cohomology of the group $K_1(N)$ with coefficients in algebraic local systems; we now discuss its cohomology in p -adic coefficient systems and introduce the eigenvariety. We write \mathscr{W} for the rigid space over L which represents the functor

$$\mathscr{W} : X \mapsto \mathrm{Hom}_{\mathrm{cts}}(\mathbf{T}(\mathbb{Z}_p), \mathcal{O}(X)^\times).$$

According to [Hanb, Theorem 1.1.2], one can associate canonically to the pair $(\mathbf{G}, K_1(N))$ the following data:

- A separated rigid space $\mathscr{X} = \mathscr{X}_{\mathbf{G}, K_1(N)}$ over L equipped with a morphism $w : \mathscr{X} \rightarrow \mathscr{W}$.
- A homomorphism $\mathbf{T}_L^{(N), p} \rightarrow \mathcal{O}(\mathscr{X})$ of L -algebras.

These data will satisfy the following conditions:

- The morphism w has discrete fibers and is finite locally on the domain.
- For any point $\lambda \in \mathscr{W}(\overline{\mathbb{Q}_p})$, there is a canonical bijection between points in the fiber $w^{-1}(\lambda)$ and the set of finite-slope eigenpackets (see below) of weight λ and level $K_1(N)$, realized by sending the point x to the composite homomorphism

$$\phi_{\mathscr{X}, x} : \mathbf{T}_L^{(N), p} \rightarrow \mathcal{O}(\mathscr{X}) \rightarrow k(x).$$

The space \mathscr{X} is what we call the eigenvariety of tame level N . We now explain what is meant by a ‘finite slope eigenpacket of weight λ ’. We must first write down certain $\mathbb{Q}_p[\Delta_p]$ -modules, whose definition we recall from [Hanb, §2.2].

If $\Omega \subset \mathscr{W}$ is any affinoid open subset, then there exists an integer $s[\Omega] \geq 1$ such that the universal character $\chi_\Omega : \mathbf{T}(\mathbb{Z}_p) \rightarrow \mathcal{O}(\Omega)^\times$ becomes analytic after restriction to the group $\ker(\mathbf{T}(\mathbb{Z}_p) \rightarrow \mathbf{T}(\mathbb{Z}_p/(p^{s[\Omega]})))$. If $s \geq 1$, then we define a normal subgroup of the Iwahori subgroup I by

$$I^s = \{g \in I \cap (1 + p^s M_n(\mathbb{Z}_p))\}.$$

We then define, for any integer $s \geq s[\Omega]$,

$$\mathbf{A}_\Omega^s = [\mathrm{Ind}_{\mathbf{B}(\mathbb{Z}_p)}^I \chi_\Omega]^{I^s\text{-an}},$$

where the superscript indicates that we are considering functions $f : I \rightarrow \mathcal{O}(\Omega)$ that are analytic on each left I^s -coset, considered as the set of \mathbb{Q}_p -points of an affinoid subspace of $\mathbf{G}_{\mathbb{Q}_p}^{\mathrm{rig}}$. The space \mathbf{A}_Ω^s is naturally a Banach $\mathcal{O}(\Omega)$ -module ([Hanb, p. 18]). We then define

$$\mathcal{A}_\Omega = \lim_{s \rightarrow \infty} \mathbf{A}_\Omega^s,$$

a topological $\mathcal{O}(\Omega)$ -module equipped with the direct limit topology, and

$$\mathcal{D}_\Omega = \mathrm{Hom}_{\mathrm{cts}, \mathcal{O}(\Omega)\text{-linear}}(\mathcal{A}_\Omega, \mathcal{O}(\Omega)).$$

It has a natural structure of $\mathcal{O}(\Omega)[\Delta_p]$ -module. If $\lambda \in \mathcal{W}(\overline{\mathbb{Q}_p})$ is a closed point with residue field $k(\lambda)$, then we define $\mathcal{D}_\lambda = \mathcal{D}_\Omega \otimes_{\mathcal{O}(\Omega)} k(\lambda)$ for some choice of Ω containing λ (this is independent of the chosen Ω). If λ is in the image of the natural embedding $X^*(\mathbf{T})^+ \rightarrow \mathcal{W}(\overline{\mathbb{Q}_p})$, then there is ([Hanb, §2.2]) a canonical surjection of $L[\Delta_p]$ -modules $i_\lambda : \mathcal{D}_\lambda \rightarrow \mathcal{L}_{\lambda, L} \otimes_L L(\lambda^{-1})$, hence a morphism of $\mathbf{T}_L^{(N), p}$ -modules

$$H^*(K_1(N; p), \mathcal{D}_\lambda) \rightarrow H^*(K_1(N; p), \mathcal{L}_{\lambda, L} \otimes_L L(\lambda^{-1})),$$

and \mathcal{D}_λ should in this case be viewed as the ‘overconvergent’ version of $\mathcal{L}_{\lambda, L}$.

We recall ([Hanb, Proposition 3.1.5]) that for any rational number h , there is a canonical decomposition of $\mathbf{T}^{(N), p}$ -modules

$$H^*(K_1(N; p), \mathcal{D}_\lambda) = H^*(K_1(N; p), \mathcal{D}_\lambda)_{\leq h} \oplus H^*(K_1(N; p), \mathcal{D}_\lambda)_{> h}, \quad (4.5)$$

where U_p acts with eigenvalues of slope (i.e. p -adic valuation) at most h on the first summand, and eigenvalues of slope strictly greater than h on the second. Moreover, the subspace $H^*(K_1(N; p), \mathcal{D}_\lambda)_{\leq h}$ is finite-dimensional.

Definition 4.5. *A finite-slope eigenpacket of weight λ and tame level N is a homomorphism $\phi : \mathbf{T}_L^{(N), p} \rightarrow \overline{\mathbb{Q}_p}$ such that the maximal ideal $\mathfrak{M} = \ker \phi \subset \mathbb{T}_L^{(N), p}$ appears in the support of $H^*(K_1(N; p), \mathcal{D}_\lambda)_{\leq h}$ for some rational number h .*

The eigenpacket ϕ is called classical if the ideal \mathfrak{M} in fact appears in the support of $H^(K_1(N; p), \mathcal{L}_{\lambda, L} \otimes_L L(\lambda^{-1}))$, and a classical eigenpacket is noncritical if the natural map*

$$H^*(K_1(N; p), \mathcal{D}_\lambda)_{\mathfrak{M}} \rightarrow H^*(K_1(N; p), \mathcal{L}_{\lambda, L} \otimes_L L(\lambda^{-1}))_{\mathfrak{M}}$$

is an isomorphism.

We next give a convenient (and optimally general) numerical criterion for a finite-slope eigenpacket to be noncritical. This result is well-known to experts, and amounts to a special case of [Urb11, Prop. 4.3.10]; for the convenience of the reader, we give a fairly detailed sketch of the proof.

Before stating the noncriticality criterion, we recall the following lemma, which plays a crucial role in the proof.

Lemma 4.6. *For any weight $\lambda \in \mathcal{W}(\overline{\mathbb{Q}}_p)$ and any $\mu \in X_*(\mathbf{T})^-$, the eigenvalues of U_μ on $H^*(K_1(N; p), \mathcal{D}_\lambda)_{\leq h}$ are all p -integral.*

Proof (Sketch). We use the notation of [Hanb]. By [Hanb, Proposition 3.1.5], it suffices to prove the same for

$$H^*(K_1(N; p), \mathbf{D}_\lambda^s)$$

for any $s \gg 0$. Lifting U_μ to \tilde{U}_μ acting on a suitable Borel–Serre complex $C^\bullet(K_1(N; p), \mathbf{D}_\lambda^s)$, one shows that any U_μ -eigenvalue on H^* is also a \tilde{U}_μ -eigenvalue on C^\bullet . On the other hand, $C^\bullet(K_1(N; p), \mathbf{D}_\lambda^s)$ is naturally a \mathbb{Q}_p -Banach space, with unit ball $C^\bullet(K_1(N; p), \mathbf{D}_\lambda^{s,0})$; one concludes by noting that the latter is preserved by \tilde{U}_μ . \square

Theorem 4.7. *Fix a weight $\lambda = (k_1 \geq k_2 \geq \dots \geq k_n) \in X^*(\mathbf{T})^+$, and fix an algebra homomorphism $\phi : \mathbf{T}_L^{(N),p} \rightarrow \overline{\mathbb{Q}}_p$ with kernel \mathfrak{M} . We say ϕ is numerically noncritical for λ if it satisfies one of the following equivalent conditions:*

1. *For each non-trivial element $w \in \mathfrak{S}_n$, we have $\text{val}(\phi(U_\mu)) < \langle w * \lambda - \lambda, \mu \rangle$ for some $\mu \in X_*(\mathbf{T})^-$.*
2. *For each simple reflection $w \in \mathfrak{S}_n$, we have $\text{val}(\phi(U_\mu)) < \langle w * \lambda - \lambda, \mu \rangle$ for some $\mu \in X_*(\mathbf{T})^-$.*
3. *$\text{val}(\phi(U_{p,i})) < 1 + k_{n-i} - k_{n+1-i}$ for each $1 \leq i \leq n - 1$.*

If ϕ is numerically noncritical, then the map

$$H^*(K_1(N; p), \mathcal{D}_\lambda)_{\mathfrak{M}} \rightarrow H^*(K_1(N; p), \mathcal{L}_{\lambda,L} \otimes_L L(\lambda^{-1}))_{\mathfrak{M}}$$

is an isomorphism; in particular, if ϕ occurs in the target of this map, then ϕ is automatically a noncritical finite-slope eigenpacket of weight λ .

Proof. We first prove the equivalence of the conditions 1–3. The implication $1 \Rightarrow 2$ is trivial. For the implication $2 \Rightarrow 1$, choose an arbitrary non-trivial element $w \in \mathfrak{S}_n$, and a reduced expression $w = s_1 \dots s_k$, where the $s_i = s_{\alpha_i}$ are simple reflections (with respect to the set Φ^+ of positive roots and root basis B corresponding to our fixed choice of Borel subgroup

B). Thus $l(w) = k$, where $l : \mathfrak{S}_n \rightarrow \mathbb{Z}$ is the length function associated to the root basis B . We then have formulae

$$w * \lambda = \lambda^w + \rho^w - \rho = \lambda^w + \sum_{\substack{\alpha \in \Phi^+ \\ \alpha^w \in \Phi^-}} \alpha^w$$

and

$$s_1 * \lambda = \lambda^{s_1} + \rho^{s_1} - \rho = \lambda^{s_1} - \alpha_1.$$

Define a partial ordering on $X^*(\mathbf{T})$ by the formula $\lambda_1 \geq \lambda_2$ if $\lambda_1 - \lambda_2 = \sum_{\alpha \in B} m_\alpha \alpha$, where the m_α are non-negative integers. The formula $l(s_1 w) < l(w)$ implies $-\alpha_1 \in w(\Phi^+)$. The equation $s_1 \leq w$ in the Bruhat ordering of W implies that $\lambda^{s_1} \geq \lambda^w$. We find that $s_1 * \lambda \geq w * \lambda$. Choosing $\mu \in X_*(\mathbf{T})^-$ such that $\text{val}(\phi(U_\mu)) < \langle s_1 * \lambda - \lambda, \mu \rangle$, we find

$$\text{val}(\phi(U_\mu)) < \langle s_1 * \lambda - \lambda, \mu \rangle \leq \langle w * \lambda - \lambda, \mu \rangle,$$

as desired.

Let w_i denote the transposition $(n - i, n - i + 1) \in \mathfrak{S}_n$. To establish the equivalence $2 \Leftrightarrow 3$, we show that for each $i = 1, \dots, n - 1$, we have $\text{val}(\phi(U_\mu)) \geq \langle w_i * \lambda - \lambda, \mu \rangle$ for all $\mu \in X_*(\mathbf{T})^-$ if and only if $\text{val}(\phi(U_{p,i})) \geq 1 + k_{n-i} - k_{n-i+1}$. The monoid $X_*(\mathbf{T})^-$ is generated by the cocharacters μ_1, \dots, μ_n , so the first statement is equivalent to asking that for each $j = 1, \dots, n$, we have $\text{val}(\phi(U_{\mu_i})) \geq \langle w_i * \lambda - \lambda, \mu_j \rangle$, or equivalently

$$\text{val}(\phi(U_{p,i})) \geq \langle \lambda^{w_i} - \lambda - \alpha_i, \mu_j \rangle, \quad (4.6)$$

where $\alpha_i(t_1, \dots, t_n) = t_{n-i}/t_{n-i+1}$. The left-hand side is always non-negative, by Lemma 4.6. The right-hand side is 0 if $i \neq j$, and if $i = j$ it equals $k_{n-i} - k_{n-i+1} + 1$, which shows the desired equivalence.

We now return to the theorem. By [Urb11, Theorem 4.4.1], there is a natural Hecke-equivariant second-quadrant spectral sequence

$$E_1^{i,j} = \bigoplus_{\substack{w \in \mathfrak{S}_n \\ l(w) = -i}} H^j(K_1(N; p), \mathcal{D}_{w*\lambda})^{\text{fs}} \otimes_L L(w * \lambda \cdot \lambda^{-1}) \Rightarrow H^{i+j}(K_1(N; p), \mathcal{L}_{\lambda, L} \otimes_L L(\lambda^{-1}))^{\text{fs}},$$

where the superscript ‘fs’ denotes the union of all finite slope subspaces. By Lemma 4.6, we see that all the eigenvalues of U_μ acting on the w -summand of the E_1 -page are of slope $\geq \langle w * \lambda - \lambda, \mu \rangle$, so localizing the spectral sequence at the maximal ideal \mathfrak{M} associated with an eigenpacket as in the theorem kills all the terms on the E_1 -page except those with $w = 0$. Therefore the localized spectral sequence degenerates to the claimed isomorphism, and the theorem is proved. \square

We now discuss the ‘completion of the eigenvariety at a point’. Let $x \in \mathcal{X}(L)$ be a closed point with residue field L , and let $\lambda = w(x) \in \mathcal{W}(L)$. We set $\mathbf{T}_x = \widehat{\mathcal{O}}_{\mathcal{X},x}$ and $\Lambda = \widehat{\mathcal{O}}_{\mathcal{W},\lambda}$.

Proposition 4.8. *Let notation be as above.*

1. *The rings \mathbf{T}_x and Λ are complete Noetherian local L -algebras with residue field L , and \mathbf{T}_x is a finite Λ -algebra.*
2. *There exists canonical data of a faithful graded \mathbf{T}_x -module H_x^* , finite over Λ and concentrated in degrees $[0, d]$, and a spectral sequence*

$$E_2^{i,j} = \mathrm{Tor}_{-i}^{\Lambda}(H_x^j, L) \Rightarrow H^{i+j}(K_1(N; p), \mathcal{D}_{\lambda})_{(\ker \phi_{\mathcal{X},x})}.$$

Proof. In [Hanb, §3.1], it is defined what it means for a pair (Ω, h) consisting of an affinoid open subset $\Omega \subset \mathcal{W}$ and a rational number h to be a *slope datum*. This implies in particular that the cohomology groups with coefficients in \mathcal{D}_{Ω} admit a decomposition as $\mathbf{T}_{\mathcal{O}(\Omega)}^{(N),p}$ -modules

$$H^*(K_1(N; p), \mathcal{D}_{\Omega}) = H^*(K_1(N; p), \mathcal{D}_{\Omega})_{\leq h} \oplus H^*(K_1(N; p), \mathcal{D}_{\Omega})_{> h}$$

in a sense generalizing that of the decomposition (4.5). Given such a slope datum (Ω, h) we define $\mathbf{T}_{\Omega,h}$ as the image of $\mathbf{T}_{\mathcal{O}(\Omega)}^{(N),p}$ in $\mathrm{End}_{\mathcal{O}(\Omega)}(H^*(K_1(N; p), \mathcal{D}_{\Omega})_{\leq h})$. This is a finite $\mathcal{O}(\Omega)$ -algebra, so in particular is an affinoid L -algebra. The space \mathcal{X} is constructed by gluing affinoids of the form $\mathrm{Sp}\mathbf{T}_{\Omega,h}$.

On the other hand, [Hanb, Theorem 3.3.1] implies that for any slope datum (Ω, h) and for any closed point $\nu \in \Omega(\overline{\mathbb{Q}}_p)$, there is a natural Hecke-equivariant spectral sequence

$$E_2^{i,j} = \mathrm{Tor}_{-i}^{\mathcal{O}(\Omega)}(H^j(K_1(N; p), \mathcal{D}_{\Omega})_{\leq h}, k(\nu)) \Rightarrow H^{i+j}(K_1(N; p), \mathcal{D}_{\nu})_{\leq h}. \quad (4.7)$$

Let us now return to our closed point $x \in \mathcal{X}(L)$, and its image $\lambda = w(x)$ in $\mathcal{W}(L)$. By the above remarks, we can find a slope datum (Ω, h) such that x has an affinoid open neighbourhood of the form $\mathrm{Sp}\mathbf{T}_{\Omega,h}$. We can moreover assume that the restriction of the weight map to $w : \mathrm{Sp}\mathbf{T}_{\Omega,h} \rightarrow \Omega$ is finite. It follows from [BGR84, 7.3.2/7] that the completed local ring $\mathbf{T}_x = \widehat{\mathcal{O}}_{\mathcal{X},x}$ is Noetherian, and it fact can be calculated as the completion of $\mathbf{T}_{\Omega,h}$ at the maximal ideal corresponding to the closed point x . The same remark applies to Λ , and this implies the first part of the proposition.

For the second part, we observe that passage to the flat extension $\mathcal{O}(\Omega) \rightarrow \Lambda$ implies the existence of a spectral sequence of $\mathbf{T}_{\Omega,h} \otimes_{\mathcal{O}(\Omega)} \Lambda$ -modules:

$$E_2^{i,j} = \mathrm{Tor}_{-i}^{\Lambda}(H^j(K_1(N; p), \mathcal{D}_{\Omega})_{\leq h} \otimes_{\mathcal{O}(\Omega)} \Lambda, L) \Rightarrow H^{i+j}(K_1(N; p), \mathcal{D}_{\lambda})_{\leq h}. \quad (4.8)$$

The finite Λ -algebra $\mathbf{T}_{\Omega, h} \otimes_{\mathcal{O}(\Omega)} \Lambda$ splits as a direct product of its localizations at its finitely many maximal ideals, one of which is canonically identified with \mathbf{T}_x . The second part of the proposition now follows on taking

$$H_x^j = H^j(K_1(N; p), \mathcal{Z}_{\Omega})_{\leq h} \otimes_{\mathcal{O}(\Omega)} \Lambda \otimes_{\mathbf{T}_{\Omega, h} \otimes_{\mathcal{O}(\Omega)} \Lambda} \mathbf{T}_x$$

and taking the projection of the spectral sequence (4.8) to this factor. \square

4.3 The eigenvariety at classical points of small slope

We now come to the main point of this paper. Let π be a regular algebraic, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$, of weight $\lambda = (k_1 \geq k_2 \cdots \geq k_n)$ and conductor $N \geq 1$ prime to p , and satisfying the parity condition (4.3). We suppose that the Satake parameter of π_p is regular semisimple. We fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \cong \mathbb{C}$ and an ordering $t = (t_1, \dots, t_n)$ of the eigenvalues of the Satake parameter of π_p .

Fix a coefficient field L which is large enough that Corollary 4.4 applies, and let $\mathfrak{M}' \subset \mathbf{T}_L^{(N), p}$ be the maximal ideal associated there with the pair (π, t) . Let $\phi : \mathbf{T}_L^{(N), p} \rightarrow L$ be the algebra homomorphism describing the action of $\mathbf{T}_L^{(N), p}$ on the line $H^{q_0 + l_0}(K_1(N; p), \mathcal{L}_{\lambda, L})_{\mathfrak{M}'} \otimes_L L(\lambda^{-1})$, and let $\mathfrak{M} = \ker \phi$. We shall assume that ϕ is numerically noncritical in the sense of Theorem 4.7, so the homomorphism $\phi : \mathbf{T}_L^{(N), p} \rightarrow L$ is a classical and noncritical finite-slope eigenpacket, and therefore determines a closed point $x = x(\pi, t) \in \mathcal{X}(L)$. Let \mathbf{T}_x and Λ be the completed local rings associated with this point, and H_x^* the graded module whose existence is asserted by Proposition 4.8.

According to Corollary 4.4, the groups $H^i(K_1(N; p), \mathcal{L}_{\lambda, L} \otimes_L L(\lambda^{-1}))_{\mathfrak{M}}$ are zero if $i \notin [q_0, q_0 + l_0]$, and have dimension $\binom{l_0}{i - q_0}$ if $i \in [q_0, q_0 + l_0]$.

Theorem 4.9. *Let assumptions be as above.*

1. We have $\dim \mathbf{T}_x \geq \dim \Lambda - l_0$.
2. If $\dim \mathbf{T}_x = \dim \Lambda - l_0$, then
 - (a) The module H_x^i vanishes for $i \neq q_0 + l_0$.
 - (b) The module $H_x := H_x^{q_0 + l_0}$ is free of rank one over \mathbf{T}_x , and there exist compatible isomorphisms $\mathbf{T}_x \otimes_{\Lambda} L \simeq L$ and

$$\mathrm{Tor}_i^{\Lambda}(H_x, L) \cong H^{q_0 + l_0 - i}(K_1(N; p), \mathcal{L}_{\lambda, L} \otimes_L L(\lambda^{-1}))[\mathfrak{M}].$$

- (c) The map $\Lambda \rightarrow \mathbf{T}_x$ is surjective and the ring \mathbf{T}_x is a complete intersection.

Proof. It follows from the construction of the spectral sequence (4.7) in [Hanb] (see in particular the proof of [Hanb, Proposition 3.1.5]) and Theorem 4.7 that we can find a complex C^\bullet of finite free Λ -modules such that $H_x^* \cong H^*(C^\bullet)$ and $H^*(C^\bullet \otimes_\Lambda L) \cong H^*(K_1(N; p), \mathcal{L}_{\lambda, L} \otimes_L L(\lambda^{-1}))[\mathfrak{M}]$. Of course, the existence of this complex is what underlies the spectral sequence constructed in Proposition 4.8.

It follows then from [Hana, Theorem 2.1.1] that $\dim_\Lambda H^*(C^\bullet) = \dim_\Lambda H_x^* \geq \dim \Lambda - l_0$, with equality if and only if H_x^* is concentrated in the single degree $H_x^* = H_x^{q_0+l_0}$. Since H_x^* is a faithful \mathbf{T}_x -module and \mathbf{T}_x is a finite Λ -algebra, we have $\dim \mathbf{T}_x = \dim_\Lambda H_x^*$. The first part of the theorem follows.

For the second part, we can now assume that $H_x^* = H_x = H_x^{q_0+l_0}$. The spectral sequence of Proposition 4.8 degenerates to a series of isomorphisms

$$\mathrm{Tor}_i^\Lambda(H_x, L) \cong H^{q_0+l_0-i}(K_1(N; p), \mathcal{L}_{\lambda, L} \otimes_L L(\lambda^{-1}))[\mathfrak{M}].$$

In particular $H_x \otimes_\Lambda L \cong H^{q_0+l_0}(K_1(N; p), \mathcal{L}_{\lambda, L} \otimes_L L(\lambda^{-1}))[\mathfrak{M}]$ is 1-dimensional, so Nakayama's lemma implies that H_x is a cyclic Λ -module, hence a cyclic \mathbf{T}_x -module. A choice of generator determines an isomorphism $H_x \simeq \Lambda/I_x$ for some ideal $I_x \subset \Lambda$ and then we see $\mathbf{T}_x \simeq \Lambda/I_x$ and H_x is free over \mathbf{T}_x of rank one. We have isomorphisms

$$H^{q_0+l_0-1}(K_1(N; p), \mathcal{L}_{\lambda, L} \otimes_L L(\lambda^{-1}))[\mathfrak{M}] \cong \mathrm{Tor}_1^\Lambda(H_x, L) \simeq \mathrm{Tor}_1^\Lambda(\mathbf{T}_x, L) \cong I_x \otimes_\Lambda L,$$

and so by Nakayama's lemma again I_x can be generated by l_0 elements. The equality $\dim \mathbf{T}_x = \dim \Lambda - l_0$ now implies that the ring \mathbf{T}_x is a complete intersection. This completes the proof of the theorem. \square

Corollary 4.10. *Let assumptions be as above, and suppose that $\dim \mathbf{T}_x = \dim \Lambda - l_0$. Let $V_x = \ker(\Lambda \rightarrow \mathbf{T}_x) \otimes_\Lambda L$. Then $H^*(K_1(N; p), \mathcal{L}_{\lambda, L} \otimes_L L(\lambda^{-1}))[\mathfrak{M}]$ has a canonical structure of free module of rank 1 over the commutative graded ring $\wedge_L^* V_x$.*

Proof. This follows immediately from Theorem 4.9, Proposition 2.1, and Lemma 2.2. \square

To go further, we need Galois representations. We recall that the main theorem of [HLTT] shows the existence (after possibly enlarging L) of a continuous, semi-simple representation $\rho_\pi : G_\mathbb{Q} \rightarrow \mathrm{GL}_n(L)$ satisfying the following conditions:

- The representation ρ_π is unramified outside primes dividing Np .
- For each prime $l \nmid Np$, the characteristic polynomial of $\rho_\pi(\mathrm{Frob}_l)$ is given by $\iota^{-1} \det(X - l^{(n-1)/2} t_l)$, where $t_l \in \mathrm{GL}_n(\mathbb{C})$ is the Satake parameter of π_l .

Moreover, it is now known that ρ_π is odd, in the sense of §3.2 [CLH].

Conjecture 4.11. *The representation ρ_π satisfies the following further conditions:*

1. The representation ρ_π is absolutely irreducible.
2. The restriction $\rho_\pi|_{G_{\mathbb{Q}_p}}$ is crystalline with Hodge-Tate weights $k_n, k_{n-1}+1, \dots, k_1+n-1$. For every prime l , we have $\iota\text{WD}(\rho_\pi|_{G_{\mathbb{Q}_l}})^{F\text{-ss}} \cong \text{rec}_l^T \pi_l$.
3. There is an isomorphism

$$R_{\rho_\pi, \alpha} \cong \mathbf{T}_x$$

of Λ -algebras, where $R_{\rho_\pi, \alpha}$ is the trianguline deformation ring of Proposition 3.8.

Assuming the first two parts of the conjecture, we see that the eigenvalues of the crystalline Frobenius on $D_{\text{crys}}(\rho_\pi|_{G_{\mathbb{Q}_p}})$ are $\alpha = (\alpha_1, \dots, \alpha_n) = \iota^{-1}p^{(n-1)/2}(t_n, \dots, t_1)$; the triple (π, t, ι) therefore determines a refinement of $\rho_\pi|_{G_{\mathbb{Q}_p}}$. The appearance of the ring $R_{\rho_\pi, \alpha}$ in the third part of this conjecture is then justified by the following lemma.

Lemma 4.12. *The first two assumptions of Conjecture 4.11 imply that $\rho|_{G_{\mathbb{Q}_p}}$ is numerically non-critical (in the sense of Definition 3.5) and very generic (in the sense of Definition 3.6).*

Proof. This is a calculation. Let $s_i = k_{n-i+1} + (i - 1)$. Then $s_1 < s_2 < \dots < s_n$ are the Hodge-Tate weights of $\rho_\pi|_{G_{\mathbb{Q}_p}}$. We first observe that according to Corollary 4.4, we have $\iota(U_{p,i} \bmod \mathfrak{M}') = t_n \dots t_{n-i+1} p^{\sum_{j=1}^i (n-1)/2 - (j-1)}$, hence

$$\iota(U_{p,i} \bmod \mathfrak{M}) = t_n \dots t_{n-i+1} p^{\sum_{j=1}^i (n-1)/2 - (j-1) - k_{n-j+1}},$$

hence

$$\text{val}(\phi(U_{p,i})) = \text{val}(\alpha_1) + \dots + \text{val}(\alpha_i) - (s_1 + \dots + s_i).$$

The ‘small slope’ condition of Theorem 4.7 is therefore equivalent to the equation (for each $i = 1, \dots, n$):

$$\text{val}(\alpha_1) + \dots + \text{val}(\alpha_i) < s_1 + \dots + s_{i-1} + s_{i+1}.$$

This is exactly the condition that the representation $\rho_\pi|_{G_{\mathbb{Q}_p}}$ with its given refinement is numerically non-critical. To check that the representation with its refinement is ‘very generic’, we must show that $H^0(\mathbb{Q}_p, \text{ad } \rho_\pi(-1)) = 0$ and that for each $1 \leq i < j \leq n$, we have $\alpha_i \alpha_j^{-1} \notin \{1, p^{-1}\}$. In fact we have $\alpha_i \alpha_j^{-1} = \iota^{-1}(t_i t_j^{-1}) \notin \{1, p^{\pm 1}\}$ for any $1 \leq i \neq j \leq n$ because t is regular semi-simple and π_p is unramified and generic. \square

Theorem 4.13. *Let assumptions be as above. Suppose that $\dim \mathbf{T}_x = \dim \Lambda - l_0$, and assume Conjecture 4.11.*

1. We have $H_f^1(\mathbb{Q}, \text{ad } \rho_\pi) = 0$, and there is a canonical exact sequence

$$0 \longrightarrow H_\alpha^1(\mathbb{Q}, \text{ad } \rho_\pi(1)) \longrightarrow H_f^1(\mathbb{Q}, \text{ad } \rho_\pi(1)) \xrightarrow{\mu_\alpha} \mathfrak{m}_\Lambda / \mathfrak{m}_\Lambda^2 \longrightarrow \mathfrak{m}_{\mathbf{T}_x} / \mathfrak{m}_{\mathbf{T}_x}^2 \longrightarrow 0.$$

2. Suppose further that eigenvariety \mathcal{X} is smooth at x (in other words, \mathbf{T}_x is a regular local ring). Then the map μ_α determines an isomorphism

$$\mu_\alpha : H_f^1(\mathbb{Q}, \text{ad } \rho_\pi(1)) \cong \ker(\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \rightarrow \mathfrak{m}_{\mathbf{T}_x}/\mathfrak{m}_{\mathbf{T}_x}^2) \cong \ker(\Lambda \rightarrow \mathbf{T}_x) \otimes_\Lambda L,$$

and consequently the graded L -vector space $H^*(K_1(N; p), \mathcal{L}_{\lambda, L})[\mathfrak{M}] \otimes_L L(\lambda^{-1})$ has a canonical structure of free module of rank 1 over the commutative graded ring $\wedge_L^* H_f^1(\mathbb{Q}, \text{ad } \rho_\pi(1))$.

Proof. If $R_{\rho_\pi, \alpha} \cong \mathbf{T}_x$, then Proposition 3.9 implies that we have

$$R_{\rho_\pi, \text{crys}} \cong R_{\rho_\pi, \alpha} \otimes_\Lambda L \cong \mathbf{T}_x \otimes_\Lambda L \cong L,$$

and hence $H_f^1(\mathbb{Q}, \text{ad } \rho_\pi)$, which can be interpreted as the (dual) Zariski tangent space of $R_{\rho_\pi, \text{crys}}$, is trivial. Proposition 3.10 then asserts the existence of an exact sequence

$$0 \rightarrow H_\alpha^1(\mathbb{Q}, \text{ad } \rho(1)) \rightarrow H_f^1(\mathbb{Q}, \text{ad } \rho(1)) \rightarrow (L^n)^\vee \rightarrow H_\alpha^1(\mathbb{Q}, \text{ad } \rho)^\vee \rightarrow 0.$$

To finish the proof of the first part of the theorem, we must show that we can identify the morphism $(L^n)^\vee \rightarrow H_\alpha^1(\mathbb{Q}, \text{ad } \rho)^\vee$ with the morphism $\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2 \rightarrow \mathfrak{m}_{\mathbf{T}_x}/\mathfrak{m}_{\mathbf{T}_x}^2$, or equivalently that we can identify the morphism $H_\alpha^1(\mathbb{Q}, \text{ad } \rho) \rightarrow L^n$ with the morphism

$$(\mathfrak{m}_{R_{\rho_\pi, \alpha}}/\mathfrak{m}_{R_{\rho_\pi, \alpha}}^2)^\vee \rightarrow (\mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2)^\vee.$$

This follows from Proposition 3.8 and Proposition 3.9.

If $\mathbf{T}_x \cong R_{\rho_\pi, \alpha}$ is regular, then $h_\alpha^1(\mathbb{Q}, \text{ad } \rho_\pi) = \dim R_{\rho_\pi, \alpha} = \dim \Lambda - l_0$, and Proposition 3.10 then implies $h_\alpha^1(\mathbb{Q}, \rho_\pi(1)) = 0$ and $h_f^1(\mathbb{Q}, \rho_\pi(1)) = l_0$. We deduce that the maps

$$H_f^1(\mathbb{Q}, \text{ad } \rho_\pi(1)) \rightarrow \mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2$$

and

$$\ker(\Lambda \rightarrow \mathbf{T}_x) \otimes_\Lambda L \rightarrow \mathfrak{m}_\Lambda/\mathfrak{m}_\Lambda^2$$

are injective. Both maps have source of dimension l_0 , and the image of the first is contained in the image of the second (because of the exactness of the sequence in the first part of the theorem). We deduce that μ_α defines an isomorphism between $H_f^1(\mathbb{Q}, \text{ad } \rho_\pi(1))$ and $\ker(\Lambda \rightarrow \mathbf{T}_x) \otimes_\Lambda L$. The last statement of theorem now follows from Corollary 4.10. \square

Remarks.

1. When the triple (π, t, ι) is ordinary (i.e. $\phi(U_{p,i})$ is a p -adic unit for each $i = 1, \dots, n$), the isomorphism $R_{\rho_\pi, \alpha} \cong \mathbf{T}_x$ can often be proved using a generalization of the Taylor–Wiles method, once it is known that the Galois representations carried by \mathbf{T}_x satisfy local-global compatibility at p ; see [KT].

2. The condition $\dim \mathbf{T}_x = \dim \Lambda - l_0$ is a non-abelian analogue of the Leopoldt conjecture, and seems out of reach at present. However, it follows from [Hanb, Theorem 4.5.1] that we at least have $\dim \mathbf{T}_x \leq \dim \Lambda - 1$ when $l_0 \geq 1$. In particular, the equality $\dim \mathbf{T}_x = \dim \Lambda - l_0$ holds unconditionally when $l_0 = 1$ (in the present context, this is equivalent to $n = 3$ or 4).

Example. In [AGG84], the authors show the existence of a regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_3(\mathbb{A})$ with the following properties:

- π is not essentially self-dual.
- π has conductor 89 and contributes to cohomology in weight $\lambda = 0$ (see Theorem 4.2).
- For each prime $l = 2, \dots, 19$, the eigenvalue a_l of the unramified Hecke operator $T_l = [\mathrm{GL}_3(\mathbb{Z}_l) \mathrm{diag}(l, 1, 1) \mathrm{GL}_3(\mathbb{Z}_l)]$ is the algebraic integer in $\mathbb{Z}[i] \subset \mathbb{C}$ given by the following table (see [AGG84, p. 433]):

l	2	3	5	7	11	13	17	19
a_l	$-1 - 2i$	$-1 - i$	$2 + 2i$	$-7 - 14i$	$-3 + 10i$	$-1 + 4i$	$-6 - 8i$	$11 + i$

If $p = 7$, then the Satake parameter equals $(-1, -i, -i)$. It is therefore not regular semi-simple, and there is no numerically non-critical refinement of π_7 . For each value $p \neq 7$ in this range, the Satake parameter t_p is regular semi-simple and for any isomorphism $\iota : \overline{\mathbb{Q}}_p \cong \mathbb{C}$, there is a unique ordering of the eigenvalues of t_p for which the corresponding finite slope eigenpacket is ordinary. In this case, we conclude that the p -adic eigenvariety for GL_3 of level $K_1(89)$ has dimension 2 and is a local complete intersection at the point corresponding to this ordering. Based on Theorem 4.13, it seems reasonable to guess that it is even smooth at this point. If $p = 3$, then [APS08, Theorem 9.4] shows that in any sufficiently small affinoid neighbourhood of this point, the Zariski closure of the classical points has dimension 1.

For a non-ordinary example, we refer again to [AGG84]. The authors also show the existence of a regular algebraic cuspidal automorphic representation π' of $\mathrm{GL}_3(\mathbb{A})$ with the following properties:

- π' is not essentially self-dual.
- π' has conductor 53 and contributes to cohomology in weight $\lambda = 0$.
- For each prime $l = 2, \dots, 29$, the eigenvalue a_l of the unramified Hecke operator T_l is an algebraic integer in $\mathbb{Z}[\omega] \subset \mathbb{C}$, where $\omega^2 - \omega + 3 = 0$. We have $a_3 = 2(\omega - 1)$.

The characteristic polynomial of the Satake parameter of π'_3 is given by

$$X^3 - \frac{2}{3}(\omega - 1)X^2 - \frac{\omega}{3}X - 1.$$

It has pairwise distinct roots. The prime 3 splits in $\mathbb{Q}(\sqrt{-11}) = \mathbb{Q}(\omega)$ as $(\omega)(\omega - 1)$, and it is easy to check (using the formula in the statement of Corollary 4.4) that for each isomorphism $\iota : \overline{\mathbb{Q}}_3 \cong \mathbb{C}$, there are exactly two orderings of the eigenvalues of the Satake parameter of π'_3 giving rise to finite-slope eigenpackets which are numerically non-critical, and that these are all non-ordinary.

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