Degenerating vector bundles in p-adic Hodge theory

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Abstract

We compute the closure relations among the individual Harder-Narasimhan strata in the moduli stack of rank n vector bundles on the Fargues-Fontaine curve. The proof combines a dynamical argument on Banach-Colmez spaces with a precise existence theorem (proved in [BFH+17]) for certain parabolic reductions of vector bundles.

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1 Introduction

1.1 Background and main results

The fields of p-adic geometry and p-adic Hodge theory have undergone tremendous development in recent years, largely on account of two parallel developments: Fargues and Fontaine's discovery of the "fundamental curve of p-adic Hodge theory" (also known as the Fargues-Fontaine curve), and Scholze's discovery of the theory of perfectoid spaces. One of the most fascinating outcomes of this development is Fargues's conjectural "geometrization" of the local Langlands correspondence for a connected reductive group G over a non-archimedean local field E, in terms of ℓ-adic sheaves on the stack Bun_G of G-bundles on the Fargues-Fontaine curve [Far16]. Even more recently, Scholze

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has announced a natural construction associating a semisimple L-parameter φ_π with any smooth irreducible representation π of $G(E)$, which relies crucially on the geometry and étale sheaf theory of Bun_G.

In this article, we study some basic geometry of this stack in the case where $G = \mathrm{GL}_n$. To explain our main result, fix an algebraic closure $\overline{\mathbf{F}_p}$, and let $\mathrm{Perf}_{\overline{\mathbf{F}_p}}$ denote the site of perfectoid spaces over $\overline{\mathbf{F}_p}$ with its v-topology. For any characteristic p perfectoid space S, let $\mathcal{X}_S$ denote the relative Fargues-Fontaine curve over S. Let $\mathrm{Bun}_n \to \mathrm{Perf}_{\overline{\mathbf{F}_p}}$ denote the fibered category whose fiber over $S \in \mathrm{Perf}_{\overline{\mathbf{F}_p}}$ is the groupoid of rank n vector bundles on $\mathcal{X}_S$. This stack is a basic example of a *small v-stack* in the sense of [Sch17, Def. 12.4]. In particular, Bun_n has enough geometric structure that it comes with a naturally associated topological space $|\mathrm{Bun}_n|$. According to a fundamental theorem of Fargues and Fontaine, the underlying point set of $|\mathrm{Bun}_n|$ is canonically identified with the set $\mathcal{P}_n$ of Harder-Narasimhan polygons of width n.

For any $P \in \mathcal{P}_n$, let $\mathrm{Bun}_n^{\geq P}$ (resp. $\mathrm{Bun}_n^{\leq P}$) denote the substack parametrizing bundles $\mathcal{E}/\mathcal{X}_S$ such that for every geometric point $x \to S$, the Harder-Narasimhan polygon of $\mathcal{E}_x$ lies above or on (resp. below or on) P with the same endpoints as P. By results of Kedlaya-Liu, $\mathrm{Bun}_n^{\geq P}$ and $\mathrm{Bun}_n^{\leq P}$ are closed and open substacks of Bun_n, respectively, and so the individual Harder-Narasimhan strata $\mathrm{Bun}_n^P = \mathrm{Bun}_n^{\geq P} \cap \mathrm{Bun}_n^{\leq P}$ are locally closed substacks. (We will see that small v-stacks admit reasonable notions of open and (locally) closed substacks, cf. Definition 2.2.) Each individual stratum Bun_n^P is a gerbe, and the associated topological spaces $|\mathrm{Bun}_n^P| \subset |\mathrm{Bun}_n|$ are singletons.

Our main result computes the closure of Bun_n^P inside Bun_n. The precise statement is as follows.

Theorem 1.1. *For any $n \geq 2$ and any $P \in \mathcal{P}_n$, we have $\overline{\mathrm{Bun}_n^P} = \mathrm{Bun}_n^{\geq P}$ as substacks of Bun_n. More precisely, $\mathrm{Bun}_n^{\geq P}$ is the minimal closed substack of Bun_n containing Bun_n^P, and $\overline{|\mathrm{Bun}_n^P|} = |\mathrm{Bun}_n^{\geq P}|$ as subsets of $|\mathrm{Bun}_n|$.*

We note that in the classical setting of vector bundles on a connected smooth projective curve over an algebraically closed field, the analogue of Theorem 1.1 holds for curves of genus zero and one, but fails in higher genus [FM02, Sch15]. Theorem 1.1 is thus related to the heuristic idea that the Fargues-Fontaine curve has genus between zero and one.

Let us sketch the proof of Theorem 1.1 in some detail. As we've already mentioned, the inclusion $\overline{\mathrm{Bun}_n^P} \subseteq \mathrm{Bun}_n^{\geq P}$ is known, so it suffices to demonstrate the opposite inclusion. This is not formal, and roughly amounts to constructing well-behaved families of vector bundles whose HN polygons degenerate from a given polygon P to any specified $Q \geq P$.

To produce the necessary families, we introduce certain auxiliary moduli spaces $\mathcal{S}_Q/\mathrm{Spd}\,\overline{\mathbf{F}_p}$ parametrized by $Q \in \mathcal{P}_n$. Precisely, for any given Q, let $\lambda_1 < \cdots < \lambda_k$ denote the slopes of Q, and let $m_i \in \mathbf{N}_{>0}$ $(1 \leq i \leq k)$ be the multiplicities such that $Q = \mathrm{HN}(\oplus_{1 \leq i \leq k} \mathcal{O}(\lambda_i)^{m_i})$. We then define $\mathcal{S}_Q \to \mathrm{Perf}_{\overline{\mathbf{F}_p}}$ as the category fibered in groupoids whose fiber category over $T \in \mathrm{Perf}_{\overline{\mathbf{F}_p}}$ has objects given by tuples

$$\left(\mathcal{E}, F_\bullet\mathcal{E} = \{0 = F_0\mathcal{E} \subset F_1\mathcal{E} \subset \cdots \subset F_k\mathcal{E} = \mathcal{E}\}, r_\bullet = \{r_i : \mathcal{O}(\lambda_i)^{m_i} \xrightarrow{\sim} F_i\mathcal{E}/F_{i-1}\mathcal{E}\}_{1 \leq i \leq k}\right)$$

where $\mathcal{E}/\mathcal{X}_T$ is a rank n vector bundle and the remaining data has the evident meaning ("a filtration together with a rigidification of its graded pieces"), and whose morphisms $(\mathcal{E}, ...) \to (\mathcal{E}', ...)$ are given by isomorphisms $f : \mathcal{E} \xrightarrow{\sim} \mathcal{E}'$ which are compatible with the filtrations and such that $\mathrm{gr}^i f \circ r_i = r_i'$. One easily checks that this fibered category is a v-stack, and that objects of $\mathcal{S}_Q$ have no automorphisms, i.e. that $\mathcal{S}_Q \to \mathrm{Perf}_{\overline{\mathbf{F}_p}}$ is a category fibered in *setoids* in the terminology of [Sta17]. There is thus no harm in replacing $\mathcal{S}_Q$ with its associated sheaf of sets. Having done this, it turns out that $\mathcal{S}_Q$ is a small v-sheaf, and the map $\mathcal{S}_Q \to \mathrm{Spd}\,\overline{\mathbf{F}_p}$ is representable in locally spatial diamonds and moreover is partially proper, cf. Proposition 3.1. Furthermore, the natural map

$$\pi_Q : \mathcal{S}_Q \to \mathrm{Bun}_n$$

given by forgetting the filtration and rigidification induces a continuous map $|\mathcal{S}_Q| \to |\mathrm{Bun}_n|$; by general properties of slope filtrations, this map factors through the inclusion $\mathrm{Bun}_n^{\leq Q} \subset \mathrm{Bun}_n$. We now appeal to the following crucial theorem, which is more or less the main result of [BFH$^+$17].

Theorem 1.2 ([BFH$^+$17, Theorem 1.1.4])**.** *For any $Q \in \mathcal{P}_n$, the map $\pi_Q : \mathcal{S}_Q \to \mathrm{Bun}_n^{\leq Q}$ induces a* surjective *map $|\mathcal{S}_Q| \to |\mathrm{Bun}_n^{\leq Q}|$.*

In particular, pulling back the HN stratification of Bun_n along π_Q induces a stratification $\mathcal{S}_Q = \cup_{P \leq Q} \mathcal{S}_Q^P$ by locally closed sub-v-sheaves such that every stratum is nonempty. We observe that $|\mathcal{S}_Q^Q|$ consists of a single point $s_Q \in |\mathcal{S}_Q|$, and in fact $\mathcal{S}_Q^Q \simeq \mathrm{Spd}\,\overline{\mathbf{F}}_p$, since the Q-filtration splits and rigidifies the HN-filtration on this stratum. The key observation is that s_Q is contained in the closure of any stratum:

Theorem 1.3. *Any open neighborhood of s_Q in $|\mathcal{S}_Q|$ meets every stratum $|\mathcal{S}_Q^P|$, $P \leq Q$. Equivalently, the closure of $|\mathcal{S}_Q^P|$ in $|\mathcal{S}_Q|$ contains s_Q for every $P \leq Q$.*

From here, the proof of Theorem 1.1 is immediate: if P and $Q \geq P$ are fixed, then either $\mathrm{Bun}_n^Q \subset \overline{\mathrm{Bun}_n^P}$ or[1] $\mathrm{Bun}_n^Q \cap \overline{\mathrm{Bun}_n^P} = \emptyset$; but if the latter holds, we can find some open subset $U \subset |\mathrm{Bun}_n|$ containing $|\mathrm{Bun}_n^Q|$ and disjoint from $|\mathrm{Bun}_n^P|$, in which case $|\pi_Q|^{-1}(U) \subset |\mathcal{S}_Q|$ is a nonempty open neighborhood of s_Q disjoint from $|\mathcal{S}_Q^P|$, contradicting Theorem 1.3.

Let us sketch the argument for Theorem 1.3. Consider the locally profinite group

$$J_Q \overset{\mathrm{def}}{=} \prod_{1 \leq i \leq k} \mathrm{GL}_{m_i}(D_{\lambda_i}).$$

Any element $j = (j_i)_{1 \leq i \leq j} \in J_Q$ defines an automorphism of $\mathcal{S}_Q$ by sending an object $(\mathcal{E}, F_\bullet \mathcal{E}, r_\bullet)$ as before to the altered object $(\mathcal{E}, F_\bullet \mathcal{E}, r_\bullet \cdot j)$ where we abbreviate

$$r_\bullet \cdot j = \{ r_i \circ j_i : \mathcal{O}(\lambda_i)^{m_i} \overset{\sim}{\to} F_i\mathcal{E}/F_{i-1}\mathcal{E} \}_{1 \leq i \leq k}.$$

This formula defines a right J_Q-action on $\mathcal{S}_Q$; note that the strata $\mathcal{S}_Q^P$ are stable under this action. The intuitive idea now is that $\mathcal{S}_Q$ is something like an iterated tower of H^1's, and the action of J_Q should move a point of $\mathcal{S}_Q$ "all around" inside these $\mathbf{Q}_p$-vector spaces. In particular, since s_Q is roughly the point corresponding to the product of the zero classes in these H^1's, one might hope that s_Q lies in the closure of the J_Q-orbit of any $x \in |\mathcal{S}_Q|$, which is a strictly stronger statement than Theorem 1.3. For example, take $n = 2$ and $Q = \mathrm{HN}(\mathcal{O} \oplus \mathcal{O}(1))$; then $\mathcal{S}_Q$ is just the sheafification of the presheaf sending $S \in \mathrm{Perf}_{\overline{\mathbf{F}_p}}$ to the $\mathbf{Q}_p$-vector space $H^1(\mathcal{X}_S, \mathcal{O}(-1))$, and an element (a, b) of $J_Q = \mathbf{Q}_p^\times \times \mathbf{Q}_p^\times$ acts by sending $f \in H^1(\mathcal{X}_S, \mathcal{O}(-1))$ to $b^{-1}a \cdot f$.

This intuition turns out to be correct in general:

Proposition 1.4. *For any point $x \in |\mathcal{S}_Q|$, the closure of the orbit $xJ_Q \subset |\mathcal{S}_Q|$ contains s_Q.*

[1] This dichotomy follows easily from the definition of stack-theoretic closure in our setting, together with the fact that each stratum Bun_n^Q is a gerbe.

Note that this is equivalent to the statement that the only J_Q-stable open neighborhood of s_Q is the entirety of $|\mathcal{S}_Q|$, cf. Lemma 3.2.

The proof of Proposition 1.4 runs by an induction on the number of slopes of Q. Note that when Q has a single slope, $\mathcal{S}_Q \cong \operatorname{Spd} \overline{\mathbf{F}_p}$ is a single point, and Proposition 1.4 is trivial. To explain the induction step, fix a general $Q = \mathrm{HN}(\oplus_{1 \leq i \leq k} \mathcal{O}(\lambda_i)^{m_i})$ as above, and let $Q' = \mathrm{HN}(\oplus_{1 \leq i \leq k-1} \mathcal{O}(\lambda_i)^{m_i})$ be the truncated polygon obtained by removing the side of largest slope from Q. There is a natural map

$$q : \mathcal{S}_Q \to \mathcal{S}_{Q'}$$

defined by sending an object $(\mathcal{E}, F_\bullet \mathcal{E}, r_\bullet)$ as before to $F_{k-1}\mathcal{E}$ equipped with the obvious truncated filtration and rigidification. We will see (in the proof of Proposition 3.1) that q is representable in locally spatial diamonds and is partially proper. Moreover, this map admits a canonical section

$$\sigma : \mathcal{S}_{Q'} \to \mathcal{S}_Q$$

sending $(\mathcal{E}', F_\bullet \mathcal{E}', r_\bullet)$ to $\mathcal{E}' \oplus \mathcal{O}(\lambda_k)^{m_k}$ equipped with the obvious k-step filtration and rigidification. Writing

$$J_Q \cong J_{Q'} \times \mathrm{GL}_{m_k}(D_{\lambda_k}),$$

the map q is then J_Q-equivariant for the evident actions on its source and target; in particular, the fibers of q are stable under the $\mathrm{GL}_{m_k}(D_{\lambda_k})$-action.

We now argue as follows. By induction, we may assume that Proposition 1.4 is known for $\mathcal{S}_{Q'}$; this implies that the J_Q-orbit closure of any point lying in the subset $|\mathcal{S}_{Q'}| \subset |\mathcal{S}_Q|$ has the desired property. The key observation is that it now suffices to check that for any point $x \in |\mathcal{S}_Q|$, *the orbit closure* $\overline{x\mathrm{GL}_{m_k}(D_{\lambda_k})}$ *meets* $|\mathcal{S}_{Q'}|$. This is a much more tractable problem, since the orbits in question lie in individual fibers of the map q, and the fibers of this map are closely related to Banach-Colmez spaces. Intuitively, the fibration structure of $\mathcal{S}_Q \to \mathcal{S}_{Q'}$ together with the product structure of the group J_Q allow us to prove the desired property of J_Q-orbit closures by a two-step procedure: first we use the $\mathrm{GL}_{m_k}(D_{\lambda_k})$-action to show that orbit closures meet $|\mathcal{S}_{Q'}|$, and then we use the induction hypothesis to show that the closure of the $J_{Q'}$-orbit of any point in $|\mathcal{S}_{Q'}|$ contains s_Q. We note that it seems hard to directly prove Theorem 1.3 by an inductive argument like this, since the fibration $\mathcal{S}_Q \to \mathcal{S}_{Q'}$ interacts quite poorly with the Harder-Narasimhan stratifications of its source and target.

1.2 Acknowledgments

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2 Preliminaries

2.1 Small v-stacks

Let Perf denote the site of characteristic p perfectoid spaces with its v-topology. In [Sch17, §12], Scholze defines the extremely general notion of a *small v-stack* on Perf. By definition, a small v-stack $\mathcal{X}$ is a stack in groupoids on Perf admitting some surjective map $U \to \mathcal{X}$ from a small v-sheaf such that $R = U \times_{\mathcal{X}} U$ is also a small v-sheaf. Equivalently, a small v-stack is a v-stack on Perf which can be presented as the quotient stack $[U/R]$ associated with some groupoid in small v-sheaves (U, R, s, t, c). Small v-stacks are presumably the most general class of v-stacks on Perf with some reasonable geometric meaning.

If $\mathcal{X}$ is a small v-stack, a *point* of $\mathcal{X}$ is an equivalence class of maps $\mathrm{Spd}(K, K^+) \to \mathcal{X}$ for some perfectoid field K with an open bounded valuation subring K^+; here two maps $\mathrm{Spd}(K_i, K_i^+) \to \mathcal{X}$ $(i = 1, 2)$ are equivalent if there exist surjective maps $\mathrm{Spd}(K_3, K_3^+) \to \mathrm{Spd}(K_i, K_i^+)$ for $i = 1, 2$ such that the diagram

$$\begin{array}{ccc} \mathrm{Spd}(K_3, K_3^+) & \longrightarrow & \mathrm{Spd}(K_2, K_2^+) \\ \downarrow & & \downarrow \\ \mathrm{Spd}(K_1, K_1^+) & \longrightarrow & \mathcal{X} \end{array}$$

is 2-commutative (as in [Sta17, Tag 04XF], one checks that this defines an equivalence relation). We write $|\mathcal{X}|$ for the set of points of $\mathcal{X}$. The set of points admits a canonical topology:

Proposition 2.1 ([Sch17, Prop. 12.7])**.** *Let $\mathcal{X}$ be a small v-stack with presentation $\mathcal{X} \simeq [U/R]$. Then $|\mathcal{X}| \cong |U|/|R|$, and the quotient topology on $|\mathcal{X}|$ induced by the surjection $|U| \to |\mathcal{X}|$ is independent of the choice of presentation. For any map $\mathcal{X} \to \mathcal{Y}$ of small v-stacks, the associated map $|\mathcal{X}| \to |\mathcal{Y}|$ is continuous.*

In [Sch17, Def. 10.7], Scholze defines open and closed immersions of small v-sheaves. We extend this notion to small v-stacks as follows.

Definition 2.2. Given a small v-stack $\mathcal{X}$, an *open* (resp. *closed*) *substack* of $\mathcal{X}$ is a strictly full subcategory $\mathcal{Z} \subset \mathcal{X}$ such that $\mathcal{Z} \times_{\mathcal{X}} W \to W$ is an open (resp. closed) immersion of small v-sheaves for any small v-sheaf W with a map $W \to \mathcal{X}$.

One easily checks that any open or closed substack of a small v-stack $\mathcal{X}$ is itself a small v-stack. Moreover, there is an equivalence between open substacks of $\mathcal{X}$ and open subsets of $|\mathcal{X}|$, cf. [Sch17, Prop. 12.9]. For closed substacks, a weaker result holds.

Proposition 2.3. *Let $\mathcal{X}$ be a small v-stack, and let $\mathcal{Z} \subset \mathcal{X}$ be a closed substack.*

i. The natural map $|\mathcal{Z}| \to |\mathcal{X}|$ is a closed embedding.

ii. There is a natural identification $\mathcal{Z} = \mathcal{X} \times_{|\mathcal{X}|} |\mathcal{Z}|$, in the sense that an arbitrary map of small v-stacks $\mathcal{Y} \to \mathcal{X}$ factors over the inclusion $\mathcal{Z} \subset \mathcal{X}$ if and only if $|\mathcal{Y}| \to |\mathcal{X}|$ factors through $|\mathcal{Z}| \to |\mathcal{X}|$.

Proof. 1. By the strict fullness of $\mathcal{Z}$ and the definition of points, one easily checks that $|\mathcal{Z}| \to |\mathcal{X}|$ is an injection. Moreover, for any small v-sheaf T with a map $T \to \mathcal{X}$, we have $|\mathcal{Z} \times_{\mathcal{X}} T| \cong |\mathcal{Z}| \times_{|\mathcal{X}|} |T|$ as subsets of $|T|$: one the one hand,

$$|\mathcal{Z} \times_{\mathcal{X}} T| \to |\mathcal{Z}| \times_{|\mathcal{X}|} |T|$$

is surjective by [Sch17, Prop. 12.10], while on the other hand the composite map

$$|\mathcal{Z} \times_{\mathcal{X}} T| \to |\mathcal{Z}| \times_{|\mathcal{X}|} |T| \to |T|$$

is a closed embedding.

Now, let $U \to \mathcal{X}$ be surjective map from a small v-sheaf. Then $|\mathcal{Z} \times_{\mathcal{X}} U| \to |U|$ is a closed embedding, since $\mathcal{Z} \times_{\mathcal{X}} U \to U$ is a closed immersion of small v-sheaves. But the map $|\mathcal{Z} \times_{\mathcal{X}} U| \to |U|$ identifies with the pullback of $|\mathcal{Z}| \to |\mathcal{X}|$ along the quotient map $|U| \to |\mathcal{X}|$, so we get the claim.

2. "Only if" is easy. For "if", consider a perfectoid space S with a map $S \to \mathcal{Y}$, corresponding to some $y \in \mathrm{Ob}(\mathcal{Y}_S)$. We need to check that the induced object of $\mathcal{X}_S$ is an object of the full subcategory $\mathcal{Z}_S$. Choose a surjective map $U \to \mathcal{X}$ from a small v-sheaf as before and set $V = U \times_{\mathcal{X}} \mathcal{Z}$ and $T = S \times_{\mathcal{X}} U$, so we get a commutative diagram

$$\begin{array}{ccccc} T & \longrightarrow & U & \longleftarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ S & \longrightarrow & \mathcal{X} & \longleftarrow & \mathcal{Z} \end{array}$$

of small v-stacks. Now, since $|S| \to |\mathcal{X}|$ factors over $|\mathcal{Z}|$, the induced map $|T| \to |U|$ factors over $|V| \cong |U| \times_{|\mathcal{X}|} |\mathcal{Z}| \subset |U|$, so $T \to U$ factors over a map $\psi : T \to V$. But $T \to S$ is a surjective map of small v-sheaves, so after passing to some v-cover $\{S_i \to S\}$ we can choose sections fitting into a diagram

$$\begin{array}{ccccccc} & & & & \overset{\psi}{\frown} & & \\ & & T & \longrightarrow & U & \longleftarrow & V \\ & \overset{s_i}{\nearrow} & \downarrow & & \downarrow & & \downarrow \\ S_i & \longrightarrow & S & \longrightarrow & \mathcal{X} & \longleftarrow & \mathcal{Z} \end{array} .$$

Going around the diagram via s_i and ψ, we see that $y|_{S_i}$ induces an object of $\mathcal{Z}_{S_i}$ for each i. Since $\mathcal{Z}$ is a stack, we conclude that y induces an object of $\mathcal{Z}_S$, as desired. $\square$

For a general small v-stack, not every closed subset of $|\mathcal{X}|$ arises as the topological space of a closed substack. For example, if X is a locally spatial diamond, the subsets of $|X|$ associated with closed sub-diamonds of X are exactly those subsets of $|X|$ which are closed and generalizing. This makes the notion of "stack-theoretic closure" slightly delicate. In particular, the existence of $\overline{\mathcal{Z}}$ in the following definition is not automatic.

Definition 2.4. Let $\mathcal{X}$ be a small v-stack, and let $\mathcal{Z} \subset \mathcal{X}$ be a small sub-v-stack. Suppose there exists a closed sub-v-stack $\overline{\mathcal{Z}} \subset \mathcal{X}$ such that the inclusion $\mathcal{Z} \to \mathcal{X}$ factors via $\mathcal{Z} \to \overline{\mathcal{Z}} \to \mathcal{X}$, such that $\overline{\mathcal{Z}}$ is initial among closed sub-v-stacks with this property. Then $\overline{\mathcal{Z}}$ (which is unique if it exists) is the *closure of* $\mathcal{Z}$ *in* $\mathcal{X}$.

Proposition 2.5. *Let $\mathcal{X}$ be a small v-stack, and let $\mathcal{Z} \subset \mathcal{X}$ be a small sub-v-stack such that $\mathcal{X} \times_{|\mathcal{X}|} \overline{|\mathcal{Z}|}$ is a closed substack of $\mathcal{X}$. Then $\mathcal{X} \times_{|\mathcal{X}|} \overline{|\mathcal{Z}|}$ is the closure of $\mathcal{Z}$ in $\mathcal{X}$.*

Proof. Let $\mathcal{Y} \subset \mathcal{X}$ be any closed substack such that the inclusion $\mathcal{Z} \to \mathcal{X}$ factors over $\mathcal{Y}$. Then $\overline{|\mathcal{Z}|} \to |\mathcal{X}|$ factors over an inclusion $\overline{|\mathcal{Z}|} \to |\mathcal{Y}|$. Since $\left|\mathcal{X} \times_{|\mathcal{X}|} \overline{|\mathcal{Z}|}\right| \cong \overline{|\mathcal{Z}|}$, Proposition 2.3.ii implies that the inclusion $\mathcal{X} \times_{|\mathcal{X}|} \overline{|\mathcal{Z}|} \to |\mathcal{X}|$ factors over a map $\mathcal{X} \times_{|\mathcal{X}|} \overline{|\mathcal{Z}|} \to \mathcal{Y}$. This shows that $\mathcal{X} \times_{|\mathcal{X}|} \overline{|\mathcal{Z}|}$ has the required universal property. $\square$

2.2 Relative Banach-Colmez spaces as diamonds

Given any perfectoid space $S/\mathbf{F}_p$, we have the (adic) relative Fargues-Fontaine curve $\mathcal{X}_S$. In this section we make a detailed study of the cohomology groups $H^i(\mathcal{X}_S, \mathcal{E})$ for $\mathcal{E}$ a vector bundle on S, in the language of diamonds. When $S = \operatorname{Spa} C^\flat$ is a tilted geometric point for some $C/\mathbf{Q}_p$, these are usually known as Banach-Colmez spaces.

A word on terminology: Suppose given S together with a vector bundle $\mathcal{E}/\mathcal{X}_S$ as above. By the *slopes* of $\mathcal{E}$, we mean the set

$$\{\lambda \in \mathbf{Q} \mid \lambda \,\mathrm{is\,a\,slope\,of\,HN}(\mathcal{E}_x) \,\mathrm{for\,some}\, x \in S\}\,.$$

When S is quasicompact, this is a finite set by [KL15, Prop. 7.4.6].

Definition 2.6. Given a perfectoid space $S \in \mathrm{Perf}$ and a vector bundle $\mathcal{E}/\mathcal{X}_S$, we define functors $\mathcal{H}^i(\mathcal{E}) \to S$ for $i = 0, 1$ as the pro-étale sheafifications of the presheaves

$$\begin{array}{rcl} \mathrm{Perf}_{/S} & \to & \mathrm{Sets} \\ (T \to S) & \mapsto & H^i(\mathcal{X}_T, \mathcal{E}_T), \end{array}$$

where $\mathcal{E}_T$ is the pullback of $\mathcal{E}$ along the canonical map $\mathcal{X}_T \to \mathcal{X}_S$.

We will sometimes write $\mathcal{H}^i_S(\mathcal{E})$ if we need to emphasize the base space S. These are sheaves of $\mathbf{Q}_p$-vector spaces over S, so the zero vector corresponds to a section $s : S \to \mathcal{H}^i(\mathcal{E})$ of the structure morphism. Note that the sheafification of $T \mapsto H^i(\mathcal{X}_T, \mathcal{E}_T)$ vanishes for any $i \geq 2$ by [KL15, Theorem 8.7.13]. In particular, applying the $\mathcal{H}^i$'s to a short exact sequence of vector bundles on $\mathcal{X}_S$ induces a six-term long exact sequence of sheaves of $\mathbf{Q}_p$-vector spaces over S in the obvious manner. Note also that $\mathcal{H}^i(\mathcal{E}_1 \oplus \mathcal{E}_2) \cong \mathcal{H}^i(\mathcal{E}_1) \times_S \mathcal{H}^i(\mathcal{E}_2)$.

Proposition 2.7. i. *If $\mathcal{E}$ has only negative slopes, then $\mathcal{H}^0(\mathcal{E}) = S$ via the zero section.*

ii. *If $\mathcal{E}$ has only nonnegative slopes, then $\mathcal{H}^1(\mathcal{E}) = S$ via the zero section.*

Proof. Part i. is immediate from Corollary 7.4.11 and Theorem 8.7.13 in [KL15].

Part ii. is local on S, so we may assume S is affinoid and that $\mathcal{E}$ has constant rank and degree. After passing to a further rational covering of S, if necessary, Lemma 8.8.13 and Corollary 8.8.14 of [KL15] guarantee the existence of a short exact sequence of vector bundles

$$0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{E} \to 0$$

over $\mathcal{X}_S$ such that $\mathcal{G} \simeq \mathcal{O}(-1)^n$ and $\mathcal{F} \simeq \mathcal{O}^m$ after pullback along any geometric point $x \to S$. In particular, $\mathcal{F}$ and $\mathcal{G}(1)$ are pointwise-étale at all points of S. By the sheaf-theoretic surjectivity of $\mathcal{H}^1(\mathcal{F}) \to \mathcal{H}^1(\mathcal{E})$, it suffices to prove that $\mathcal{H}^1(\mathcal{F}) = S$ via the zero section. This can be checked pro-étale-locally on S. After passing to an affinoid pro-étale cover $S' \to S$, we can choose isomorphisms $\mathcal{F}_{S'} \simeq \mathcal{O}^m$ and $\mathcal{G}_{S'} \simeq \mathcal{O}(-1)^n$. By [Sch17, Lemma 7.18], we may assume, after passing to a further affinoid pro-étale cover of S', that any surjective étale map $V \to S'$ admits a section.

By Theorems 8.7.13 and 9.4.5 in [KL15],

$$H^1(\mathcal{X}_{S'}, \mathcal{F}_{S'}) \simeq H^1_{\mathrm{proet}}(S', \underline{\mathbf{Q}_p^m}),$$

so we're reduced to the claim that $H^1_{\mathrm{proet}}(S', \underline{\mathbf{Q}_p}) = 0$ for S' chosen as above. Since

$$H^1_{\mathrm{proet}}(S', \underline{\mathbf{Q}_p}) \cong \left(\varprojlim_n H^1_{\mathrm{proet}}(S', \underline{\mathbf{Z}/p^n\mathbf{Z}})\right)[\tfrac{1}{p}],$$

this reduces further to the vanishing of $H^1_{\mathrm{proet}}(S', \underline{\mathbf{Z}/p^n\mathbf{Z}})$. The sheaf $\underline{\mathbf{Z}/p^n\mathbf{Z}}$ on S'_{proet} is pulled back from S'_{et}, so [Sch17, Prop. 14.8] gives an isomorphism

$$H^1_{\mathrm{proet}}(S', \underline{\mathbf{Z}/p^n\mathbf{Z}}) \simeq H^1_{\mathrm{et}}(S', \underline{\mathbf{Z}/p^n\mathbf{Z}}).$$

But $H^1_{\mathrm{et}}(S', \underline{\mathbf{Z}/p^n\mathbf{Z}}) = 0$, since any étale cover of S' splits, and the result follows. $\square$

It turns out that $\mathcal{H}^0(\mathcal{E})$ is well-behaved in all generality.

Proposition 2.8. *The functor $\mathcal{H}^0(\mathcal{E})$ is a locally spatial diamond, and the structure map $\mathcal{H}^0(\mathcal{E}) \to S$ is partially proper.*

Proof. This is local on S, so we can assume S is affinoid. Applying [KL15, Theorem 8.8.15], we can choose (locally on some rational covering of S) an exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}(m_1)^{N_1} \xrightarrow{i} \mathcal{O}(m_2)^{N_2}$$

for some $N_1, N_2 \geq 0$ and $0 \ll m_1 \ll m_2$ (we learned this device from [Far16]). Applying $\mathcal{H}^0$ then presents $\mathcal{H}^0(\mathcal{E})$ as the fiber product

$$\mathcal{H}^0(\mathcal{E}) \cong \mathcal{H}^0\left(\mathcal{O}(m_1)^{N_1}\right) \times_{i,\mathcal{H}^0\left(\mathcal{O}(m_2)^{N_2}\right),s} S,$$

so it suffices to prove the result in the case where $\mathcal{E} = \mathcal{O}(m)^N$. This reduces further to $\mathcal{E} = \mathcal{O}(m)$, which can be proved as in e.g. [BFH$^+$17, Prop. 3.3.2].

For partial properness, the valuative criterion is obvious, so we need to check that the relative diagonal is closed. Writing it as the pullback of the zero section $S \to \mathcal{H}^0(\mathcal{E})$ along

$$\mathcal{H}^0(\mathcal{E}) \times_S \mathcal{H}^0(\mathcal{E}) \xrightarrow{(f,g)\mapsto f-g} \mathcal{H}^0(\mathcal{E}),$$

it suffices to check that $S \to \mathcal{H}^0(\mathcal{E})$ is closed. Again, choose an injection $\mathcal{E} \to \mathcal{O}(m)^N$ for some large m and N, so we get an injective map $\mathcal{H}^0(\mathcal{E}) \to \mathcal{H}^0(\mathcal{O}(m)^N)$ compatible with the zero sections of the source and target. This reduces us to the case where $\mathcal{E} = \mathcal{O}(m)^N$, which again reduces to the case $\mathcal{E} = \mathcal{O}(m)$, in which case the result follows from [Far17, Lemme 2.10]. ☐

Proposition 2.9. *If $\mathcal{E}$ has only negative slopes, the functor $\mathcal{H}^1(\mathcal{E})$ is a diamond, and the structure map $\mathcal{H}^1(\mathcal{E}) \to S$ is partially proper.*

Proof. To check that $\mathcal{H}^1(\mathcal{E})$ is a diamond, we can (by [Sch17, Prop. 11.6]) work pro-étale-locally on S. Arguing as in the proof of Proposition 2.7.ii, we can find a short exact sequence

$$0 \to \mathcal{O}(-1)^n \to \mathcal{O}^m \to \mathcal{E}^\vee \to 0$$

of vector bundles locally on some pro-étale cover of S. Dualizing this sequence, passing to the associated long exact sequence of $\mathcal{H}^i$'s, and applying Proposition 2.7, we get a short exact sequence

$$0 \to \mathcal{H}^0(\mathcal{O}^m) \simeq \underline{\mathbf{Q}_p^m} \to \mathcal{H}^0(\mathcal{O}(1)^n) \to \mathcal{H}^1(\mathcal{E}) \to 0$$

of $\mathbf{Q}_p$-vector diamonds over S. Thus we get an isomorphism

$$\mathcal{H}^1(\mathcal{E}) \simeq \mathcal{H}^0(\mathcal{O}(1)^n)/\underline{\mathbf{Q}_p^m},$$

which presents $\mathcal{H}^1(\mathcal{E})$ as the quotient of a diamond by a quasi-pro-étale equivalence relation. Therefore $\mathcal{H}^1(\mathcal{E})$ is a diamond by [Sch17, Prop. 11.8].

For partial properness, the valuative criterion is obvious, so we again need to check that the diagonal is closed. Writing it as the pullback of the zero section $S \to \mathcal{H}^1(\mathcal{E})$ along

$$\mathcal{H}^1(\mathcal{E}) \times_S \mathcal{H}^1(\mathcal{E}) \xrightarrow{(f,g)\mapsto f-g} \mathcal{H}^1(\mathcal{E})$$

as before, it suffices to check that the zero section $s : S \to \mathcal{H}^1(\mathcal{E})$ is closed. For this we first argue on presheaves. More precisely, suppose given a perfectoid space $T \to S$ and an element $c \in H^1(\mathcal{X}_T, \mathcal{E}_T)$, with associated extension bundle $\mathcal{F}/\mathcal{X}_T$. Using [KL15, Corollary 7.4.11], one checks that the presheaf $H^1(\mathcal{E}) : T \mapsto H^1(\mathcal{X}_T, \mathcal{E}_T)$ is separated in the sense of [Sta17, Tag 00WA]. Moreover, if $x = \mathrm{Spa}(K, K^+) \to T$ is any point, then the pullback of c to $H^1(\mathcal{X}_x, \mathcal{E}_x)$ vanishes if and only if the point $(1, 0)$ lies on or below the HN polygon of $\mathcal{F}_x$. By semicontinuity, the locus of such points is closed and generalizing in $|T|$; it therefore corresponds to a closed immersion of diamonds $X \to T$. It's then easy to see (using separatedness) that $X \to T$ satisfies the correct universal property: if $g : T' \to T$ is any map of perfectoid spaces, the pullback of c to $H^1(\mathcal{X}_{T'}, \mathcal{E}_{T'})$ vanishes if and only if g factors through a map $T' \to X$. Therefore $T \times_{c,H^1(\mathcal{E}),s} S$ is representable by a closed subdiamond of T, and in particular is already a sheaf. Thus

$$X = T \times_{c,H^1(\mathcal{E}),s} S \cong T \times_{c,\mathcal{H}^1(\mathcal{E}),s} S \to T$$

is a closed immersion of diamonds. (Here we use that sheafification commutes with finite limits.) In other words, we've shown that zero section $s : S \to \mathcal{H}^1(\mathcal{E})$ pulls back to a closed immersion of diamonds along any map $T \to \mathcal{H}^1(\mathcal{E})$ which factors through the canonical map $H^1(\mathcal{E}) \to \mathcal{H}^1(\mathcal{E})$.

To conclude the general case, choose any perfectoid space T with a map $f : T \to \mathcal{H}^1(\mathcal{E})$. Since $H^1(\mathcal{E})$ is separated, we can find (by [Sta17, Tags 00W9 & 00WB]) a pro-étale cover $\tilde{T} \to T$ such that the composite map $\tilde{f} : \tilde{T} \to \mathcal{H}^1(\mathcal{E})$ factors (uniquely) as

$$\tilde{T} \to H^1(\mathcal{E}) \to \mathcal{H}^1(\mathcal{E}),$$

in which case the base change

$$\left(T \times_{f,\mathcal{H}^1(\mathcal{E}),s} S \to T\right) \times_T \tilde{T} = \left(\tilde{T} \times_{\tilde{f},\mathcal{H}^1(\mathcal{E}),s} S \to \tilde{T}\right)$$

is a closed immersion by our arguments so far. Since $\tilde{T} \to T$ is surjective as a map of v-sheaves, we deduce that $T \times_{\mathcal{H}^1(\mathcal{E})} S \to T$ is a closed immersion by [Sch17, Prop. 10.11.i], so the result follows. □

Our next goal is the local spatiality of $\mathcal{H}^1(\mathcal{E})$ for any $\mathcal{E}$ with only negative slopes. Although one can likely deduce this directly from the presentation of $\mathcal{H}^1(\mathcal{E})$ used in the proof of Proposition 2.9, it seems a little bit tricky to write this deduction out transparently. Our strategy instead is to show that $\mathcal{H}^1(\mathcal{E})$ admits a separated quasi-pro-étale map to a diamond whose local spatiality can be checked by hand, which implies the desired result by [Sch17, Corollary 11.28]. This argument relies on some auxiliary results which turn out to be very useful in their own right.

Theorem 2.10. *Suppose that $\mathcal{E}$ has only nonnegative slopes, and let $X \subset \mathcal{H}^0(\mathcal{E})$ be any open subdiamond such that $|X| \to |S|$ is surjective. Then $X \to S$ is surjective as a map of pro-étale sheaves.*

This result is extremely useful in practice: the surjectivity of $|X| \to |S|$ more or less amounts to the non-emptiness of $X \times_S \overline{s}$ for all geometric points $\overline{s} = \mathrm{Spa}(C, C^+) \to S$, and many natural X's of interest are much more accessible when the base is a geometric point.

Proof. Passing to an open-closed decomposition of S, we may assume $\mathcal{E}$ has constant rank and degree. The claim is clearly pro-étale-local on S, so replacing S by a suitable pro-étale cover and arguing as in the proof of Proposition 2.7.ii, we can then find a short exact sequence

$$0 \to \mathcal{O}(-1)^m \to \mathcal{O}^n \to \mathcal{E} \to 0.$$

Taking the associated long exact sequence of $\mathcal{H}^i$'s and applying Proposition 2.7, we get a short exact sequence

$$0 \to \mathcal{H}^0(\mathcal{O}^n) \simeq \underline{\mathbf{Q}_p^n} \xrightarrow{i} \mathcal{H}^0(\mathcal{E}) \xrightarrow{q} \mathcal{H}^1\left(\mathcal{O}(-1)^m\right) \to 0$$

of $\mathbf{Q}_p$-vector diamonds over S. Clearly

$$\mathcal{H}^0(\mathcal{E}) \times \underline{\mathbf{Q}_p^n} \overset{(f,a)\mapsto(f,f+i(a))}{\longrightarrow} \mathcal{H}^0(\mathcal{E}) \times_{\mathcal{H}^1(\mathcal{O}(-1)^m)} \mathcal{H}^0\left(\mathcal{E}\right)$$

is an isomorphism, so we get a cartesian diagram

$$\begin{array}{ccc} \mathcal{H}^0(\mathcal{E}) \times \underline{\mathbf{Q}_p^n} & \xrightarrow{(f,a)\mapsto f} & \mathcal{H}^0(\mathcal{E}) \\ {\scriptstyle (f,a)\mapsto f+i(a)}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{H}^0(\mathcal{E}) & \xrightarrow{q} & \mathcal{H}^1\left(\mathcal{O}(-1)^m\right) \end{array}$$

of diamonds; note that the upper horizontal arrow is clearly a $\underline{\mathbf{Q}_p^n}$-torsor in the sense of [Sch17, Def. 10.12], and the vertical copy of q is surjective as a map of pro-étale sheaves. Thus the map q is a $\underline{\mathbf{Q}_p^n}$-torsor by [Sch17, Lemma 10.13]; in particular, q is universally open and quasi-pro-étale. Let $Y \subset \mathcal{H}^1(\mathcal{O}(-1)^m)$ be the open subdiamond associated with the open subset

$$\mathrm{im}\left(|\mathcal{H}^0(\mathcal{E})| \to |\mathcal{H}^1(\mathcal{O}(-1)^m)|\right)(|X|) \subset |\mathcal{H}^1(\mathcal{O}(-1)^m)|.$$

The map $\mathcal{H}^0(\mathcal{E}) \to \mathcal{H}^1(\mathcal{O}(-1)^m)$ then induces a map $X \to Y$ which is universally open and quasi-pro-étale, and such that $|X| \to |Y|$ is surjective. It's easy to check that any such map of locally spatial diamonds is surjective as a map of pro-étale sheaves.

It thus remains to check that the induced map $Y \to S$ is surjective as a map of pro-étale sheaves. In fact, we claim this map admits sections étale-locally on S. This is local on S, so we may assume that S is affinoid, in which case it admits a map $a : S \to \mathrm{Spd}\, \mathbf{Q}_p^{\mathrm{cyc}}$. The sheafification of the presheaf

$$\begin{array}{rcl} \mathrm{Perf}_{/\mathrm{Spd}\, \mathbf{Q}_p^{\mathrm{cyc}}} & \to & \mathrm{Sets} \\ T & \mapsto & H^1(\mathcal{X}_T, \mathcal{O}(-1)) \end{array}$$

is then representable by the sheaf quotient $\mathbf{A}^{1,\lozenge}_{\mathbf{Q}_p^{\mathrm{cyc}}}/\underline{\mathbf{Q}_p}$, cf. the proof of Proposition 2.13 below. Base changing back to S along a gives

$$\mathcal{H}^1_S(\mathcal{O}(-1)) \cong \mathbf{A}^{1,\lozenge}_{\mathbf{Q}_p^{\mathrm{cyc}}}/\underline{\mathbf{Q}_p} \times_{\mathrm{Spd}\, \mathbf{Q}_p^{\mathrm{cyc}}} S \cong \mathbf{A}^{1,\lozenge}_{S^\sharp}/\underline{\mathbf{Q}_p}$$

where $S^\sharp$ is the untilt of S specified by a. Pulling back the inclusion $Y \subset \mathcal{H}^1_S(\mathcal{O}(-1))$ along $\mathbf{A}^{1,\lozenge}_{S^\sharp} \to \mathbf{A}^{1,\lozenge}_{S^\sharp}/\underline{\mathbf{Q}_p}$ gives an open subdiamond of $\mathbf{A}^{1,\lozenge}_{S^\sharp}$, corresponding to an open adic subspace $W \subset \mathbf{A}^1_{S^\sharp}$ such that $|W| \to |S^\sharp|$ is surjective. We're now reduced to the existence of a section $\tilde{S}^\sharp \to W$ over some étale cover $\tilde{S}^\sharp \to S^\sharp$. This is local on $S^\sharp$, so we can assume that $S^\sharp = \mathrm{Spa}(A, A^+)$ is affinoid perfectoid.

Assume for the moment that $f : \mathbf{A}^1_{S^\sharp} \to S^\sharp$ is open. Writing W as a union of its quasicompact open subspaces W_i, the quasicompact open subsets $f(W_i)$ cover $S^\sharp$, and by quasicompacity of $S^\sharp$ this covering can be refined to a cover by $f(W_i)$ for some finite set of i's. Replacing W with the union of these finitely many W_i's reduces us to the case where W is quasicompact. Rescaling $W \subset \mathbf{A}^1_{S^\sharp}$ by a large power of p, we can assume that W is contained in

$$\mathbf{B}^1_{S^\sharp} \stackrel{\text{def}}{=} \mathrm{Spa}(A\langle T\rangle, A^+\langle T\rangle) \subset \mathbf{A}^1_{S^\sharp}.$$

As in [KL15, §2.6], we can write A as the completed direct limit of a directed system of strongly Noetherian Tate $\mathbf{Q}_p$-algebras A_i along submetric transition maps. Setting $X_i = \mathrm{Spa}(A_i, A_i^+)$ and $\mathbf{B}^1_{X_i} = \mathrm{Spa}(A_i\langle T\rangle, A_i^+\langle T\rangle)$, we have natural compatible homeomorphisms $|S^\sharp| \cong \lim_{\leftarrow} |X_i|$ and $|\mathbf{B}^1_{S^\sharp}| \cong \lim_{\leftarrow i} |\mathbf{B}^1_{X_i}|$. By spectrality of all spaces and maps, the subset $|W| \subset |\mathbf{B}^1_{S^\sharp}|$ is the preimage of a quasicompact open $|W_i| \subset |\mathbf{B}^1_{X_i}|$ for some large i; by taking i sufficiently large, we can also assume that $|W_i| \to |X_i|$ is surjective. Then $W_i \to X_i$ is a smooth surjective map of locally Noetherian adic spaces, so there exists an étale cover $\tilde{X}_i \to X_i$ and a section $s : \tilde{X}_i \to W_i$. Base changing the diagram

$$\begin{array}{ccc} & & W_i \\ & \overset{s}{\nearrow} & \downarrow \\ \tilde{X}_i & \longrightarrow & X_i \end{array}$$

along the map $S^\sharp \to X_i$ then gives the desired section.

It still remains to check that $\mathbf{A}^1_{S^\sharp} \to S^\sharp$ is open. Since $\mathbf{A}^1_{S^\sharp} = \cup_{n\geq 1}\mathrm{Spa}(A\langle p^nT\rangle, A^+\langle p^nT\rangle)$, it suffices to check that $\mathbf{B}^1_{S^\sharp} \to S^\sharp$ is open. By the limit argument above, this reduces to openness of $\mathbf{B}^1_{X_i} \to X_i$, which is a special case of [Hub96, Prop. 1.7.8]. □

As a first application, we have the following result.

Theorem 2.11. *Suppose S is an affinoid perfectoid space over $\mathbf{F}_p$, and let $\mathcal{E}/\mathcal{X}_S$ be a vector bundle of constant rank r and degree d whose slopes are all positive. Let N be any positive integer such that $\frac{1}{N}$ is less than the smallest slope of $\mathcal{E}$. Then, after pullback along some affinoid pro-étale cover $S' \to S$, we can find a short exact sequence*

$$0 \to \mathcal{O}^m \to \mathcal{O}(\tfrac{1}{N})^d \to \mathcal{E}_{S'} \to 0.$$

Note that necessarily $m = Nd - r$.

Proof. Fix N as in the theorem. Regarding $\mathcal{H}^0\left((\mathcal{O}(\frac{1}{N})^d)^\vee \otimes \mathcal{E}\right)$ as parametrizing bundle maps $\mathcal{O}(\frac{1}{N})^d \to \mathcal{E}$, let

$$\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\tfrac{1}{N})^d, \mathcal{E}\right) \subset \mathcal{H}^0\left((\mathcal{O}(\tfrac{1}{N})^d)^\vee \otimes \mathcal{E}\right)$$

be the subfunctor whose T-points parametrize surjective bundle maps $\mathcal{O}(\frac{1}{N})^d \to \mathcal{E}_T$ on $\mathcal{X}_T$, and let $\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)^{\mathrm{ss}}$ be the subfunctor of $\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)$ whose T-points parametrize surjective bundle maps $\mathcal{O}(\frac{1}{N})^d \to \mathcal{E}_T$ whose kernel is pointwise-semistable at all points of T. These are clearly pro-étale sheaves. We claim that it suffices to prove that the structure map

$$\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\tfrac{1}{N})^d, \mathcal{E}\right)^{\mathrm{ss}} \to S$$

is surjective as a map of pro-étale sheaves. Indeed, if this map is surjective, then we can choose a pro-étale cover $S'' \to S$ together with an S''-point of $\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)^{\mathrm{ss}}$ lying over $S'' \to S$ and corresponding to a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}(\tfrac{1}{N})^d \to \mathcal{E}_{S''} \to 0$$

of vector bundles on $\mathcal{X}_{S''}$ such that $\mathcal{F}$ is pointwise-semistable of slope zero and degree m. Passing to a further pro-étale cover $S' \to S''$ such that $\mathcal{F}_{S'} \simeq \mathcal{O}^m$ and $S'' \to S$ is affinoid pro-étale, the theorem follows.

By [BFH+17, Prop. 3.3.5], $\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)$ is an open subfunctor of $\mathcal{H}^0\left((\mathcal{O}(\frac{1}{N})^d)^\vee \otimes \mathcal{E}\right)$ (strictly speaking, this result is only proved in loc. cit. when the base is a geometric point, but the proof works verbatim in our more general setup). By semicontinuity, $\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)^{\mathrm{ss}}$ is an open subfunctor of $\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)$. Thus $\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)^{\mathrm{ss}}$ is an open subdiamond of $\mathcal{H}^0\left((\mathcal{O}(\frac{1}{N})^d)^\vee \otimes \mathcal{E}\right)$. By the previous theorem, it now suffices to check that $\left|\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)^{\mathrm{ss}}\right| \to |S|$ is surjective.

We first prove this in the special case where $S = \mathrm{Spa}(C, C^+)$ is a geometric point. For this, choose a decomposition $\mathcal{E} \simeq \oplus_i \mathcal{E}_i$ where $\mathcal{E}_i$ is semistable of rank r_i and degree d_i. By [BFH+17, Theorem 1.1.2], we can choose short exact sequences

$$0 \to \mathcal{O}^{Nd_i - r_i} \to \mathcal{O}(\tfrac{1}{N})^{d_i} \to \mathcal{E}_i \to 0$$

for each i, and taking the termwise direct sum of these sequences over i gives a short exact sequence as in the theorem. In particular, this shows that $\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)^{\mathrm{ss}} \to S$ admits a section, so $\left|\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)^{\mathrm{ss}}\right| \to |S|$ is surjective.

Now let S be any affinoid perfectoid space. For any given point $s \in |S|$, we can find a map from a geometric point $\tilde{x} = \mathrm{Spa}(C, C^+) \to S$ such that the topological image of the closed point in $\tilde{x}$ is s. For any such s and $\tilde{x}$, we get a cartesian diagram

$$\begin{array}{ccc} \mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)^{\mathrm{ss}} \times_S \tilde{x} & \xrightarrow{t} & \tilde{x} \\ \downarrow & & \downarrow \\ \mathcal{S}\mathrm{urj}\left(\mathcal{O}(\frac{1}{N})^d, \mathcal{E}\right)^{\mathrm{ss}} & \longrightarrow & S \end{array}$$

of diamonds. By the argument in the previous paragraph, t admits a section r. Passing to topological spaces and going around the resulting diagram via $|r|$ implies that

$$s \in \mathrm{im}\left(\left|\mathcal{S}\mathrm{urj}\left(\mathcal{O}(\tfrac{1}{N})^d, \mathcal{E}\right)^{\mathrm{ss}}\right| \to |S|\right),$$

and so the claim follows upon varying s and $\tilde{x}$. □

Corollary 2.12. *Let S be any perfectoid space, and let $\mathcal{E}$ be any vector bundle on $\mathcal{X}_S$ with only positive slopes. Then the structure morphism $\mathcal{H}^0(\mathcal{E}) \to S$ is smooth.*

Proof. This is pro-étale-local on S, so we can assume that S is affinoid and that $\mathcal{E}$ has constant rank r and degree d and moreover fits into a short exact sequence

$$0 \to \mathcal{O}^m \to \mathcal{O}(\tfrac{1}{N})^d \to \mathcal{E} \to 0$$

as in the previous theorem. Passing to $\mathcal{H}^i$'s and applying Proposition 2.7, we get a short exact sequence

$$0 \to \mathcal{H}^0(\mathcal{O}^m) \simeq \underline{\mathbf{Q}_p^m} \to \mathcal{H}^0(\mathcal{O}(\tfrac{1}{N})^d) \xrightarrow{q} \mathcal{H}^0(\mathcal{E}) \to 0$$

of $\mathbf{Q}_p$-vector diamonds over S. Arguing as in the proof of Theorem 2.10 gives that q is a pro-étale $\underline{\mathbf{Q}_p^m}$-torsor. In particular, choosing any open compact subgroup $K \subset \mathbf{Q}_p^m$, we get (by [Sch17, Lemma 10.13]) a surjective separated étale map

$$\mathcal{H}^0(\mathcal{O}(\tfrac{1}{N})^d)/\underline{K} \xrightarrow{q'} \mathcal{H}^0(\mathcal{O}(\tfrac{1}{N})^d)/\underline{\mathbf{Q}_p^m} \simeq \mathcal{H}^0(\mathcal{E}).$$

The diamond $\mathcal{H}^0(\mathcal{O}(\frac{1}{N})^d)$ is smooth over S, since it's represented by an open d-variable polydisk over S. Since K is a pro-p group acting freely and continuously on $\mathcal{H}^0(\mathcal{O}(\frac{1}{N})^d)$, we deduce that $\mathcal{H}^0(\mathcal{O}(\frac{1}{N})^d)/\underline{K}$ is smooth over S, and thus the target of q' is smooth as well, since smoothness is étale-local on the source. □

Finally, we need the following "seed" result.

Proposition 2.13. *For any $N \geq 1$ and any S, the functor $\mathcal{H}^1(\mathcal{O}(\frac{-1}{N}))$ is a locally spatial diamond.*

Proof. Arguing locally on S, we can assume that S admits a map $a : S \to \mathrm{Spa}\,\mathbf{Q}_p^{\mathrm{cyc}}$, so it's enough to show that $\mathcal{H}^1_{\mathrm{Spa}\,\mathbf{Q}_p^{\mathrm{cyc}}}(\mathcal{O}(\frac{-1}{N}))$ is a locally spatial diamond. By [Sch17, Prop. 13.4.ii-iv], it suffices to show that $\mathcal{H}^1_{\mathrm{Spa}\,\mathbf{C}_p}(\mathcal{O}(\frac{-1}{N}))$ is a locally spatial diamond.

Next, observe that over $\mathcal{X}_{\mathrm{Spa}\,\mathbf{C}_p}$ we can choose a short exact sequence of coherent sheaves

$$0 \to \mathcal{O}(-\tfrac{1}{N}) \to \mathcal{O}^N \to i_*\mathbf{C}_p \to 0,$$

where $i : \mathrm{Spa}\mathbf{C}_p \to \mathcal{X}_{\mathrm{Spa}\,\mathbf{C}_p}$ is the inclusion of the point at infinity. Note that for a general S with a map $S \to \mathrm{Spd}\,\mathbf{C}_p$, the pullback of $i_*\mathbf{C}_p$ along $\mathcal{X}_S \to \mathcal{X}_{\mathrm{Spa}\,\mathbf{C}_p}$ is $i_*\mathcal{O}_{S^\sharp}$, where $i : S^\sharp \to \mathcal{X}_S$ is the closed immersion of the specified untilt of S into $\mathcal{X}_S$, cf. [Han16, §2.3]. In particular, the functor $\mathcal{H}^0(i_*\mathbf{C}_p)$ on perfectoid spaces over $\mathrm{Spd}\,\mathbf{C}_p$ is represented by the diamond $(\mathbf{A}^1_{\mathbf{C}_p})^\diamondsuit$. Therefore, passing to $\mathcal{H}^i$'s and applying 2.7, we get a short exact sequence

$$0 \to \mathcal{H}^0(\mathcal{O}^N) \simeq \underline{\mathbf{Q}_p^N} \xrightarrow{\alpha} \mathcal{H}^0(i_*\mathbf{C}_p) \simeq (\mathbf{A}^1_{\mathbf{C}_p})^\diamondsuit \xrightarrow{\beta} \mathcal{H}^1(\mathcal{O}(-\tfrac{1}{N})) \to 0.$$

Note that α corresponds to an embedding of $\mathbf{Q}_p^N$ as a closed subgroup of $|\mathbf{A}^1_{\mathbf{C}_p}|$ consisting of classical points; here of course we give $\mathbf{A}^1_{\mathbf{C}_p}$ the usual additive group structure. Now, writing

$$\mathbf{A}^1_{\mathbf{C}_p} = \cup_{n\geq 1}\mathrm{Spa}(\mathbf{C}_p\langle p^nT\rangle, \mathcal{O}_{\mathbf{C}_p}\langle p^nT\rangle) = \cup U_n$$

as a union of quasicompact open subgroups, we get an associated covering of $\mathcal{H}^1_{\mathrm{Spd}\,\mathbf{C}_p}(\mathcal{O}(-\frac{1}{N}))$ by open subdiamonds

$$\mathcal{H}^1_{\mathrm{Spd}\,\mathbf{C}_p}(\mathcal{O}(-\tfrac{1}{N})) \simeq \cup_{n\geq 1} U_n^\diamondsuit/(\alpha(\underline{\mathbf{Q}_p^N}) \cap U_n^\diamondsuit).$$

Since $\alpha(\underline{\mathbf{Q}_p^N}) \cap U_n \simeq \underline{\mathbf{Z}_p^N}$ is profinite and $U_n^\diamondsuit$ is spatial, we're now reduced to the following lemma, whose proof we leave as an exercise for the reader. □

Lemma 2.14. *Let G be a profinite group, and let X be a spatial diamond with a $\underline{G}$-action. Then $X/\underline{G}$ is a spatial diamond.*

Theorem 2.15. *Suppose that $\mathcal{E}$ has only negative slopes. Then $\mathcal{H}^1(\mathcal{E})$ is a locally spatial diamond.*

Proof. This is local on S, so we can assume that S is affinoid and that $\mathcal{E}$ has constant rank r and degree $-d$. We already know that $\mathcal{H}^1(\mathcal{E})$ is a diamond with separated structure map $\mathcal{H}^1(\mathcal{E}) \to S$, so by [Sch17, Prop. 13.4], the local spatiality of $\mathcal{H}^1(\mathcal{E})$ can be checked pro-étale-locally on S. Applying Theorem 2.11, we may choose (after replacing S by some affinoid pro-étale cover) a short exact sequence

$$0 \to \mathcal{O}^m \to \mathcal{O}(\tfrac{1}{N})^d \to \mathcal{E}^\vee \to 0$$

of vector bundles on S for some fixed large N. Dualizing this sequence, passing to the associated long exact sequence of $\mathcal{H}^i$'s, and applying Proposition 2.7, we get a short exact sequence

$$0 \to \mathcal{H}^0(\mathcal{O}^m) \simeq \underline{\mathbf{Q}_p^m} \xrightarrow{i} \mathcal{H}^1(\mathcal{E}) \xrightarrow{q} \mathcal{H}^1\left(\mathcal{O}(\tfrac{-1}{N})^d\right) \to 0$$

of $\mathbf{Q}_p$-vector diamonds over S. Clearly

$$\mathcal{H}^1(\mathcal{E}) \times \underline{\mathbf{Q}_p^m} \overset{(f,a)\mapsto(f,f+i(a))}{\longrightarrow} \mathcal{H}^1(\mathcal{E}) \times_{\mathcal{H}^1\left(\mathcal{O}(\frac{-1}{N})^d\right)} \mathcal{H}^1(\mathcal{E})$$

is an isomorphism, so we get a cartesian diagram

$$\begin{array}{ccc} \mathcal{H}^1(\mathcal{E}) \times \underline{\mathbf{Q}_p^m} & \xrightarrow{(f,a)\mapsto f} & \mathcal{H}^1(\mathcal{E}) \\ {\scriptstyle (f,a)\mapsto f+i(a)}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ \mathcal{H}^1(\mathcal{E}) & \xrightarrow{q} & \mathcal{H}^1\left(\mathcal{O}(\frac{-1}{N})^d\right) \end{array}$$

of diamonds; note that the upper horizontal arrow is clearly separated and quasi-pro-étale. In particular, (the lower horizontal copy of) q becomes separated and quasi-pro-étale after base changing along the v-cover given by (the right vertical copy of) q, so q is separated and quasi-pro-étale by [Sch17, Prop. 10.11.v]. Since the target of q is locally spatial, we can now conclude the local spatiality of its source by applying [Sch17, Corollary 11.28]. □

2.3 The stack Bun_n

In this section we check that the stack Bun_n is a small v-stack.

Proposition 2.16. *The fibered category* Bun_n *is a v-stack.*

Proof. This follows from Proposition 20.2.1 and Lemma 20.2.2 of [SW15] □

Proposition 2.17. *The diagonal map* $\Delta : \mathrm{Bun}_n \to \mathrm{Bun}_n \times \mathrm{Bun}_n$ *is representable in locally spatial diamonds.*

Proof. Let S be a perfectoid space with a map $S \to \mathrm{Bun}_n \times \mathrm{Bun}_n$, corresponding to a pair of vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ over $\mathcal{X}_S$. We need to check that $\mathrm{Bun}_n \times_{\mathrm{Bun}_n \times \mathrm{Bun}_n} S$ is a locally spatial diamond. By definition, this fiber product is the sheaf

$$\begin{array}{rcl} \mathcal{I}\mathrm{som}_S(\mathcal{E}_1, \mathcal{E}_2) : \mathrm{Perf}_{/S} & \to & \mathrm{Sets} \\ T \to S & \mapsto & \mathcal{O}_{\mathcal{X}_T}\text{-module isomorphisms } \mathcal{E}_{1,T} \xrightarrow{\sim} \mathcal{E}_{2,T}. \end{array}$$

For any two bundles $\mathcal{E}, \mathcal{F}$ on $\mathcal{X}_S$, let $\mathcal{H}\mathrm{om}_S(\mathcal{E}, \mathcal{F}) \cong \mathcal{H}^0_S(\mathcal{E}^\vee \otimes \mathcal{F})$ be the functor on $\mathrm{Perf}_{/S}$ sending $T \to S$ to the set of $\mathcal{O}_{\mathcal{X}_T}$-module maps $\mathcal{E}_T \to \mathcal{F}_T$. By Proposition 2.8, this is a locally spatial diamond. Note that the identity map $\mathcal{E} \to \mathcal{E}$ defines a distinguished section $\mathrm{id} : S \to \mathcal{H}\mathrm{om}_S(\mathcal{E}, \mathcal{E})$ of the structure morphism to S. We then conclude by observing the isomorphism

$$\mathcal{I}\mathrm{som}_S(\mathcal{E}_1, \mathcal{E}_2) \cong (\mathcal{H}\mathrm{om}_S(\mathcal{E}_1, \mathcal{E}_2) \times_S \mathcal{H}\mathrm{om}_S(\mathcal{E}_2, \mathcal{E}_1)) \times_{\gamma, \mathcal{H}\mathrm{om}_S(\mathcal{E}_1,\mathcal{E}_1) \times_S \mathcal{H}\mathrm{om}_S(\mathcal{E}_2,\mathcal{E}_2), \mathrm{id}^2} S,$$

where γ is the map sending $(f, g) \in \mathcal{H}\mathrm{om}_S(\mathcal{E}_1, \mathcal{E}_2) \times_S \mathcal{H}\mathrm{om}_S(\mathcal{E}_2, \mathcal{E}_1)$ to $(g \circ f, f \circ g) \in \mathcal{H}\mathrm{om}_S(\mathcal{E}_1, \mathcal{E}_1) \times_S \mathcal{H}\mathrm{om}_S(\mathcal{E}_2, \mathcal{E}_2)$. □

It remains to construct reasonable charts for Bun_n. To facilitate this, note that Bun_n decomposes as the disjoint union of open and closed substacks $\mathrm{Bun}_n^d \subset \mathrm{Bun}_n$ parametrizing rank n vector bundles of constant degree d. It thus suffices to find small v-sheaves X_d together with surjective maps $X_d \to \mathrm{Bun}_n^d$ for each d. There are several options for how to do this; in particular, one can build suitable X_d's from affine Grassmannians and prove a Beauville-Laszlo type uniformization, or one can build X_d's inspired by the theory of Quot schemes. We take the latter approach, following an idea of Fargues.

For any fixed $m \gg 0$, consider the functor

$$X_{d,m} = \mathcal{S}\mathrm{urj}\left(\mathcal{O}(m)^{mn+n-d}, \mathcal{O}(m+1)^{mn-d}\right)$$

on perfectoid spaces over $\breve{\mathbf{Q}}_p$. By the proof of Proposition 2.11, this is a locally spatial diamond. An easy calculation shows that for any complete algebraically closed field $C/\mathbf{F}_p$ and any surjection $q : \mathcal{O}(m)^{mn+n-d} \twoheadrightarrow \mathcal{O}(m+1)^{mn-d}$ of vector bundles over $\mathcal{X}_{\mathrm{Spa}\, C}$, the bundle $\ker q$ has rank n, degree d, and maximal HN slope at most m. Moreover, *every* vector bundle $\mathcal{E}$ satisfying these three numerical conditions arises as the kernel of such a surjection: after replacing $\mathcal{E}$ by $\mathcal{E}^\vee(m)$, this becomes the statement that any vector bundle of rank n with (positive) degree e and with all HN slopes non-negative can be realized as the cokernel of an injection $\mathcal{O}(-1)^e \to \mathcal{O}^{e+n}$, which again follows from Lemma 8.8.13 and Corollary 8.8.14 of [KL15]. In particular, the natural map

$$\begin{array}{rcl} \pi_m : X_{d,m} & \to & \mathrm{Bun}_n^d \\ (q : \mathcal{O}(m)^{mn+n-d} \twoheadrightarrow \mathcal{O}(m+1)^{mn-d}) & \mapsto & \ker q \end{array}$$

factors through the inclusion of the open substack $\mathrm{Bun}_n^{d,\mathrm{max.slope}\leq m}$ parametrizing bundles with maximal slope $\leq m$. Let S be a perfectoid space with a map $f : S \to \mathrm{Bun}_n^{d,\mathrm{max.slope}\leq m}$, corresponding to a bundle $\mathcal{E}/\mathcal{X}_S$. Replacing $\mathcal{E}$ with $\mathcal{E}^\vee(m)$ and arguing as in the proof of Proposition 2.7.ii, we can find a pro-étale cover $S' \to S$ such that the composite map $S' \to \mathrm{Bun}_n^{d,\mathrm{max.slope}\leq m}$ lifts to an S'-point of $X_{d,m}$. In particular, the map

$$\pi_m : X_{d,m} \to \mathrm{Bun}_n^{d,\mathrm{max.slope}\leq m}$$

is surjective as a map of v-stacks. Setting $X = \coprod_{m>|d|} X_{d,m}$, the evident map $X \to \mathrm{Bun}_n^d$ is then surjective as a map of v-stacks, and the source is a locally spatial diamond, so we conclude.

3 Dynamics on Banach-Colmez spaces

3.1 The space $\mathcal{S}_Q$

Proposition 3.1. *The map $\mathcal{S}_Q \to \mathrm{Spd}\, \overline{\mathbf{F}_p}$ is representable in locally spatial diamonds and is partially proper.*

Proof. We argue by induction on the number of slopes of Q. When Q has one slope, $\mathcal{S}_Q \cong \mathrm{Spd}\, \overline{\mathbf{F}_p}$, so we may assume Q has two or more slopes. Write $Q = \mathrm{HN}(\oplus_{1\leq i\leq k}\mathcal{O}(\lambda_i)^{m_i})$ as in the introduction. Notation as in the introduction, it then suffices to show that

$$q : \mathcal{S}_Q \to \mathcal{S}_{Q'}$$

is representable in locally spatial diamonds and partially proper. Let T be a perfectoid space with a map $f : T \to \mathcal{S}_{Q'}$, corresponding to a bundle $\mathcal{E}'/\mathcal{X}_T$ with filtration and rigidification. One then checks directly from the definitions that the sheaf of sets

$$\mathcal{S}_Q \times_{\mathcal{S}_{Q'}} T$$

on $\mathrm{Perf}_{/T}$ is represented by the functor

$$\mathcal{H}^1\left((\mathcal{O}(\lambda_k)^{m_k})^\vee \otimes \mathcal{E}'\right).$$

By [BFH+17, Corollary 2.2.13], the maximal slope of $\mathcal{E}'_x$ at any point $x \in T$ is at most λ_{k-1}, so $(\mathcal{O}(\lambda_k)^{m_k})^\vee \otimes \mathcal{E}'$ has only negative slopes. We then conclude by Proposition 2.9 and Theorem 2.15. □

3.2 Orbit closures

In this section we fill in the details of the proof of Proposition 1.4. We begin with some easy lemmas.

Lemma 3.2. *Let X be a topological space with an action of a group G, and let $x \in X$ be a G-fixed point. Then $x \in \overline{yG}$ for all $y \in X$ if and only if X is the unique G-stable open neighborhood of x.*

Proof. The existence of a G-stable open neighborhood U of x with $U \subsetneq X$ is clearly equivalent to the existence of a non-empty G-stable closed subset $V \subset X$ with $x \notin V$. But the existence of such a V is clearly equivalent to the existence of a G-orbit yG with $x \notin \overline{yG}$ (one direction is obvious; for the other direction, write $V = \cup_{y\in V}\overline{yG}$). □

Lemma 3.3. *Let X and Y be topological spaces with actions of a group G, and let $f : Y \to X$ be a continuous G-equivariant map. Then for any G-fixed point $y \in Y$ and any $y' \in Y$ such that $y \in \overline{y'G}$, we have $f(y) \in \overline{f(y')G}$.*

Proof. Observe that

$$f(y) \in f(\overline{y'G}) \subseteq \overline{f(y'G)} = \overline{f(y')G},$$

where the middle containment follows from continuity. □

Lemma 3.4. *Let G be a group with a product decomposition $G = H \times K$, and let X be a topological space with a G-action. Let $x \in X$ be a G-fixed point, and let $S \subset X$ be a K-stable subspace containing x. Suppose that every H-orbit closure in X meets S and that every K-orbit closure of a point of S contains x. Then every G-orbit closure in X contains x.*

Proof. Let $x' \in X$ be any point. By assumption, we may choose some $s \in S$ with $s \in \overline{x'H}$. Then $sK \subseteq \overline{x'H}K$, so

$$\overline{sK} \subseteq \overline{\overline{x'H}K} = \overline{x'HK} = \overline{x'G},$$

where the middle equality follows from the general identity

$$\overline{\cup_{i\in I}\overline{V_i}} = \overline{\cup_{i\in I}V_i}$$

for any collection of subsets V_i of any topological space X. Since $x \in \overline{sK}$ by assumption, the result follows. □

We now return to the problem at hand.

Proof of Proposition 1.4. Let $x \subset |\mathcal{S}_Q|$ be any point. We need to prove that $s_Q \in \overline{xJ_Q}$. As in the introduction, we have the fibration $q : \mathcal{S}_Q \to \mathcal{S}_{Q'}$ with its canonical section $\sigma : \mathcal{S}_{Q'} \to \mathcal{S}_Q$, so we can regard $|\mathcal{S}_{Q'}|$ as a closed subspace of $|\mathcal{S}_Q|$ via σ; note also that $\sigma(s_{Q'}) = s_Q$. By induction, we can assume that the the $J_{Q'}$-orbit closure of any point in $|\mathcal{S}_{Q'}| \subset |\mathcal{S}_Q|$ contains s_Q. By Lemma 3.4, it then suffices to check that

$$q(x) \in \overline{x\mathrm{GL}_{m_k}(D_{\lambda_k})}.$$

To verify this, choose a complete algebraically closed extension $C/\mathbf{Q}_p$ and some open bounded valuation subring $C^+ \subset C$ together with a map

$$\mathrm{Spd}(C, C^+) \to \mathcal{S}_{Q'}$$

such that the topological image of the unique closed point of $|\mathrm{Spd}(C, C^+)|$ is $q(x)$. Let $\mathcal{E}'/\mathcal{X}_{\mathrm{Spa}(C^\flat, C^{+\flat})}$ be the vector bundle (with $k-1$-step filtration and rigidication) defined by this map. Set $\mathcal{S} = \mathcal{S}_Q \times_{q, \tilde{\mathcal{S}}_{Q'}} \mathrm{Spd}(C, C^+)$, so

$$\mathcal{S} \cong \mathcal{H}^1(\mathcal{E}' \otimes \mathcal{O}(-\lambda_k)^{m_k})$$

is a locally spatial $\mathbf{Q}_p$-vector diamond over $\mathrm{Spd}(C, C^+)$ by the arguments in the previous sections. Let $0 \in |\mathcal{S}|$ be the topological image of the unique closed point in $\mathrm{Spd}(C, C^+)$ along the zero section, so we get a natural $\mathrm{GL}_{m_k}(D_{\lambda_k})$-equivariant commutative diagram

$$\begin{array}{ccc} |\mathcal{S}| & \xrightarrow{\pi} & |\mathcal{S}_Q| \\ \uparrow & & \uparrow \\ \{0\} & \longrightarrow & \{q(x)\} \end{array}$$

such that the image of π contains x. By Lemma 3.3, it suffices to check that any $\mathrm{GL}_{m_k}(D_{\lambda_k})$-orbit closure in $|\mathcal{S}|$ contains 0.

To proceed, note it suffices to prove that for any given $\mathcal{F}/\mathcal{X}_{\mathrm{Spa}(C^\flat, C^{+\flat})}$ with only negative slopes, the $p^{\mathbf{Z}}$-orbit (for the scaling action of $p^{\mathbf{Z}} \subset \mathbf{Q}_p^\times$) of any point $x \in |\mathcal{H}^1(\mathcal{F})|$ has the point 0 in its closure. Indeed, for the particular $\mathcal{F}$ of interest to us, the scaling action of $a \in \mathbf{Q}_p^\times$ corresponds to the action of the element $\mathrm{diag}(a, a, \ldots, a) \in \mathrm{GL}_{m_k}(D_{\lambda_k})$, and so the $p^{\mathbf{Z}}$-orbit of any $x \in |\mathcal{H}^1(\mathcal{F})|$ is contained in the $\mathrm{GL}_{m_k}(D_{\lambda_k})$-orbit of x.

To check this claim about $p^{\mathbf{Z}}$-orbit closures, we observe that $\mathcal{F}$ can be written as the kernel of a surjection $\mathcal{O}^m \to \mathcal{O}(1)^n$ for some m, n. Taking cohomology of the associated short exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}^m \to \mathcal{O}(1)^n \to 0,$$

we get a surjection of vector diamonds $\mathcal{H}^0(\mathcal{O}(1)^n) \to \mathcal{H}^1(\mathcal{F})$ over $\mathrm{Spd}(C, C^+)$. Applying Lemma 3.3 again reduces us to checking that the closure of any $p^{\mathbf{Z}}$-orbit in $|\mathcal{H}^0(\mathcal{O}(1)^n)|$ contains 0. This statement, finally, can be checked by hand. Indeed, there is a natural identification of $\mathcal{H}^0(\mathcal{O}(1)^n)$ with the n-variable open perfectoid unit disk

$$\tilde{\mathbf{D}}^n = \mathrm{Spa}\left(C^+[[T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}]], C^+[[T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}]]\right)_\eta$$

over $\mathrm{Spa}(C, C^+)$, matching the scaling action of p with the Frobenius operator $\varphi : T_i \mapsto T_i^p$. Moreover, the point 0 identifies with the (unique) point x_0 lying over the closed point of $\mathrm{Spa}(C, C^+)$ and whose associated valuation sends each T_i to 0.

By Lemma 3.2, we're now reduced to checking that the only φ-stable open neighborhood of x_0 in $|\tilde{\mathbf{D}}^n|$ is the entirety of $|\tilde{\mathbf{D}}^n|$, which is easy. Indeed, it suffices to check that if $U \subset |\tilde{\mathbf{D}}^n|$ is an open neighborhood of x_0, then $\cup_{j\gg 0}\varphi^{-j}(U) = |\tilde{\mathbf{D}}^n|$; but the subsets

$$V_m = \left\{ x \in |\tilde{\mathbf{D}}^n| \mid |T_i|_x \leq |p|_x^m \,\forall 1 \leq i \leq n \right\}$$

are cofinal among open neighborhoods of x_0, and clearly $\cup_{j\gg 0}\varphi^{-j}(V_m) = |\tilde{\mathbf{D}}^n|$ for any m. □

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