Voronoi formulas on $GL(n)$

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Abstract

In this paper, we give a new, simple, purely analytic proof of the Voronoi formula for Maass forms on $GL(3)$ first derived by Miller and Schmid. Our method is based on two lemmas of the first author and Thillainatesan which appear in their recent non-adelic proof of the converse theorem on $GL(3)$. Using a different, even simpler method we derive Voronoi formulas on $GL(n)$ twisted by additive characters of prime conductors. We expect that this method will work in general. In the final section of the paper Voronoi formulas on $GL(n)$ are obtained, but in this case, the twists are by automorphic forms from lower rank groups.

1 Introduction

The classical Poisson summation formula states that for any function f in the Schwartz class $\mathbb{S}(\mathbb{R}^l)$, we have

$$\sum_{n\in\mathbb{Z}^l} f(n) = \sum_{m\in\mathbb{Z}^l} \hat{f}(m),$$

where

$$\hat{f}(x) = \int\limits_{\mathbb{R}^l} f(y) e\left(-\sum_{i=1}^{l} x_i y_i\right) dy$$

is the Fourier transform of f, and $e(x) = e^{2\pi i x}$ throughout the paper.

Voronoi formulas associated to automorphic forms are Poisson summation formulas weighted by Fourier coefficients of automorphic forms with possible twists by characters or other arithmetic weights. They usually serve as tools to study the shifted sum $\sum\limits_{n\leqslant N} \alpha(n)\beta(n+k)$ where $\alpha(n), \beta(n)$ are Fourier coefficients of automorphic forms. Such problems are often encountered when evaluating power moments of L-functions, see, for example, [DFI], [LS], [Sa]. Voronoi formulas for automorphic forms on $GL(2)$ were well-established in the past (see [Me]) and play important roles in the theory of $GL(2)$ L-functions (see the excellent survey papers [IS], [MS1]).

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Voronoi formulas associated to Maass forms on $GL(3)$ which have twists by additive characters were first derived by Miller and Schmid [MS2] using the theory of automorphic distributions. In section 3, we give a new, simple, purely analytic proof of Miller and Schmid's formula. The proof is based on [JPS] and two lemmas proved by the first author and Thillainatesan, see [Go, Chapter VII] which they use to give a new proof of the converse theorem on $GL(3)$.

In section 4, we start from the functional equation of L-functions on $GL(n)$ twisted by Dirichlet characters to derive Voronoi formulas for Maass forms on $GL(n)$ twisted by additive characters of prime conductors. The method is expected to work in general modulo some technical difficulties involving imprimitive characters. Our main result is Theorem 4.1.

In the last section, we derive Voronoi formulas for Maass forms on $GL(n)$ with $n \geqslant 3$ twisted by Fourier coefficients of automorphic forms on lower rank groups. The results are direct consequences of the functional equations of the Rankin-Selberg L-functions ([Go, Chapter XII], [JPS]).

2 Background on automorphic forms

The facts in this section can be found in [Go].

For $n \geqslant 2$, let $G = GL(n, \mathbb{R}), \Gamma = SL(n, \mathbb{Z})$ and

$$\mathfrak{h}^n = GL(n, \mathbb{R})/\langle O(n, \mathbb{R}) \cdot \mathbb{R}^\times \rangle$$

be the generalized upper half plane. Every element $z \in \mathfrak{h}^n$ has the form $z = xy$ where

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \dots & x_{1,n} \\ & 1 & x_{2,3} & \dots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix},$$

$$y = \mathrm{diag}(y_1 y_2 ... y_{n-1},\ y_1 y_2 ... y_{n-2},\ ..., y_1,\ 1),$$

with $x_{ij} \in \mathbb{R}$ for $1 \leqslant i < j \leqslant n$ and $y_i > 0$ for $1 \leqslant i \leqslant n-1$.

Let $\nu = (\nu_1, \nu_2, ..., \nu_{n-1}) \in \mathbb{C}^{n-1}$. The function

$$I_\nu(z) = \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} y_i^{b_{n-i,j}\nu_j} \tag{2.1}$$

with

$$b_{i,j} = \begin{cases} ij & \text{if } i + j \leqslant n, \\ (n-i)(n-j) & \text{otherwise,} \end{cases} \tag{2.2}$$

is an eigenfunction of every differential operator D in $\mathcal{D}^n$, the center of the universal enveloping algebra of $gl(n, \mathbb{R})$. Here $gl(n, \mathbb{R})$ is the Lie algebra of $GL(n, \mathbb{R})$. Let us write

$$DI_\nu(z) = \lambda_D I_\nu(z) \tag{2.3}$$

for every $D \in \mathcal{D}^n$. An automorphic form of type ν for $\Gamma = SL(n,\mathbb{Z})$ is a smooth function on $\mathfrak{h}^n$which satisfies

1) $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$;

2) $Df(z) = \lambda_D f(z)$ for all $D \in \mathcal{D}^n$.

If f also satisfies

3) $\int\limits_{\Gamma\cap U\backslash U} f(uz) d^*u = 0$

where $d^*u = \prod\limits_{1\leqslant i<j\leqslant n} du_{i,j}$ for all upper triangular matrices of the form

$$u = \begin{pmatrix} I_{r_1} & & & \\ & I_{r_2} & & * \\ & & \ddots & \\ & & & I_{r_m} \end{pmatrix},$$

with $r_1 + r_2 + \cdots + r_m = n$, then f is called a Maass form of type ν.

For $z \in \mathfrak{h}^n$, let $U_n(\mathbb{R})$ denote the group of $n \times n$ upper triangular matrices with ones on the diagonal. Let

$$W_{\text{Jacquet}}(z;\nu,\psi_m) = \int\limits_{U_n(\mathbb{R})} I_\nu(w_n uz)\overline{\psi_m(u)} d^*u \tag{2.4}$$

be Jacquet's Whittaker function which has rapid decay as $y_i \to \infty, 1 \leqslant i \leqslant n-1$. Here

$$\psi_m(u) = e(m_1 u_{1,2} + m_2 u_{2,3} + \cdots + m_{n-1} u_{n-1,n})$$

and

$$w_n = \begin{pmatrix} & & & \pm 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}.$$

Every Maass form $f(z)$ of type $\nu = (\nu_1, ..., \nu_{n-1})$ has the following Fourier-Whittaker expansion:

$$f(z) = \sum_{\gamma \in U_{n-1}(\mathbb{Z})\backslash SL(n-1,\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1}\neq 0} \frac{A(m_1,\ldots,m_{n-1})}{\prod\limits_{k=1}^{n-1} |m_k|^{\frac{k(n-k)}{2}}} \cdot W_{\text{Jacquet}}\left(M \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, \nu_f, \psi_{1,\cdots 1,1} \right), \tag{2.5}$$

where $U_n(\mathbb{Z})$ is the group of unipotent $n \times n$ upper triangular matrices with coefficients in $\mathbb{Z}$, and $M = \operatorname{diag}(m_1 \cdots m_{n-2}|m_{n-1}|, \cdots, m_1 m_2, m_1, 1)$. It is easy to prove that (see Chapter 9 in [Go]) the dual Maass form $\tilde{f}(z) := f(w_n{}^t z^{-1} w_n)$ is a Maass form of type $(\nu_{n-1}, \cdots, \nu_1)$ with Fourier coefficients $A(m_{n-1}, \ldots, m_1)$.

Next let's recall some facts about Hecke operators. Let $\mathcal{L}^2(\Gamma \setminus \mathfrak{h}^{\mathfrak{n}})$ be the space of square integrable automorphic forms on Γ equipped with the inner product:

$$\langle f, g \rangle = \int\limits_{\Gamma \setminus \mathfrak{h}^{\mathfrak{n}}} f(z)\overline{g(z)}\, d^*(z),$$

for all $f, g \in \mathcal{L}^2(\Gamma \setminus \mathfrak{h}^{\mathfrak{n}})$, where $d^*(z) = \prod\limits_{1 \leqslant i < j \leqslant n} dx_{i,j} \prod\limits_{k=1}^{n-1} y_k^{-k(n-k)-1} dy_k$ is the G left invariant measure. For every integer $N \geqslant 1$, we define a Hecke operator T_N acting on $\mathcal{L}^2(\Gamma \setminus \mathfrak{h}^{\mathfrak{n}})$ by the following formula:

$$T_N f(z) = \frac{1}{N^{\frac{n-1}{2}}} \sum_{\substack{\prod_{l=1}^{n} c_l = N \\ 0 \leqslant c_{i,l} < c_l \ (1 \leqslant i, < l \leqslant n)}} f\left(\begin{pmatrix} c_1 & c_{1,2} & \ldots & c_{1,n} \\ & c_2 & \ldots & c_{2,n} \\ & & \ddots & \vdots \\ & & & c_n \end{pmatrix} \cdot z \right).$$

The Hecke operators are normal operators. They commute with each other as well as with the G invariant differential operators. So we may simultaneously diagonalize the space $\mathcal{L}^2(\Gamma \setminus \mathfrak{h}^{\mathfrak{n}})$ by all these operators. Let f be a Maass form with Fourier expansion (2.5) which is also an eigenfunction of all the Hecke operators. We normalize $A(1, \ldots, 1)$ to be 1. Then we have the following multiplicativity relations:

$$A(m_1 m_1', \ \ldots, \ m_{n-1} m_{n-1}') = A(m_1, \ \ldots, \ m_{n-1}) \cdot A(m_1', \ \ldots, \ m_{n-1}'),$$

if $(m_1 \ldots m_{n-1}, m_1' \ldots m_{n-1}') = 1$, and

$$A(m, 1, \ldots, 1) A(m_1, \ldots, m_{n-1}) = \sum_{\substack{\prod_{l=1}^{n} c_l = m \\ c_1 | m_1, \ c_2 | m_2, \ \ldots, \ c_{n-1} | m_{n-1}}} A\left(\frac{m_1 c_n}{c_1}, \frac{m_2 c_1}{c_2}, \ldots, \frac{m_{n-1} c_{n-2}}{c_{n-1}} \right).$$

3 Voronoi formulas on $GL(3)$

In this section, we will give a new poof of the Voronoi formula for Maass forms on $GL(3)$ first proved by Miller and Schmid [MS2].

Suppose f is a Maass form of type (ν_1, ν_2) for $SL(3,\mathbb{Z})$. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ \frac{h}{q} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \delta := (h,q),$$

$$u = \begin{pmatrix} 1 & 0 & u_3 \\ & 1 & u_1 \\ & & 1 \end{pmatrix}, \qquad w_2 = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix},$$

$$z = \begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix} \in \mathfrak{h}^3.$$

If f is automorphic then

$$f(Auz) = \tilde{f}(w_2{}^t(Auz)^{-1}). \tag{3.1}$$

For $k = 0, 1$, let

$$F_k(y,h,q) := \left(\frac{\partial}{\partial x_2}\right)^k \int_0^1 \int_0^1 f(Auz) e(-qu_1) du_1 du_3 \Bigg|_{x_1 = x_2 = 0}, \tag{3.2}$$

with $y = \mathrm{diag}(y_1 y_2,\ y_1,\ 1)$.

Lemma 3.1. *For fixed y_1, the function $F_k(y,h,q)$ has rapid decay as $y_2 \to \infty$ or $y_2 \to 0$.*

Proof [Go, Chapter VII, pp. 8]. Suppose f has the following Fourier expansion:

$$\begin{aligned} f(z) = & \sum_{\gamma \in U_2(\mathbb{Z}) \backslash SL(2,\mathbb{Z})} \sum_{m_1=1}^{\infty} \sum_{m_2 \neq 0} \frac{A(m_1,m_2)}{m_1 |m_2|} \\ & \cdot W_{\text{Jacquet}}\left(\mathrm{diag}(m_1|m_2|, m_1, 1) \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} z, (\nu_1,\nu_2), \psi_{1,1}\right) \\ = & \sum_{(c,d)=1} \sum_{m_1=1}^{\infty} \sum_{m_2 \neq 0} \frac{A(m_1,m_2)}{m_1|m_2|} e\left(m_1(cx_3 + dx_1) + m_2 \Re \frac{az_2+b}{cz_2+d}\right) \\ & \cdot W_{\text{Jacquet}}\left(\mathrm{diag}\left(\frac{m_1|m_2|y_1y_2}{|cz_2+d|}, m_1 y_1 |cz_2+d|, 1\right), (\nu_1,\nu_2), \psi_{1,1}\right), \end{aligned} \tag{3.3}$$

where $z_2 = x_2 + iy_2$. For any $n_1 \neq 0, n_2 \neq 0$, we may compute the Fourier coefficient:

$$\begin{aligned} & \frac{A(m_1,m_2)}{|m_1 m_2|} W_{\text{Jacquet}}\left(\mathrm{diag}(|m_1 m_2| y_1 y_2, m_1 y_1, 1), (\nu_1,\nu_2), \psi_{1,1}\right) \\ & \qquad = \int_0^1 \int_0^1 \int_0^1 f(z) e(-m_1 x_1 - m_2 x_2) dx_1 dx_2 dx_3. \end{aligned}$$

Since every Maass form for $SL(3,\mathbb{Z})$ is even (see [Bu, (4.13)] or [Go, Chapter IX]) it follows that $A(m_1,m_2)=A(\pm m_1,\pm m_2)$.

A simple matrix computation shows

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ h/q & 1 \end{pmatrix}=\begin{pmatrix} a+bh/q & b \\ c+dh/q & d \end{pmatrix}=\begin{pmatrix} a' & b \\ c' & d \end{pmatrix}.$$

Applying this to (3.3), yields

$$\begin{aligned}
&\int_0^1\int_0^1 f(Auz)e(-qu_1)du_1du_3 \\
&\quad=\sum_{(c,d)=1}\sum_{m_1=1}^{\infty}\sum_{m_2\neq 0}\frac{A(m_1,m_2)}{m_1|m_2|} \\
&\quad\cdot\int_0^1\int_0^1 e(m_1(c'(x_3+u_3)+d'(x_1+u_1)))e(-qu_1)e\left(m_2\Re\frac{a'z_2+b}{c'z_2+d}\right)
\end{aligned}\tag{3.4}$$
$$\cdot W_{\text{Jacquet}}\left(\operatorname{diag}(m_1|m_2|y_1y_2/|c'z_2+d|,m_1y_1|c'z_2+d|,1),(\nu_1,\nu_2),\psi_{1,1}\right)\cdot du_1du_3.$$

Because of the simple fact

$$\int_0^1 e(x\alpha)dx=\begin{cases} 1 & \text{if } \alpha=0, \\ 0 & \text{otherwise,}\end{cases}$$

the integrals over u_1,u_3 are zero unless $c=-\frac{dh}{q},\ m_1d=q$. It follows that $m_1=\delta$. Setting $q_\delta=q\delta^{-1},\ h_\delta=h\delta^{-1}$, we have

$$F_k(y,h,q)=\sum_{m_2\neq 0}\frac{A(\delta,m_2)}{\delta|m_2|}e\left(\frac{m_2\bar{h}_\delta}{q_\delta}\right)\left(\frac{2\pi i m_2}{q_\delta^2}\right)^k \cdot W_{\text{Jacquet}}\left(\operatorname{diag}(\delta|m_2|y_1y_2q_\delta^{-1},\delta y_1q_\delta,1),(\nu_1,\nu_2),\psi_{1,1}\right).\tag{3.5}$$

Obviously $F_k(y,h,q)$ has rapid decay as $y_2\to\infty$ because of the decay property of the Jacquet-Whittaker function. On the other hand, using the following Fourier expansion of $\tilde{f}(z)$:

$$\tilde{f}(z)=\sum_{(c,d)=1}\sum_{m_1=1}^{\infty}\sum_{m_2\neq 0}\frac{A(m_2,m_1)}{m_1|m_2|}e\left(m_1(cx_3+dx_1)+m_2\Re\frac{az_2+b}{cz_2+d}\right) \cdot W_{\text{Jacquet}}\left(\operatorname{diag}(m_1|m_2|y_1y_2/|cz_2+d|,m_1y_1|cz_2+d|,1),(\nu_2,\nu_1),\psi_{1,1}\right),\tag{3.6}$$

toghether with the identity

$$w_2 \cdot \left({}^t(Auz)^{-1}\right) \cdot w_2^{-1} = \begin{pmatrix} 1 & -u_3 - x_3 & -u_1 - x_1 + \frac{x_2(u_3+x_3)}{x_2^2+y_2^2} \\ & 1 & -\frac{h}{q} - \frac{x_2}{x_2^2+y_2^2} \\ & & 1 \end{pmatrix}$$
$$\cdot \begin{pmatrix} \frac{y_1 y_2}{\sqrt{x_2^2+y_2^2}} & & \\ & \frac{y_2}{x_2^2+y_2^2} & \\ & & 1 \end{pmatrix} \mathrm{mod}(O(3,\mathbb{R})\mathbb{R}^\times),$$

and the identity (3.1), we obtain

$$\int_0^1 \int_0^1 f(Auz) e(-qu_1) du_1 du_3$$
$$= \sum_{(c,d)=1} \sum_{m_1=1}^{\infty} \sum_{m_2 \neq 0} \frac{A(m_1,m_2)}{m_1|m_2|} \int_0^1 \int_0^1 e\left(m_1 c\left(-u_1 - x_1 + \frac{x_2(u_3+x_3)}{x_2^2+y_2^2}\right)\right)$$
$$\cdot e\left(-m_1 d\left(\frac{h}{q} + \frac{x_2}{x_2^2+y_2^2}\right) - qu_1 + m_2 \Re \frac{az_2'+b}{cz_2'+d}\right)$$
$$\cdot W_{\text{Jacquet}}\left((\mathrm{diag}\frac{m_1|m_2|y_1y_2}{|cz_2'+d|}, \frac{m_1 y_2|cz_2'+d|}{x_2^2+y_2^2}, 1), \nu'\right) \cdot du_1 du_3. \tag{3.7}$$

Here

$$z_2' = -u_3 - x_3 + iy_1\sqrt{x_2^2+y_2^2}, \tag{3.8}$$

$\nu' = (\nu_2, \nu_1)$, and we have supressed the character $\psi_{1,1}$ in the Jacquet-Whittaker function in order to simplify notation. For the same reason as before, the u_1 intgeral disappears unless $m_1 c = -q$. Note that $\frac{az+b}{cz+d} = \frac{a}{c} - \frac{1}{c(cz+d)}$. Let $d = lc + r$ with $l \in \mathbb{Z}$ and $1 \leqslant r < |c|$ with $(r,c) = 1$. After changing variables $u_3 \to u_3 + l - \frac{rm_1}{q}$ in formula (3.7), it follows that the right side of (3.7) becomes

$$e(qx_1) \sum_{m_1|q} \sum_{m_2 \neq 0} \frac{A(m_2,m_1)}{m_1|m_2|} S(h, m_2; qm_1^{-1})$$
$$\cdot \sum_{l \in \mathbb{Z}} \int_{l-\frac{rm_1}{q}}^{1+l-\frac{rm_1}{q}} e\left(\frac{-qx_2(u_3+x_3)}{x_2^2+y_2^2} + m_2 \Re\left(\frac{-1}{c^2 z_2'}\right)\right)$$
$$\cdot W_{\text{Jacquet}}\left(\left(\mathrm{diag}\frac{m_1|m_2|y_1y_2}{\sqrt{x_2^2+y_2^2}|cz_2'|}, \frac{m_1 y_2|cz_2'|}{x_2^2+y_2^2}, 1\right), (\nu_2,\nu_1), \psi_{1,1}\right) du_3, \tag{3.9}$$

where

$$S(m,n;c)=\sum_{\substack{d\ (\mathrm{mod} c)\\ d\bar{d}\equiv 1(c)}} e\left(\frac{md+n\bar{d}}{c}\right) \tag{3.10}$$

is the classical Kloosterman sum. Making succesive transformtaions

$$u_3\to u_3-x_3,\qquad u_3\to u_3y_1\sqrt{x_2^2+y_2^2},$$

(3.9) becomes

$$e(qx_1)\sum_{m_1|q}\sum_{m_2\neq 0}\frac{A(m_2,m_1)}{m_1|m_2}S(h,m_2;qm_1^{-1})$$
$$\cdot\int_{-\infty}^{\infty} e\left(\frac{-qx_2y_1u_3}{\sqrt{x_2^2+y_2^2}}+\frac{m_1^2m_2u_3}{q^2y_1\sqrt{x_2^2+y_2^2}(u_3^2+1)}\right)$$
$$\cdot W_{\mathrm{Jacquet}}\left(\mathrm{diag}\left(\frac{m_1^2|m_2|y_2}{q(x_2^2+y_2^2)\sqrt{u_3^2+1}},\frac{qy_1y_2\sqrt{u_3^2+1}}{\sqrt{x_2^2+y_2^2}},1\right),(\nu_2,\nu_1),\psi_{1,1}\right)$$
$$\cdot y_1\sqrt{x_2^2+y_2^2}du_3.$$

Taking partial derivatives with respect to x_1,x_2 and setting $x_1=x_2=0$, we have

$$F_k(y,h,q)=e(qx_1)\sum_{m_1|q}\sum_{m_2\neq 0}\frac{A(m_2,m_1)}{m_1|m_2}S(h,m_2;qm_1^{-1})$$
$$\cdot\int_{-\infty}^{\infty}\left(\frac{-2\pi iqy_1u_3}{y_2}\right)^k e\left(\frac{m_1^2m_2u_3}{q^2y_1y_2(u_3^2+1)}\right)$$
$$\cdot W_{\mathrm{Jacquet}}\left(\mathrm{diag}\left(\frac{m_1^2|m_2|}{qy_2\sqrt{u_3^2+1}},qy_1\sqrt{u_3^2+1},1\right),(\nu_2,\nu_1),\psi_{1,1}\right) \tag{3.11}$$
$$\cdot y_1y_2du_3,$$

which has rapid decay as $y_2\to 0$ because of the decay property of the Jacquet-Whittaker function. □

It follows from the above lemma, for $k=0,1,\Re s_1$ large, that

$$\mathcal{F}_k(h,q,s):=\int_0^{\infty}\int_0^{\infty}F_k(y_1,y_2,h,q)y_1^{s_1-1}y_2^{s_2-1}\frac{dy_1}{y_1}\frac{dy_2}{y_2} \tag{3.12}$$

is absolutely convergent for all $s_2\in\mathbb{C}$ and hence an entire function of s_2. The following two lemmas on $\mathcal{F}_k(h,q,s)$ were used by the first author and Thillainatesan to give a new proof of the $GL(3)$ converse theorem (see [Go, Chapter VII]). These lemmas are crucial for the Voronoi formula:

Lemma 3.2. ([Go, Lemma 7.1.12]) *For* $\Re s_2$ *large,*

$$\mathcal{F}_k(h,q,s) = \frac{1}{q_\delta^{s_1-2s_2+1}\delta^{s_1}} \sum_{m_2\neq 0} \frac{A(\delta,m_2)}{|m_2|^{s_2}} \left(\frac{2\pi i m_2}{q_\delta^2}\right)^k e\left(\frac{m_2\bar{h}_\delta}{q_\delta}\right) G_1(s_1,s_2,\nu)$$

with

$$G_1(s_1,s_2,\nu) = \int_0^\infty\int_0^\infty W_{\text{Jacquet}}(\mathrm{diag}(\mathrm{y}_1\mathrm{y}_2,\mathrm{y}_1,1),(\nu_1,\nu_2),\psi_{1,1})\mathrm{y}_1^{\mathrm{s}_1-1}\mathrm{y}_2^{\mathrm{s}_2-1}\frac{d\mathrm{y}_1}{\mathrm{y}_1}\frac{d\mathrm{y}_2}{\mathrm{y}_2}.$$

Remark: This follows directly from (3.5).

Lemma 3.3. ([Go, Lemma 7.1.13]) *For* $-\Re s_2$ *large,*

$$\mathcal{F}_k(h,q,s) = \frac{(-2\pi i q)^k \pi^{\frac{s_1+s_2}{2}}}{q^{s_1+s_2}\Gamma(\frac{s_1+s_2}{2})} \sum_{m_1|q}\sum_{m_2\neq 0} \frac{A(m_2,m_1)}{m_1^{2k+1-2s_2}|m_2|^{k+1-s_2}} \left(\frac{im_2}{|m_2|}\right)^k \cdot S(h,m_2;qm_1^{-1})G_2(s,\nu,k)$$

with

$$G_2(s,\nu,k) = \int_0^\infty\int_0^\infty W_{\text{Jacquet}}(y,(\nu_2,\nu_1),\psi_{1,1})K_{\frac{s_1+s_2-1-2k}{2}}(2\pi y_2) \cdot y_1^{2k+s_1-s_2} y_2^{\frac{2k+s_1-s_2-1}{2}}\frac{dy_1}{y_1}\frac{dy_2}{y_2},$$

and $K_\nu(x)$ *is the* K*-Bessel function.*

Remark: Lemma 3.3 follows from (3.11) by making the transformations

$$y_1 \to \frac{y_1}{q\sqrt{u_3^2+1}}, \qquad y_2 \to \frac{m_1^2|m_2|}{q\sqrt{u_3^2+1}y_1y_2},$$

and invoking the following formulas:

$$\int_{-\infty}^\infty e(uy_2)(u^2+1)^{-s}du = \frac{2\pi^s}{\Gamma(s)}|y_2|^{s-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|y_2|),$$

$$\int_{-\infty}^\infty e(uy_2)(u^2+1)^{-s}udu = 2\left(i\frac{y_2}{|y_2|}\right)\frac{\pi^s}{\Gamma(s)}|y_2|^{s-\frac{1}{2}}K_{s-\frac{3}{2}}(2\pi|y_2|).$$

Now, for $\Re s_2 > 3$, we define

$$L_k(\bar{h},q,s_2) = \sum_{m_2>0}\frac{A(\delta,m_2)}{m_2^{s_2-k}}\left[e\left(\frac{m_2\bar{h}_\delta}{q_\delta}\right) + (-1)^k e\left(\frac{-m_2\bar{h}_\delta}{q_\delta}\right)\right] \tag{3.15}$$

and

$$\hat{L}_k(h,q,s_2) = \sum_{m_1|q}\sum_{m_2>0} \frac{A(m_2,m_1)}{m_1^{2k+2s_2-1}m_2^{k+s_2}} \cdot \left[S(h,m_2;qm_1^{-1}) + (-1)^k S(h,-m_2;qm_1^{-1})\right]. \tag{3.16}$$

By Lemmas 3.2 and 3.3, these functions have analytic continuation to the whole complex plane and satisfy the following functional equation

$$\frac{i^k}{q_\delta^{s_1-2s_2+1+2k}\delta^{s_1}} L_k(\bar{h},q,s_2)G_1(s_1,s_2,\nu) = \frac{\pi^{\frac{s_1+s_2}{2}}}{q^{s_1+s_2-k}\Gamma(\frac{s_1+s_2}{2})}\hat{L}_k(h,q,1-s_2)G_2(s_1,s_2,\nu,k). \tag{3.17}$$

Define

$$W^*(y,(\nu_1,\nu_2)) = \pi^{\frac{1}{2}-3\nu_1-3\nu_2}\Gamma\left(\frac{3\nu_1}{2}\right)\Gamma\left(\frac{3\nu_2}{2}\right) \cdot\Gamma\left(\tfrac{3\nu_1+3\nu_2-1}{2}\right)W(y,(\nu_1,\nu_2),\psi_{1,1}). \tag{3.18}$$

Then by [Bu, (10.1)],

$$\int_0^\infty\int_0^\infty W^*(y,(\nu_1,\nu_2))y_1^{s_1-1}y_2^{s_2-1}\frac{dy_1}{y_1}\frac{dy_2}{y_2} = \frac{\pi^{-s_1-s_2}}{4}G_1^*(s_1,s_2,\nu), \tag{3.19}$$

where

$$G_1^*(s_1,s_2,\nu) = \frac{\Gamma(\frac{s_1+\alpha}{2})\Gamma(\frac{s_1+\beta}{2})\Gamma(\frac{s_1+\gamma}{2})\Gamma(\frac{s_1-\alpha}{2})\Gamma(\frac{s_1-\beta}{2})\Gamma(\frac{s_1-\gamma}{2})}{\Gamma(\frac{s_1+s_2}{2})},$$

and

$$\alpha = \nu_1 - 2\nu_2 + 1, \quad \beta = -\nu_1+\nu_2, \quad \gamma = 2\nu_1+\nu_2-1. \tag{3.20}$$

By ([St, pp. 357]), we have

$$\int_0^\infty\int_0^\infty W^*(y,(\nu_2,\nu_1))K_{\frac{s_1+s_2-1-2k}{2}}(2\pi y_2)y_1^{2k+s_1-s_2}y_2^{\frac{2k+s_1-s_2-1}{2}}\frac{dy_1}{y_1}\frac{dy_2}{y_2} = \pi^{\frac{-3s_1-6k+3s_2-3}{2}}\hat{G}_2(s_1,s_2,\nu,k) \tag{3.21}$$

where $\hat{G}_2(s_1,s_2,\nu,k)$ is equal to

$$\frac{1}{4}\Gamma\left(\frac{1-s_2+2k+\alpha}{2}\right)\Gamma\left(\frac{1-s_2+2k+\beta}{2}\right)\Gamma\left(\frac{1-s_2+2k+\gamma}{2}\right) \cdot\Gamma\left(\frac{s_1+\alpha}{2}\right)\Gamma\left(\frac{s_1+\beta}{2}\right)\Gamma\left(\frac{s_1+\gamma}{2}\right).$$

From the above formulas, we obtain

(3.22)
$$L_k(\bar{h},q,s_2)=\hat{L}_k(h,q,1-s_2)i^{-k}q^{-3s_2+1+3k}\pi^{3s_2-3k-\frac{3}{2}}\delta^{2s_2-1-2k}G(s_2,k,\nu),$$

where

$$G(s_2,k,\nu)=\frac{\Gamma(\frac{1-s_2+2k+\alpha}{2})\Gamma(\frac{1-s_2+2k+\beta}{2})\Gamma(\frac{1-s_2+2k+\gamma}{2})}{\Gamma(\frac{s_2-\alpha}{2})\Gamma(\frac{s_2-\beta}{2})\Gamma(\frac{s_2-\gamma}{2})}. \tag{3.23}$$

Assume $\phi(x)\in C_c^\infty(0,\infty)$ and $\tilde{\phi}(s)=\int\limits_0^\infty\phi(x)x^s\frac{dx}{x}$ is its Mellin transform. Then for $\sigma>3$, by Mellin inversion, we have

$$\sum_{m_2\in\mathbb{Z}}A(\delta,m_2)\left[e\left(\frac{m_2\bar{h}_\delta}{q_\delta}\right)+(-1)^ke\left(-\frac{m_2\bar{h}_\delta}{q_\delta}\right)\right]\phi(m_2) \tag{3.24}$$
$$=\frac{1}{2\pi i}\int\limits_{\Re s_2=\sigma}\tilde{\phi}(s_2-k)L_k(\bar{h},q,s_2)ds_2.$$

Moving the line of integration to $-\sigma$, applying (3.22) and letting $s_2\to -s_2$, it follows that (3.24) is equal to

$$\frac{\pi^{-3k-\frac{5}{2}}q_\delta^{1+3k}\delta^k}{2i^{1+k}}\sum_{m_1|q_\delta\delta}\sum_{m_2>0}\frac{A(m_2,m_1)}{m_1^{2k+1}m_2^{k+1}}$$
$$\cdot[S(\delta h_\delta,m_2;\delta q_\delta m_1^{-1})+(-1)^kS(\delta h_\delta,-m_2;\delta q_\delta m_1^{-1})]\Phi_k\left(\frac{m_2m_1}{q_\delta^3\delta}\right),$$

where

(3.25)
$$\Phi_k(x)=\int\limits_{\Re s_2=\sigma}(\pi^3x)^{-s_2}\frac{\Gamma(\frac{1+s_2+2k+\alpha}{2})\Gamma(\frac{1+s_2+2k+\beta}{2})\Gamma(\frac{1+s_2+2k+\gamma}{2})}{\Gamma(\frac{-s_2-\alpha}{2})\Gamma(\frac{-s_2-\beta}{2})\Gamma(\frac{-s_2-\gamma}{2})}\tilde{\phi}(-s_2-k)ds_2.$$

Set $a:=h_\delta$ and $c:=q_\delta$, we end up with the Voronoi formula on $GL(3)$:

Theorem 3.1. *Let $k=0,1$, and $\phi(x)\in C_c^\infty(0,\infty)$. Let $A(m,n)$ denote the (m,n)-th Fourier coefficient of a Maass form for $SL(3,\mathbb{Z})$ as in (3.3). Let $a,\bar{a},c,\delta\in\mathbb{Z}$ with $\delta>0,c\neq 0,(a,c)=1$, and $a\bar{a}\equiv 1(\mathrm{mod\ c})$. Then we have*

$$\sum_{m>0}A(\delta,m)\left[e\left(\frac{m\bar{a}}{c}\right)+(-1)^ke\left(\frac{-m\bar{a}}{c}\right)\right]\phi(m)$$
$$=\frac{\pi^{-3k-\frac{5}{2}}c^{1+3k}\delta^k}{2i^{1+k}}\sum_{m_1|c\delta}\sum_{m_2>0}\frac{A(m_2,m_1)}{m_1^{2k+1}m_2^{k+1}}$$
$$\cdot\Big(S(\delta a,m_2;\delta cm_1^{-1})+(-1)^kS(\delta a,-m_2;\delta cm_1^{-1})\Big)\Phi_k\left(\frac{m_2m_1^2}{c^3\delta}\right).$$

Next let

$$\Phi_{0,1}^0(x) = \Phi_0(x) + \frac{\pi^{-3}c^3\delta}{m_1^2 m_2 i}\Phi_1(x)$$

and

$$\Phi_{0,1}^1(x) = \Phi_0(x) - \frac{\pi^{-3}c^3\delta}{m_1^2 m_2 i}\Phi_1(x).$$

We obtain the following:

Corollary 3.1. *Let* $k = 0, 1,$ *and* $\phi(x) \in C_c^\infty(0,\infty)$. *Let* $A(m,n)$ *denote the* (m,n)*-th Fourier coefficient of a Maass form for* $SL(3,\mathbb{Z})$ *as in (3.3). Let* $a, \bar{a}, c, \delta \in \mathbb{Z}$ *with* $\delta > 0, c \neq 0, (a,c) = 1,$ *and* $a\bar{a} \equiv 1(\mathrm{mod\ c})$. *Then we have*

$$\sum_{m>0} A(\delta, m) e\left(\frac{m\bar{a}}{c}\right)\phi(m)$$
$$= \frac{c\pi^{-\frac{5}{2}}}{4i} \sum_{m_1|c\delta}\sum_{m_2>0} \frac{A(m_2,m_1)}{m_1 m_2} S(\delta a, m_2; \delta c m_1^{-1}) \Phi_{0,1}^0\left(\frac{m_2 m_1^2}{c^3\delta}\right)$$
$$+ \frac{c\pi^{-\frac{5}{2}}}{4i} \sum_{m_1|c\delta}\sum_{m_2>0} \frac{A(m_2,m_1)}{m_1 m_2} S(\delta a, -m_2; \delta c m_1^{-1}) \Phi_{0,1}^1\left(\frac{m_2 m_1^2}{c^3\delta}\right).$$

4 Voronoi formulas on $GL(n)$ twisted by additive characters

In this section we will derive Voronoi formulas on $GL(n)$ twisted by additive characters. The method of proof is much simpler, even for $GL(3)$, than the method presented in section 3, at least in the case of additive characters where the conductor q is a prime. We expect this method to work in the most general case modulo some technical difficulties involving imprimitive characters. For simplicity we assume q is a prime. Without loss of generality, let f be an even Maass form on $GL(n)$ of type $\nu = (\nu_1, \cdots, \nu_{n-1}) \in \mathbb{C}^{n-1}$, which implies its Fourier coefficients $A(\pm m_1, \pm m_2, \ldots, \pm m_{n-1}) = A(m_1, m_2, \ldots, m_{n-1})$. We also assume f is a Hecke eigenform with normalized Fourier coefficient $A(1, \ldots, 1)$ to be 1. The Godement-Jacquet L-function defined for $\Re s$ large by

$$L_f(s) := \sum_{m=1}^{\infty} \frac{A(m,1,\ldots,1)}{m^s}$$

has analytic continuation to the whole complex plane and satifies the following functional equation:

(4.1)
$$\pi^{-\frac{ns}{2}} \prod_{i=1}^{n} \Gamma\left(\frac{s-\lambda_i(\nu)}{2}\right) L_f(s) = \pi^{-\frac{n(1-s)}{2}} \prod_{i=1}^{n} \Gamma\left(\frac{1-s-\tilde{\lambda}_i(\nu)}{2}\right) L_{\tilde{f}}(1-s),$$

where $\tilde{f}$ is the dual Maass form of f and where $\lambda_i(\nu)$ and $\tilde{\lambda}_i(\nu)$ are linear forms in ν as defined in ([Go, Remark 10.8.7]). Let χ be an even primitive character modulo q. Then, for $\Re s$ sufficiently large,

$$L_f(s,\chi) := \sum_{m\neq 0} \frac{A(m,1,\ldots,1)}{|m|^s}\chi(m)$$

has analytic continuation to the entire plane and satisfies the following functional equation:

$$\begin{aligned}\left(\frac{q}{\pi}\right)^{\frac{ns}{2}} \prod_{i=1}^{n} \Gamma\left(\frac{s-\lambda_i(\nu)}{2}\right) L_f(s,\chi) \\ = \left(\frac{\tau(\chi)}{\sqrt{q}}\right)^n \left(\frac{q}{\pi}\right)^{\frac{n(1-s)}{2}} \prod_{i=1}^{n} \Gamma\left(\frac{1-s-\tilde{\lambda}_i(\nu)}{2}\right) L_{\tilde{f}}(1-s,\bar{\chi}),\end{aligned} \tag{4.2}$$

where

$$\tau(\chi) = \sum_{l (\mathrm{mod}\ q)} \chi(l) e\left(\frac{l}{q}\right)$$

is the Gauss sum. Let

$$G(s) = \prod_{i=1}^{n} \Gamma\left(\frac{s-\lambda_i(\nu)}{2}\right), \quad \tilde{G}(1-s) = \prod_{i=1}^{n} \Gamma\left(\frac{1-s-\tilde{\lambda}_i(\nu)}{2}\right). \tag{4.3}$$

Then (4.2) implies that

$$L_f(s,\chi) = \tau(\chi)^n q^{-ns} \pi^{-\frac{n}{2}+ns} \frac{\tilde{G}(1-s)}{G(s)} L_{\tilde{f}}(1-s,\bar{\chi}). \tag{4.4}$$

For $(m,q)=1, (q,h)=1$ we have the following identities:

$$\sum_{\substack{\chi\ (\mathrm{mod}\ q) \\ \chi\neq\chi_0 \\ \chi(-1)=1}} \tau(\bar{\chi})^{n-1}\chi(m) = \frac{q-1}{2}\Big(K_{n-1}(m,q)+K_{n-1}(-m,q)\Big) + (-1)^n, \tag{4.5}$$

where

$$K_{n-1}(m,q) = \sum_{x_1 x_2 \cdots x_{n-1} \equiv m (q)} e\left(\frac{x_1+\cdots+x_{n-1}}{q}\right) \tag{4.6}$$

is the hyper-Kloosterman sum and

$$e\left(\frac{mh}{q}\right) = \frac{1}{q-1} \sum_{\substack{\chi\ (\mathrm{mod}\ q) \\ \chi\neq\chi_0}} \bar{\chi}(mh)\tau(\chi) - \frac{1}{q-1}. \tag{4.7}$$

Let

(4.8)

$$L_f(q,h,s) := \sum_{m\neq 0} \frac{A(m,1,\ldots,1)}{|m|^s} e\left(\frac{mh}{q}\right)$$

$$= \sum_{m\equiv 0 (\mathrm{mod}\ q)} \frac{A(m,1,\ldots,1)}{|m|^s} e\Big(\frac{mh}{q}\Big) + \sum_{(m,q)=1} \frac{A(m,1,\ldots,1)}{|m|^s} e\Big(\frac{mh}{q}\Big).$$

In the above, by (4.7), we have

(4.9)

$$\sum_{(m,q)=1} \frac{A(m,1,\ldots,1)}{|m|^s} e\left(\frac{mh}{q}\right)$$

$$= \frac{1}{q-1} \sum_{m\neq 0} \frac{A(m,1,\ldots,1)}{|m|^s} \cdot \sum_{\substack{\chi\ (\mathrm{mod}\ q)\\ \chi\neq\chi_0\\ \chi(-1)=1}} \bar{\chi}(mh)\tau(\chi)$$

$$-\frac{1}{q-1} \sum_{(m,q)=1} \frac{A(m,1,\ldots,1)}{|m|^s},$$

where the odd characters don't contribute due to f being an even Maass form.

Applying the functional equation (4.4) and the identity (4.5), the first term in (4.9) becomes

$$\frac{q^{-ns+1}}{q-1} \pi^{-\frac{n}{2}+ns} \frac{\tilde{G}(1-s)}{G(s)} \sum_{(m,q)=1} \frac{A(1,\ldots,1,m)}{|m|^{1-s}} \tag{4.10}$$

$$\cdot \left[\frac{q-1}{2}(K_{n-1}(m\bar{h},q) + K_{n-1}(-m\bar{h},q)) + (-1)^n\right]$$

after analytic continuation. Let

$$\tilde{\phi}_q(s) = \sum_{k=0}^{\infty} \frac{A(1,\ldots,1,q^k)}{q^{ks}}$$

$$= \left[1 + \sum_{1\leqslant l\leqslant n-1} (-1)^{n-l} q^{-(n-l)s} A(\underbrace{1,\ldots,1,q}_{\text{position } l},1,\ldots,1) + (-1)^n q^{-ns}\right]^{-1}. \tag{4.11}$$

Then for $\Re w$ sufficiently large,

$$L_{\tilde{f}}(w) = \sum_{m=1}^{\infty} \frac{A(1,\ldots,1,m)}{m^w} = \prod_p \tilde{\phi}_p(w).$$

So the following identity is obvious

$$\sum_{\substack{m\neq 0\\(m,q)=1}} \frac{A(1,\ \dots,\ 1,m)}{m^w} = \sum_{m\neq 0} \frac{A(1,\ \dots,\ 1,m)}{m^w}$$

$$- \sum_{m\neq 0} \frac{A(1,\ \dots,\ 1,m)}{m^w}\Big(1-\tilde{\phi}_q(w)^{-1}\Big). \tag{4.12}$$

When $q|m$, we have $K_{n-1}(m\bar{h},q)=(-1)^{n-2}$, so using (4.12) with $w=1-s$, we have

$$\frac{q^{-ns+1}\pi^{-\frac{n}{2}+ns}}{2}\frac{\tilde{G}(1-s)}{G(s)}\sum_{(m,q)=1}\frac{A(1,\dots,1,m)}{|m|^{1-s}}\left[K_{n-1}(m\bar{h},q)+K_{n-1}(-m\bar{h},q)\right]$$

$$= \frac{q^{-ns+1}\pi^{-\frac{n}{2}+ns}}{2}\frac{\tilde{G}(1-s)}{G(s)}\sum_{m\neq 0}\frac{A(1,\dots,1,m)}{|m|^{1-s}}\left[K_{n-1}(m\bar{h},q)+K_{n-1}(-m\bar{h},q)\right]$$

$$+(-1)^{n-1}q^{-ns+1}\pi^{-\frac{n}{2}+ns}\frac{\tilde{G}(1-s)}{G(s)}\sum_{m\neq 0}\frac{A(1,\dots,1,m)}{|m|^{1-s}}\left(1-\tilde{\phi}_q(1-s)^{-1}\right). \tag{4.13}$$

It is obvious that

$$(-1)^n\frac{q^{-ns+1}}{q-1}\pi^{-\frac{n}{2}+ns}\frac{\tilde{G}(1-s)}{G(s)}\sum_{(m,q)=1}\frac{A(1,\dots,1,m)}{|m|^{1-s}} \tag{4.14}$$

$$= (-1)^n\frac{q^{-ns+1}}{q-1}\pi^{-\frac{n}{2}+ns}\frac{\tilde{G}(1-s)}{G(s)}\sum_{m\neq 0}\frac{A(1,\dots,1,m)}{|m|^{1-s}}\tilde{\phi}_q(1-s)^{-1}.$$

It follows from (4.13) and (4.14) that we can write (4.10) as

$$\frac{q^{-ns+1}\pi^{-\frac{n}{2}+ns}}{2}\frac{\tilde{G}(1-s)}{G(s)}\sum_{m\neq 0}\frac{A(1,\dots,1,m)}{|m|^{1-s}}\left[K_{n-1}(m\bar{h},q)+K_{n-1}(-m\bar{h},q)\right]$$

$$+(-1)^{n-1}q^{-ns+1}\pi^{-\frac{n}{2}+ns}\frac{\tilde{G}(1-s)}{G(s)}\sum_{m\neq 0}\frac{A(1,\dots,1,m)}{|m|^{1-s}}\left(1-\tilde{\phi}_q(1-s)^{-1}\right). \tag{4.15}$$

$$+(-1)^n\frac{q^{-ns+1}}{q-1}\pi^{-\frac{n}{2}+ns}\frac{\tilde{G}(1-s)}{G(s)}\sum_{m\neq 0}\frac{A(1,\dots,1,m)}{|m|^{1-s}}\tilde{\phi}_q(1-s)^{-1}.$$

The second term in (4.9) is equal to

$$-\frac{1}{q-1}\sum_{m\neq 0}\frac{A(m,1,\dots,1)}{|m|^s}\phi_q(s)^{-1} \tag{4.16}$$

$$= -\frac{\pi^{ns-\frac{n}{2}}}{q-1}\frac{\tilde{G}(1-s)}{G(s)}\sum_{m\neq 0}\frac{A(1,\dots,1,m)}{|m|^{1-s}}\phi_q(s)^{-1},$$

after analytic continuation, where

$$(4.17)\qquad \phi_q(s) = \sum_{k=0}^{\infty} \frac{A(q^k,1,\dots,1)}{q^{ks}}$$
$$= \left[1 + \sum_{1\leqslant l\leqslant n-1} (-1)^l q^{-ls} A(\underbrace{1,\dots,1,q}_{\text{position } l},1,\dots,1) + (-1)^n q^{-ns}\right]^{-1}$$

by the Hecke relations. The first term in (4.8) is equal to

$$(4.18)\qquad \sum_{m\equiv 0 (\mathrm{mod}\ q)} \frac{A(m,1,\dots,1)}{|m|^s} = \sum_{k\geqslant 1}\sum_{(m,q)=1} \frac{A(q^k,1,\dots,1)A(m,1,\dots,1)}{q^{ks}|m|^s}$$
$$= \left[1-\phi_q(s)^{-1}\right]\cdot \pi^{ns-\frac{n}{2}} \frac{\tilde{G}(1-s)}{G(s)} \sum_{m\neq 0} \frac{A(1,\dots,1,m)}{|m|^{1-s}}$$

after analytic continuation. By combination of (4.15), (4.16) and (4.18), it follows that (4.8) may be written in the form:

$$(4.19)\qquad L_f(q,h,s) = \pi^{ns-\frac{n}{2}} \frac{\tilde{G}(1-s)}{G(s)} \sum_{m\neq 0} \frac{A(1,\dots,1,m)}{|m|^{1-s}}$$
$$\cdot\Big(\frac{q^{-ns+1}}{2}(K_{n-1}(m\bar{h},q)+K_{n-1}(-m\bar{h},q))+(-1)^{n-1}q^{-ns+1}(1-\tilde{\phi}_q(1-s)^{-1})$$
$$+(-1)^n \frac{q^{-ns+1}}{q-1}\tilde{\phi}_q(1-s)^{-1} - \frac{1}{q-1}\phi_q(s)^{-1} + 1 - \phi_q(s)^{-1}\Big),$$

after analytic continuation. It is easy to check that

$$(-1)^n \frac{q^{-ns+1}}{q-1}\tilde{\phi}_q(1-s)^{-1} - \frac{1}{q-1}\phi_q(s)^{-1} + 1 - \phi_q(s)^{-1}$$
$$= \sum_{1\leqslant l\leqslant n-1} (-1)^{l+1} A(\underbrace{1,\dots,1,q}_{\text{position } l},1,\dots,1) \sum_{0\leqslant j\leqslant n-l-1} q^{-ls-j} - \sum_{0\leqslant j\leqslant n-2} q^{j-n+1}.$$

Applying the Hecke relation

$$A(1,\dots,1,m)A(\underbrace{1,\dots,1,q}_{\text{position } l},1,\dots,1) = A(\underbrace{1,\dots,1,q}_{\text{position } l},1,\dots,1,m)$$
$$(4.20)\qquad + A\Big(\underbrace{1,\dots,1,q}_{\text{position } l-1},1,\dots,1,\frac{m}{q}\Big),$$

where the second term is nonzero only if $q|m$, we have

(4.21)
$$\sum_{m\neq 0}\frac{A(1,\ldots,1,m)}{|m|^{1-s}}\left[(-1)^n\frac{q^{-ns+1}}{q-1}\tilde{\phi}_q(1-s)^{-1}-\frac{1}{q-1}\phi_q(s)^{-1}+1-\phi_q(s)^{-1}\right]$$
$$=\sum_{1\leqslant l\leqslant n-1}\sum_{m\neq 0}\frac{A(\overbrace{1,\ldots,1,q}^{\text{position } l},1,\ldots,1,m)}{|m|^{1-s}}(-1)^{l+1}\sum_{0\leqslant j\leqslant n-l-1}q^{-ls-j}$$
$$+\sum_{2\leqslant l\leqslant n-1}\sum_{m\neq 0}\frac{A(\overbrace{1,\ldots,1,q}^{\text{position } l-1},1,\ldots,1,m)}{|m|^{1-s}}(-1)^{l+1}\sum_{0\leqslant j\leqslant n-l-1}q^{(-l+1)s-1-j}.$$

Similarly by the Hecke relation (4.20), one can verify the following

(4.22)
$$\sum_{m\neq 0}\frac{A(1,\ldots,1,m)}{|m|^{1-s}}(-1)^{n-1}q^{-ns+1}\Big(1-\tilde{\phi}_q(1-s)^{-1}\Big)$$
$$=\sum_{1\leqslant l\leqslant n-1}(-1)^l q^{1-n+l(1-s)}\sum_{m\neq 0}\frac{A(\overbrace{1,\ldots,1,q}^{\text{position } l},1,\ldots,1,m)}{|m|^{1-s}}$$
$$+\sum_{2\leqslant l\leqslant n-1}(-1)^l q^{s-n+l(1-s)}\sum_{m\neq 0}\frac{A(\overbrace{1,\ldots,1,q}^{\text{position } l-1},1,\ldots,1,m)}{|m|^{1-s}}.$$

Combining (4.19), (4.21) and (4.22) we arrive at

$$L_f(q,h,s)=\pi^{ns-\frac{n}{2}}\frac{\tilde{G}(1-s)}{G(s)}\hat{L}_f(q,\bar{h},1-s), \tag{4.23}$$

where

(4.24)
$$\hat{L}_f(q,\bar{h},s)=\frac{q^{n(s-1)+1}}{2}\sum_{m\neq 0}\frac{A(1,\ldots,1,m)}{|m|^s}\Big(K_{n-1}(m\bar{h},q)+K_{n-1}(-m\bar{h},q)\Big)$$
$$+\sum_{1\leqslant l\leqslant n-2}(-1)^{l+1}q^{-l(-s+1)}\sum_{m\neq 0}\frac{A(\overbrace{1,\ldots,1,q}^{\text{position } l},1,\ldots,1,m)}{|m|^s}.$$

Note that when $n=2$ the second term on the right side of the above formula doesn't exist.

After analytic continuation, we obtain the following functional equation

$$\pi^{-\frac{ns}{2}}G(s)L_f(q,h,s)=\pi^{-\frac{n(1-s)}{2}}\tilde{G}(1-s)\hat{L}_f(q,\bar{h},1-s). \tag{4.25}$$

Assume $\omega(x)\in C_c^\infty(0,\infty)$ and $\tilde{\omega}(s)=\int\limits_0^\infty \omega(x)x^s\frac{dx}{x}$ is the Mellin transform of $\omega(x)$. Then for σ large, we have

$$\sum_{m\in\mathbb{Z}}A(m,1,\ldots,1)\left[e\left(\frac{mh}{q}\right)+e\left(\frac{-mh}{q}\right)\right]\omega(m)=\frac{1}{2\pi i}\int\limits_{\Re s=\sigma}\tilde{\omega}(s)L_f(q,h,s)ds.$$

If we shift the line of integration to $-\sigma$ and apply the functional equation (4.23), we end up with the Voronoi formula. To state it, let

$$\Omega_1(x)=\frac{1}{2\pi i}\int\limits_{\Re s=-\sigma}\frac{q}{2}\tilde{\omega}(s)\pi^{\frac{-n}{2}}\frac{\tilde{G}(1-s)}{G(s)}x^s ds, \tag{4.26}$$

$$\Omega_2(x)=\frac{1}{2\pi i}\int\limits_{\Re s=-\sigma}\tilde{\omega}(s)\pi^{\frac{-n}{2}}\frac{\tilde{G}(1-s)}{G(s)}x^s ds. \tag{4.27}$$

We now state the main theorem in this section.

Theorem 4.1. (Voronoi formula on $GL(n)$): *Let f be an even Maass Hecke eigenform for $SL(n,\mathbb{Z})$ with $n\geqslant 2$. Let $A(m_1,\ \ldots,\ m_{n-1})$ be the Fourier coefficient of f as in (2.5). We assume $A(1,\ \ldots,\ 1)=1$. Let $\omega(x)\in C_c^\infty(0,\ \infty)$, $q=$ prime, and $h\bar{h}\equiv 1(\mathrm{mod\ q})$. Then*

$$\begin{aligned}\sum_{m>0}&A(m,1,\ldots,1)\left[e\left(\frac{mh}{q}\right)+e\left(\frac{-mh}{q}\right)\right]\omega(m)\\&=\sum_{m\neq 0}\frac{A(1,\ldots,1,m)}{|m|}\Big(K_{n-1}(m\bar{h},q)+K_{n-1}(-m\bar{h},q)\Big)\Omega_1\left(\frac{|m|\pi^n}{q^n}\right)\\&\quad+\sum_{1\leqslant l\leqslant n-2}(-1)^{l+1}\sum_{m\neq 0}\frac{A(\overbrace{1,\ldots,1,q}^{\text{position } l},1,\ldots,1,m)}{|m|}\Omega_2\left(\frac{|m|\pi^n}{q^l}\right)\end{aligned}$$

where $K_{n-1}(m,q)$ is the hyper-Kloosterman sum (4.6) and $\Omega_1(x),\Omega_2(x)$ are defined by (4.26) and (4.27) respectively.

Remark. When $n=2$ the second term on the right side of the above formula doesn't exist.

5 More general Voronoi formulas on $GL(n)$

For $2 \leqslant l < n$, let f, g be Maass forms of type $\nu_f \in \mathbb{C}^{n-1}, \nu_g \in \mathbb{C}^{l-1}$ for $SL(n,\mathbb{Z})$ and $SL(l,\mathbb{Z})$, respectively, with Fourier expansions:

$$f(z) = \sum_{\gamma \in U_{n-1}(\mathbb{Z})\backslash SL(n-1,\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{n-2}=1}^{\infty} \sum_{m_{n-1}\neq 0} \frac{A_f(m_1,\ldots,m_{n-1})}{\prod_{k=1}^{n-1} |m_k|^{\frac{k(n-k)}{2}}} \cdot W_{\text{Jacquet}}\left(M\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} z, \nu_f, \psi_{1,\ldots 1,1}\right), \tag{5.1}$$

$$g(z) = \sum_{\gamma \in U_{l-1}(\mathbb{Z})\backslash SL(l-1,\mathbb{Z})} \sum_{m_1=1}^{\infty} \cdots \sum_{m_{l-2}=1}^{\infty} \sum_{m_{l-1}\neq 0} \frac{B_g(m_1,\ldots,m_{l-1})}{\prod_{k=1}^{l-1} |m_k|^{\frac{k(l-k)}{2}}} \cdot W_{\text{Jacquet}}\left(M\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} z, \nu_g, \psi_{1,\cdots,1,1}\right). \tag{5.2}$$

Set

$$L_f(s) = \sum_{m\geqslant 1} \frac{A_f(m,1,\ldots,1)}{m^s} \tag{5.3}$$

$$L_g(s) = \sum_{m\geqslant 1} \frac{B_g(m,1,\ldots,1)}{m^s} \tag{5.4}$$

to be the associated L-functions which satisfy the following functional equations:

$$\Lambda_f(s) := \prod_{i=1}^{n} \pi^{\frac{-s+\lambda_i(\nu_f)}{2}} \Gamma\left(\frac{s-\lambda_i(\nu_f)}{2}\right) L_f(s) = \Lambda_{\tilde{f}}(1-s), \tag{5.5}$$

$$\Lambda_g(s) := \prod_{i=1}^{n} \pi^{\frac{-s+\lambda_i(\nu_g)}{2}} \Gamma\left(\frac{s-\lambda_i(\nu_g)}{2}\right) L_g(s) = \Lambda_{\tilde{g}}(1-s), \tag{5.6}$$

where $\tilde{f}, \tilde{g}$ are the dual forms. Then the Rankin-Selberg L-functions $L_{g\times f}(s)$ defined by

$$L_{g\times f}(s) = \sum_{m_1=1}^{\infty} \cdots \sum_{m_l=1}^{\infty} \frac{B_g(m_2,\ldots,m_l)\overline{A_f(m_1,\ldots,m_l,1,\ldots,1)}}{(m_1^l m_2^{l-1}\ldots m_l)^s} \tag{5.7}$$

for $\Re s > nl$ has a holomorphic continuation to all $s \in \mathbb{C}$. It also satisfies the functional equation

$$\Lambda_{g\times f}(s) := \prod_{i=1}^{n}\prod_{j=1}^{l} \pi^{\frac{-s+\lambda_i(\nu_g)+\overline{\lambda_j(\nu_f)}}{2}} \Gamma\left(\frac{s-\lambda_i(\nu_g)-\overline{\lambda_j(\nu_f)}}{2}\right) L_{g\times f}(s) = \Lambda_{\tilde{g}\times\tilde{f}}(1-s). \tag{5.8}$$

For $\phi(x) \in C_c^\infty(0,\infty)$, let

$$\tilde{\phi}(s) = \int_0^\infty \phi(x) x^s \frac{dx}{x} \tag{5.9}$$

be its Mellin transform. Then by Mellin inversion

$$\begin{aligned} &\sum_{m_1=1}^\infty \cdots \sum_{m_l=1}^\infty B_g(m_2,\ldots,m_l)\overline{A_f(m_1,\ldots,m_l,1,\ldots,1)}\phi(m_1^l m_2^{l-1}\cdots m_l) \\ &\quad = \frac{1}{2\pi i}\int_{\Re s=\sigma} \sum_{m_1=1}^\infty \cdots \sum_{m_l=1}^\infty B_g(m_2,\ldots,m_l)\overline{A_f(m_1,\ldots,m_l,1,\ldots,1)} \\ &\qquad\qquad \cdot \tilde{\phi}(s)(m_1^l m_2^{l-1}\cdots m_l)^{-s} ds \\ &\qquad\qquad = \frac{1}{2\pi i}\int_{(\sigma)} \psi(s) L_{g\times f}(s) ds. \end{aligned} \tag{5.10}$$

Moving the line of integration to $\Re s = -\sigma$ and applying the functional equation, it follows that (5.10) equals

$$\frac{1}{2\pi i}\int_{(-\sigma)} \psi(s) L_{\tilde{g}\times\tilde{f}}(1-s) G_s(\nu_f,\nu_g) ds$$

where

$$G_s(\nu_f,\nu_g) = \prod_{i=1}^l \prod_{j=1}^n \frac{\pi^{\frac{-nl(1-s)}{2}}\Gamma\left(\frac{1-s-\tilde{\lambda}_i(\nu_g)-\overline{\tilde{\lambda}_j(\nu_f)}}{2}\right)}{\pi^{\frac{-nls}{2}}\Gamma\left(\frac{s-\lambda_i(\nu_g)-\overline{\lambda_j(\nu_f)}}{2}\right)}. \tag{5.11}$$

Expanding $L_{\tilde{g}\times\tilde{f}}(1-s)$, it follows that (5.10) is equal to

$$\sum_{m_1=1}^\infty \cdots \sum_{m_l=1}^\infty \frac{B_{\tilde{g}}(m_1,\ldots,m_l)\overline{A_f(m_1,\ldots,m_l,1,\ldots,1)}}{m_1^l m_2^{l-1}\cdots m_l}\Phi(m_1^l m_2^{l-1}\cdots m_l),$$

where

$$\Phi(x) = \frac{1}{2\pi i}\int_{(-\sigma)} \psi(s) G_s(\nu_f,\nu_g) x^s ds. \tag{5.12}$$

Since

$$B_{\tilde{g}}(m_2,\ldots,m_l) = B_g(m_l,\ldots,m_2), \tag{5.13}$$

and

$$A_{\tilde{f}}(m_1,\ldots,m_l,1,\ldots,1) = A_f(1,\ldots,1,m_l,\ldots,m_1). \tag{5.14}$$

We end up with the following theorem.

Theorem 5.1. *(Voronoi formula on $GL(n)$): Let f, g be Maass forms for $SL(n,\mathbb{Z})$, $SL(l,\mathbb{Z})$, respectively where $2 \leqslant l < n$. Let $A(m_1, \ldots, m_{n-1})$, $B(m_1, \ldots, m_{l-1})$ denote the Fourier coefficients of f and g as in (5.1) and (5.2) .Then for $\phi(x) \in C_c^\infty(0,\infty)$, we have*

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_l=1}^{\infty} B_g(m_2,\ldots,m_l)\overline{A_f(m_1,\ldots,m_l,1,\ldots,1)}\phi(m_1^l m_2^{l-1}\cdots m_l)$$
$$= \sum_{m_1=1}^{\infty} \cdots \sum_{m_l=1}^{\infty} \frac{B_g(m_l,\ldots,m_2)\overline{A_f(1,\ldots,1,m_l,\ldots,m_1)}}{m_1^l m_2^{l-1}\cdots m_l}\Phi(m_1^l m_2^{l-1}\cdots m_l).$$

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