1. Trace Formula for finite groups

Let $G$ be a finite group and let $\Gamma$ be a subgroup of $G$. We consider the $\mathbb{C}$-vector space of functions

$$V := \left\{ \phi : G \to \mathbb{C} \mid \phi(\gamma x) = \phi(x), \ \forall \gamma \in \Gamma, \ \forall x \in G \right\}.$$ 

We can think of $V$ as the $\mathbb{C}$-vector space of automorphic forms on $\Gamma \backslash G$.

**Definition 1.1. (Inner product on $V$)** For $\phi_1, \phi_2 \in V$ we define

$$\langle \phi_1, \phi_2 \rangle := \sum_{x \in \Gamma \backslash G} \phi_1(x) \overline{\phi_2(x)}.$$ 

**Definition 1.2. (Orthonormal basis for $V$)** Let $\Gamma z \in \Gamma \backslash G$ and define $\mathbf{1}_{\Gamma z} \in V$ by

$$\mathbf{1}_{\Gamma z}(x) := \begin{cases} 1 & \text{if } x \in \Gamma z, \\ 0 & \text{otherwise}. \end{cases}$$

The functions $\mathbf{1}_{\Gamma z}$ for distinct cosets $\Gamma z$ form an orthonormal basis for $V$ with respect to the above inner product.

**Definition 1.3. (The kernel function $K_f(x, y)$)** Let $G$ be a finite group and $f : G \to \mathbb{C}$. Let $\Gamma$ be a subgroup of $G$. For $x, y \in G$ we define the kernel function

$$K_f(x, y) := \sum_{\gamma \in \Gamma} f(x^{-1} \gamma y).$$

**Definition 1.4. (The linear map $K_f$)** Let $\phi \in V$. We define a linear map $K_f : V \to V$ where for $\phi \in V$ we have $K_f(\phi) \in V$ where

$$(K_f \phi)(x) := \sum_{y \in \Gamma \backslash G} K_f(x, y) \cdot \phi(y).$$

**Definition 1.5. (Trace of a linear map on a finite dimensional vector space)** Let $V$ be a complex vector space with basis $v_1, v_2, \ldots, v_n$. A linear map $L : V \to V$ satisfies

$$L(v_i) = \sum_{j=1}^{n} a_{i,j} v_j, \quad \text{(for all } 1 \leq i \leq n \text{ and } a_{i,j} \in \mathbb{C}).$$

Then associated to $L$ we may define the matrix $A_L := (a_{i,j})_{1 \leq i,j \leq n}$. The trace of $L$ (denoted $\text{Tr}(L)$) is defined to be

$$\text{Tr}(L) = \sum_{i=1}^{n} a_{i,i},$$

i.e., it is the trace of the matrix $A_L$.
Remarks on eigenvalues of a matrix: It is well known that for a matrix $A$ with complex coefficients $a_{i,j}$, $(1 \leq i, j \leq n)$ that

- $\text{Tr}(A) = \text{sum of the eigenvalues of } A$;
- $\text{Det}(A) = \text{product of the eigenvalues of } A$.

We now compute the action of the linear map $K_f : V \to V$ on the orthonormal basis given in definition 1.2. Let $z \in \Gamma \setminus G$. We have

$$
\left( K_f \mathbf{1}_{\Gamma z} \right)(x) = \sum_{y \in \Gamma \setminus G} K_f(x, y) \cdot \mathbf{1}_{\Gamma z}(y) \\
= K_f(x, z) \\
= \sum_{x_1 \in \Gamma \setminus G} K_f(x_1, z) \cdot \mathbf{1}_{\Gamma x_1}(x).
$$

It follows that the matrix $A_{K_f}$ associated to the linear map $K_f$ is given by $\left( K_f(x_1, z) \right)_{x_1, z \in \Gamma \setminus G}$, and

$$
\text{Tr}(K_f) = \sum_{z \in \Gamma \setminus G} K_f(z, z).
$$

Proposition 1.6. (Trace formula for a finite group $G$) Let $G$ be a finite group and let $\Gamma$ be a subgroup of $G$. Let $\text{Cl}[\Gamma]$ denote the set of conjugacy classes $\{ \sigma^{-1} \gamma \sigma \mid \sigma \in \Gamma \}$ with $\gamma \in \Gamma$. For $\gamma \in \Gamma$ we also define

$$
\Gamma_\gamma := \{ \sigma \in \Gamma \mid \sigma^{-1} \gamma \sigma = \gamma \},
$$
$$
G_\gamma := \{ g \in G \mid g^{-1} \gamma g = \gamma \}.
$$

It follows that

$$
\text{Tr}(K_f) = \sum_{\gamma \in \text{Cl}[\Gamma]} \frac{\#(G_\gamma)}{\#(\Gamma_\gamma)} \sum_{z \in G_\gamma \setminus G} f(z^{-1} \gamma z).
$$
Proof. We have

\[
\text{Tr}(K_f) = \sum_{z \in \Gamma \setminus G} K_f(z, z)
\]

\[
= \sum_{z \in \Gamma \setminus G} \sum_{\gamma \in \Gamma} f(z^{-1} \gamma z)
\]

\[
= \sum_{z \in \Gamma \setminus G} \sum_{\gamma \in \text{Cl}([\Gamma])} \sum_{\sigma \in \Gamma \setminus \Gamma} f(z^{-1} \cdot \sigma^{-1} \gamma \cdot z)
\]

\[
= \sum_{\gamma \in \text{Cl}([\Gamma])} \sum_{z \in \Gamma \setminus G} \sum_{\sigma \in \Gamma \setminus \Gamma} f(z^{-1} \cdot \sigma^{-1} \gamma \cdot z)
\]

\[
= \sum_{\gamma \in \text{Cl}([\Gamma])} \sum_{z \in \Gamma \setminus G} \sum_{\sigma \in \Gamma \setminus \Gamma} f(z^{-1} \gamma z)
\]

\[
= \sum_{\gamma \in \text{Cl}([\Gamma])} \sum_{z \in \Gamma \setminus G} \sum_{\sigma \in \Gamma \setminus \Gamma} f((gz)^{-1} \cdot \gamma \cdot gz)
\]

\[
= \sum_{\gamma \in \text{Cl}([\Gamma])} \sum_{z \in \Gamma \setminus G} \sum_{\sigma \in \Gamma \setminus \Gamma} f(z^{-1} \cdot g^{-1} \gamma g \cdot z)
\]

\[
= \sum_{\gamma \in \text{Cl}([\Gamma])} \sum_{z \in \Gamma \setminus G} \sum_{\sigma \in \Gamma \setminus \Gamma} f(z^{-1} \gamma z)
\]

\[
= \sum_{\gamma \in \text{Cl}([\Gamma])} \# (G_{\gamma}) \# (\Gamma_{\gamma}) \sum_{z \in G_{\gamma} \setminus G} f(z^{-1} \gamma z)
\]

\[
\square
\]

The trace formula has two sides:

\[
\text{Tr}(K_f)_{\text{spectral side}} = \sum_{\gamma \in \text{Cl}([\Gamma])} \# (G_{\gamma}) \# (\Gamma_{\gamma}) \sum_{z \in G_{\gamma} \setminus G} f(z^{-1} \gamma z).
\]

where the geometric side consists of a sum over conjugacy classes.

2. Trace Formula for the infinite additive group $\mathbb{R}$ and subgroup $\mathbb{Z}$

Consider the additive group $G = \mathbb{R}$ and subgroup of rational integers $\Gamma = \mathbb{Z}$. Following the recipe in §1, we define the $\mathbb{C}$ vector space of smooth functions

\[
V := \left\{ \phi : \mathbb{R} \to \mathbb{C} \mid \phi(x + n) = \phi(x), \ \forall n \in \mathbb{Z}, \ \forall x \in \mathbb{R} \right\}.
\]
Definition 2.1. (Inner product on $V$) Let $\phi_1, \phi_2 \in V$. We define the inner product
\[
\langle \phi_2, \phi_2 \rangle := \int_{\mathbb{Z} \setminus \mathbb{R}} \phi_1(x) \overline{\phi_2(x)} \, dx = \int_0^1 \phi_1(x) \overline{\phi_2(x)} \, dx.
\]

Definition 2.2. (Orthonormal basis for $V$) Let $e_n(x) := e^{2\pi inx}$. It is well known that every periodic function is a linear combination of $e_n(x)$ with $n \in \mathbb{Z}$. Furthermore
\[
\langle e_m, e_n \rangle = \int_0^1 e^{2\pi i(m-n)x} \, dx = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise}. \end{cases}
\]
This establishes that the $\{e_n\}_{n \in \mathbb{Z}}$ is an orthonormal basis of $V$.

Definition 2.3. (The kernel function $K_f(x, y)$) Let $f : \mathbb{R} \to \mathbb{C}$ be a Schwartz function, i.e., a smooth function all of whose derivatives have rapid decay. For $x, y \in \mathbb{R}$ we define the kernel function
\[
K_f(x, y) := \sum_{n \in \mathbb{Z}} f(-x + n + y).
\]

Definition 2.4. (The linear map $K_f$) Let $\phi \in V$. We define the linear map $K_f : V \to V$ by
\[
(K_f \phi)(x) := \int_0^1 K_f(x, y) \cdot \phi(y) \, dy.
\]

Proposition 2.5. (The $e_n$ are eigenfunctions of $K_f$) Let $n \in \mathbb{Z}$. Then
\[
(K_f e_n)(x) = \hat{f}(-n) \cdot e_n(x),
\]
where $\hat{f}$ is Fourier transform of $f$ given by $\hat{f}(\xi) := \int_\mathbb{R} f(x) e^{-2\pi ix\xi} \, dx$.

Proof. \[
(K_f e_n)(x) = \int_0^1 K_f(x, y) \cdot e_n(y) \, dy
\]
\[
= \int_0^1 \sum_{n \in \mathbb{Z}} f(-x + n + y) \cdot e_n(y) \, dy
\]
\[
= \int_{-\infty}^\infty f(-x + y) \cdot e_n(y) \, dy = \int_{-\infty}^\infty f(y) \cdot e_n(x + y) \, dy
\]
\[
= \hat{f}(-n) \cdot e_n(x),
\]
since $e_n(x + y) = e_n(x) \cdot e_n(y)$. \qed
Proposition 2.6. (The trace formula for the additive group \( \mathbb{R} \))

The trace formula for the group \( \mathbb{R} \) and subgroup \( \Gamma = \mathbb{Z} \) is given by

\[
\sum_{n \in \mathbb{Z}} \hat{f}(n) = \sum_{n \in \mathbb{Z}} f(n).
\]

Remark: The trace formula is the well known Poisson summation formula.

Proof. We have shown that \( e_n \) is an eigenfunction of the linear map \( K_f \) with the eigenvalue \( \hat{f}(n) \). The trace of \( K_f \) is given by the sum of the eigenvalues which is just \( \sum_{n \in \mathbb{Z}} \hat{f}(n) \). To find the geometric side we must compute

\[
\int_0^1 K(x, x) \, dx = \int_0^1 \sum_{n \in \mathbb{Z}} f(n) \, dx = \sum_{n \in \mathbb{Z}} f(n).
\]

Note that since the additive group \( \mathbb{Z} \) is abelian there is only one conjugacy class, namely \( \mathbb{Z} \) itself.

\[\square\]

3. The Selberg Trace Formula for \( SL(2, \mathbb{R}) \) (Spectral Side)

Let \( G = SL(2, \mathbb{R}) \) and \( \Gamma = SL(2, \mathbb{Z}) \). We also require the maximal compact subgroup

\[
K = SO(2, \mathbb{R}) = \left\{ \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \ \bigg| \ 0 \leq \theta < 2\pi \right\}.
\]

In this case we define the vector space \( V \) as a space of smooth functions \( \phi \) given by

\[
V := \left\{ \phi : G \to \mathbb{C} \ \big| \ \phi(\gamma g k) = \phi(g), \ \forall \gamma \in \Gamma, \ \forall k \in K, \ \forall g \in G \right\}.
\]

By the Iwasawa decomposition it is known that every \( g \in G/K \) can be uniquely expressed in the form

\[
g = \begin{pmatrix} y^{\frac{1}{2}} & xy^{\frac{1}{2}} \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}
\]

for some \( 0 \leq \theta < 2\pi \).

The action of \( SL(2, \mathbb{R}) \) on the upper half plane \( \mathfrak{h} := \{ x + iy \mid x \in \mathbb{R}, y > 0 \} \) given by

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az + b}{cz + d}, \quad \left( \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}), \ \forall z \in \mathfrak{h} \right),
\]
establishes a one-to-one correspondence between $G/K$ and $\mathfrak{h}$ given by

$\begin{pmatrix} y^{\frac{1}{2}} & xy^{-\frac{1}{2}} \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} i = \begin{pmatrix} y^{\frac{1}{2}} & xy^{-\frac{1}{2}} \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} i = x + iy.$

**Notation involving $z$:** We shall use the notation $z = \begin{pmatrix} y^{\frac{1}{2}} & xy^{-\frac{1}{2}} \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \in G/K$ and also $z = x + iy \in \mathfrak{h}$ interchangeably. The usage will depend on the context of the discussion.

**Definition 3.1. (Petersson inner product on $V$)** For $\phi_1, \phi_2 \in V$ we define the inner product

$$\langle \phi_1, \phi_2 \rangle := \int_{\Gamma \setminus G/K} \phi_1(g) \overline{\phi_2(g)} \, d^\times g$$

where $d^\times g$ denotes the Haar measure on $G$. If $z = \begin{pmatrix} y^{\frac{1}{2}} & xy^{-\frac{1}{2}} \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \in G/K$ then the Petersson inner product can be written in a simple explicit manner as

$$\langle \phi_1, \phi_2 \rangle := \iint_{z \in \Gamma \setminus \mathfrak{h}} \phi_1(z) \overline{\phi_2(z)} \frac{dx dy}{y^2}.$$

We may then define $L^2(\Gamma \setminus G/K)$ as the Hilbert space completion of $V$. This space had been studied by Maass and elements of $L^2(\Gamma \setminus G/K)$ which are eigenfunctions of the Laplacian are termed Maass forms for $SL(2, \mathbb{Z})$.

**Definition 3.2. (The space $L^2_{\text{cusp}}$ of cusp forms)** We let $L^2_{\text{cusp}}$ denote the Hilbert space of Maass forms for $SL(2, \mathbb{Z})$. It can be shown that each Maass form $\phi$ vanishes at the cusp, i.e., $\lim_{y \to \infty} \phi(x + iy) = 0$.

It was not known before Selberg’s work on the trace formula whether infinitely many such Maass forms existed or not. In addition to the Maass forms there are also Eisenstein series which are eigenfunctions of the Laplacian but are not in $L^2$.

**Definition 3.3.** Let $\Gamma_\infty := \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \right\}_{m \in \mathbb{Z}}$. Let $s \in \mathbb{C}$ with $\text{Re}(s) > 1$. We define the Eisenstein series

$$E(z, s) := \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \text{Im}(\gamma z)^s.$$
Theorem 3.4. (Fourier expansion of Eisenstein series) The Eisenstein series $E(z, s)$ has the Fourier expansion

$$E(z, s) = y^s + M(s)y^{1-s} + \frac{2\pi^s \sqrt{y}}{\Gamma(s)\zeta(2s)} \sum_{n \neq 0} \sigma_1(n)|n|^{-\frac{s}{2}} K_{s-\frac{1}{2}}(2\pi|y|) e^{2\pi i n x}$$

where

$$M(s) = \sqrt{\frac{\pi}{\Gamma(s)}} \frac{(s - \frac{1}{2}) \zeta(2s - 1)}{\zeta(2s)}, \quad \sigma_s(n) = \sum_{d|n \atop d > 0} d^s,$$

$$K_s(y) = \frac{1}{2} \int_0^\infty e^{-\frac{y}{2}(u^{s+\frac{1}{2}})} u^s \frac{du}{u}.$$

Proof. See [Gol06]. □

Corollary 3.5. (Growth of the Eisenstein series at the cusp) Let $s \in \mathbb{C}$ with $\text{Re}(s) > 1$. Then $|E(z, s)| \ll y^{\text{Re}(s)}$ and $|E(z, s) - y^s| \ll y^{1-\text{Re}(s)}$ as $y \to \infty$.

Definition 3.6. (The kernel function $K_f$) Let $f : K \backslash G / K \to \mathbb{C}$. For $z, z' \in \text{SL}(2, \mathbb{R})$ we define the kernel function

$$K_f(z, z') := \sum_{\gamma \in \Gamma} f(z^{-1}\gamma z'),$$

provided the sum converges absolutely.

Proposition 3.7. (Properties of $K_f$) For all $z, z' \in \text{SL}(2, \mathbb{R})$ the Kernel function $K_f$ (given in the above definition) satisfies the following properties:

- $K_f(zk, z'k') = K_f(z, z')$, \quad ($\forall \ k, k' \in K$),
- $K_f(\gamma z, \gamma' z) = K_f(z, z')$, \quad ($\forall \ \gamma, \gamma' \in \Gamma$).

Proof. Exercise for the reader. □

To facilitate the computation of the trace of the linear operator $K_f$ Selberg chose a class of especially nice test functions $f$ which we now describe.

Definition 3.8. (The function $\tau$) We define the function $\tau : G \to \mathbb{C}$ by

$$\tau(g) := a^2 + b^2 + c^2 + d^2 - 2, \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$
Proposition 3.9. (Properties of $\tau$) Let $z = \begin{pmatrix} y^\frac{1}{2} & xy^{-\frac{1}{2}} \\ 0 & y^{-\frac{1}{2}} \end{pmatrix}$, and let $z' = \begin{pmatrix} y'^\frac{1}{2} & x'y'^{-\frac{1}{2}} \\ 0 & y'^{-\frac{1}{2}} \end{pmatrix} \in G/K$. Then we have

- $\tau(k) = 0$, $(\forall \ k \in K)$,
- $\tau(kz) = \tau(zk)$, $(\forall \ k \in K)$,
- $\tau(z^{-1}z') = \frac{(x-x')^2 + (y-y')^2}{yy'} = \frac{|z-z'|^2}{yy'}$.

Proof. Exercise for the reader. □

Definition 3.10. (Selberg's kernel function) Let $f : \mathbb{R}^+ \to \mathbb{C}$ be a smooth function satisfying $f(t) \ll \epsilon |t + 2|^{-1-\epsilon}$. Then for $g, g' \in SL(2, \mathbb{R})$, we define

$$K_f(g, g') = \sum_{\gamma \in \Gamma} f(\tau(g^{-1}\gamma g')).$$

Theorem 3.11. (Growth of Selberg's kernel function at the cusps) Let $z = \begin{pmatrix} y^\frac{1}{2} & xy^{-\frac{1}{2}} \\ 0 & y^{-\frac{1}{2}} \end{pmatrix}$, $z' = \begin{pmatrix} y'^\frac{1}{2} & x'y'^{-\frac{1}{2}} \\ 0 & y'^{-\frac{1}{2}} \end{pmatrix} \in G/K$. Then, for every $\epsilon > 0$ and $y, y' > 1$, we have

- $|K_f(z, z')| \leq \sum_{\gamma \in \Gamma} |f(\tau(z^{-1}\gamma z'))| \ll_{\epsilon} y^{-\epsilon}(y')^{1+\epsilon}$
- $\sum_{\gamma \in \Gamma - \Gamma_{\infty}} |f(\tau(z^{-1}\gamma z'))| \ll_{\epsilon} (yy')^{-\epsilon}$.

Proof. We compute

$$|f(\tau(z^{-1}\gamma z'))| = f\left(\frac{(x-x')^2 + (y-y')^2}{yy'}\right) \ll_{\epsilon} \left(\frac{(x-x')^2}{yy'} + \frac{y}{y'} + \frac{y'}{y}\right)^{-1-\epsilon}$$

$$\ll_{\epsilon} \left(\frac{(x-x')^2}{yy'} + \frac{y}{y'}\right)^{-1-\epsilon}.$$
Next, for $\alpha \in \Gamma$, we adopt the notation that $z_\alpha = \alpha z' = x_\alpha + iy_\alpha$. It follows that
\[
K_f(z, z') = \sum_{\alpha \in \Gamma \setminus \Gamma} \sum_{\delta \in \Gamma} f \left( \frac{|z - z_\delta|}{y y_\delta} \right)
= \sum_{\alpha \in \Gamma \setminus \Gamma} \sum_{m \in \mathbb{Z}} f \left( \frac{|z - z_\alpha - m|^2}{y y_\alpha} \right)
= \sum_{\alpha \in \Gamma \setminus \Gamma} \sum_{m \in \mathbb{Z}} f \left( \frac{(x - x_\alpha - m)^2 + (y - y_\alpha)^2}{y y_\alpha} \right)
\leq \epsilon \sum_{\alpha \in \Gamma \setminus \Gamma} \sum_{m \in \mathbb{Z}} \left( \frac{y_\alpha^{1+\epsilon}}{y_\alpha} \right) \left( \frac{(x - x_\alpha - m)^2 + y^2}{y y_\alpha} \right)^{-1-\epsilon}
\leq \epsilon y^{1+\epsilon} \left( \sum_{\alpha \in \Gamma \setminus \Gamma} y_\alpha^{1+\epsilon} \right) \sum_{m \in \mathbb{Z}} \left( \frac{(x - x_\alpha - m)^2 + y^2}{y y_\alpha} \right)^{-1-\epsilon}
\]
Now, we can bound the latter sum above as follows.
\[
\sum_{m \in \mathbb{Z}} \left( \frac{(x - x_\alpha - m)^2 + y^2}{y y_\alpha} \right)^{-1-\epsilon} \ll y^{-2-2\epsilon} + \int_0^\infty (u^2 + y^2)^{-1-\epsilon} \, du \ll y^{-1-2\epsilon}.
\]
It immediately follows from corollary 3.5 and the above calculations that
\[
K_f(z, z') \ll \epsilon \, y^{-\epsilon} E(z', 1 + \epsilon) \ll \epsilon \, y^{-\epsilon} (y')^{1+\epsilon}.
\]
This proves the first assertion.

For the second assertion we calculate the partial sums over $\Gamma \setminus \Gamma$
\[
\sum_{\gamma \in \Gamma \setminus \Gamma} \left| f \left( \tau(z^{-1}y_\gamma') \right) \right| \ll \epsilon \, y^{-\epsilon} \sum_{\Gamma \setminus \Gamma} y_\gamma^{1+\epsilon} \ll (yy')^{-\epsilon}
\]
since $|E(z, 1 + \epsilon) - y^{1+\epsilon}| \ll y^{-\epsilon}$ by corollary 3.5.

**Definition 3.12. (The integral operator $K_f$)** Let $z \in \mathfrak{h}$. Then for $\phi \in L^2(\Gamma \setminus \mathfrak{h})$ we define the integral operator $K_f : L^2(\Gamma \setminus \mathfrak{h}) \to L^2(\Gamma \setminus \mathfrak{h})$ by
\[
(K_f \phi)(z) = \int_{\Gamma \setminus \mathfrak{h}} K_f(z, z') \phi(z') \frac{dx'dy'}{(y')^2}.
\]

**Remarks** Since $K_f$ is symmetric, i.e., $K_f(z, z') = K_f(z', z)$, we see that $K_f$ also satisfies the bound $|K_f(z, z')| \ll \epsilon \, y^{1+\epsilon}(y')^{-\epsilon}$. It follows
that for every $z \in \mathfrak{h}$ that

$$| (K_f \phi)(z) | \leq \left( \int_{\Gamma \setminus b} |K_f(z, z')|^2 \frac{dx'dy'}{(y')^2} \cdot \int_{\Gamma \setminus b} |\phi(z')|^2 \frac{dx'dy'}{(y')^2} \right)^{\frac{1}{2}}$$

$$\ll_{z, \epsilon} \left( \int_{\Gamma \setminus b} (y')^{-2\epsilon} \frac{dx'dy'}{(y')^2} \cdot \int_{\Gamma \setminus b} |\phi(z')|^2 \frac{dx'dy'}{(y')^2} \right)^{\frac{1}{2}}$$

$$= O_{z, \epsilon}(1).$$

This shows that the integral defining the integral operator $K_f$ converges absolutely.

**Definition 3.13. (Hilbert-Schmidt Operator)** Let $X$ be a locally compact space with a positive Borel measure. Assume that $L^2(X)$ is a separable Hilbert space. Let $K : X \times X \to \mathbb{C}$. We define the integral operator

$$(K \phi)(x) := \int_X K(x, y) \phi(y) \, dy, \quad (\phi \in L^2(X)).$$

The integral operator $K$ is said to be of Hilbert-Schmidt type if

$$\int_{X \times X} |K(x, y)|^2 \, dx \, dy < \infty.$$  

**Theorem 3.14. (Hilbert-Schmidt)** A Hilbert-Schmidt operator as in definition 3.13 is a compact operator. If $K$ is self adjoint then the space $L^2(X)$ has an orthonormal basis of eigenfunctions $\phi_1, \phi_2, \ldots$ where $K \phi_i = \lambda_i \phi_i$, $(i = 1, 2, 3, \ldots)$ and $\lambda_i \to 0$ as $i \to \infty$. If the sum of the eigenvalues converges absolutely, then we say $K$ is of trace class and the trace of the operator $K$ is given by

$$\text{Tr}(k) = \sum_{i=1}^{\infty} \lambda_i = \int_X K(x, x) \, dx.$$  

**Proof.** See [Bum97] □

**Warning:** The integral operator $K_f$ given in definition 3.12 is not Hilbert-Schmidt since it can be shown that

$$\int_{\Gamma \setminus b} \int_{\Gamma \setminus b} |K_f(z, z')|^2 \frac{dx'dy'}{(y)^2} \frac{dx'dy'}{(y')^2} = \infty.$$  

We shall now construct a modification of the integral operator $K_f$. The modification will be done in two steps.
Definition 3.15. (First modification of $K_f$) We define
\[
\tilde{K}_f(z, z') := K_f(z, z') - \int_0^1 K_f(z, z' + t) \, dt.
\]

Proposition 3.16. Let $z \in \mathfrak{h}$. Fix a fundamental domain for $\Gamma \backslash \mathfrak{h}$ given by
\[
\mathcal{D} = \left\{ z \in \mathfrak{h} \left| -1/2 \leq \text{Re}(z) \leq 1/2, \ |z| \geq 1 \right. \right\}
\]
If $\phi \in L^2_{\text{cusp}}$ then we have
\[
\left( \tilde{K}_f \phi \right)(z) = \int_{\mathcal{D}} \tilde{K}_f(z, z') \phi(z') \, \frac{dx'dy'}{(y')^2} = (K_f \phi)(z).
\]

Proof. We calculate
\[
\int_{\mathcal{D}} \left( \int_0^1 K_f(z, z' + t) \, dt \right) \phi(z') \, \frac{dx'dy'}{(y')^2} = \int_{\mathcal{D}} K_f(z, z') \left( \int_0^1 \phi(z' - t) \, dt \right) \, \frac{dx'dy'}{(y')^2} = 0.
\]

Definition 3.17. (Second modification of $K_f$) We define
\[
K_f^\#(z, z') := K_f(z, z') - \sum_{n \in \mathbb{Z}} f \left( \frac{|z - z' + n|^2}{yy'} \right).
\]

Theorem 3.18. (Growth of the Modified Kernel at the Cusps) Let $z = x + iy, z' = x' + iy' \in \mathfrak{h}$. Then for every $\epsilon > 0$ and all $y, y' > 1$
\[
\left| \tilde{K}_f(z, z') \right| \ll (yy')^{-\epsilon} + \int_0^\infty |f'(r)| \, dr.
\]

Proof. We calculate
\[
K_f^\#(z, z') - \tilde{K}_f(z, z') = \int_0^1 K_f(z, z' + t) \, dt - \sum_{n \in \mathbb{Z}} f \left( \frac{|z - z' + n|^2}{yy'} \right)
\]
\[
= \int_0^1 K_f^\#(z, z' + t) \, dt + \int_0^1 \left[ \sum_{n \in \mathbb{Z}} f \left( \frac{|z - z' + n + t|^2}{yy'} \right) - \sum_{n \in \mathbb{Z}} f \left( \frac{|z - z' + n|^2}{yy'} \right) \right] \, dt
\]
\[
= \int_0^1 K_f^\#(z, z' + t) \, dt + \int_{-\infty}^\infty f \left( \frac{|z - z' + t|^2}{yy'} \right) \, d(t - [t]).
\]
The first term above is bounded by \((yy')^{-\epsilon}\) by theorem 3.11. For the second term we apply integration by parts.

\[
\int_{-\infty}^{\infty} f \left( \frac{(x-x'+t)^2 + (y-y')^2}{yy'} \right) \, d(t - [t])
\]

\[
= - \int_{-\infty}^{\infty} (t - [t]) \cdot df \left( \frac{(x-x'+t)^2 + (y-y')^2}{yy'} \right)
\]

\[
\ll \int_{-\infty}^{\infty} df \left( \frac{t^2 + (y-y')^2}{yy'} \right) = \int_{-\infty}^{\infty} \left| f' \left( \frac{t^2 + (y-y')^2}{yy'} \right) \right| \, dt
\]

\[
\ll \int_{0}^{\infty} |f'(r)| \, dr.
\]

We have proved that \(|K_f^#(z,z') - \tilde{K}_f(z,z')| \ll (yy')^{-\epsilon}\) + \(\int_{0}^{\infty} |f'(r)| \, dr\).

The theorem follows from theorem 3.11 which says \(|K_f^#(z,z')| \ll (yy')^{-\epsilon}\).

**Theorem 3.19. (The Kernel function \(\tilde{K}_f\) is Hilbert-Schmidt and of Trace Class)** The kernel function \(\tilde{K}_f\) defines an integral operator \(\tilde{K}_f : \mathcal{L}_{\text{cusp}}^2 \to \mathcal{L}_{\text{cusp}}^2\) which is Hilbert-Schmidt i.e., it satisfies

\[
\int_{\mathcal{D}} \int_{\mathcal{D}} \left| \tilde{K}_f(z,z') \right|^2 \, \frac{dxdy}{(y)^2} \, \frac{dx'dy'}{(y')^2} < \infty.
\]

Furthermore, when restricted to the space \(\mathcal{L}_{\text{cusp}}^2\) we have \(K_f = \tilde{K}_f\). For \(f\) real valued the integral operator \(\tilde{K}_f : \mathcal{L}_{\text{cusp}}^2 \to \mathcal{L}_{\text{cusp}}^2\) is self-adjoint and of trace class with trace given by

\[
\text{Tr}(\tilde{K}_f) = \int_{\mathcal{D}} \tilde{K}_f(z,z) \, \frac{dxdy}{(y)^2}.
\]

**Proof.** It follows from theorem 3.18 that

\[
\int_{\mathcal{D}} \int_{\mathcal{D}} \left| \tilde{K}_f(z,z') \right|^2 \, \frac{dxdy}{(y)^2} \, \frac{dx'dy'}{(y')^2} \ll \int_{\mathcal{D}} \int_{\mathcal{D}} \left( (yy')^{-\epsilon} + 1 \right) \, \frac{dxdy}{(y)^2} \, \frac{dx'dy'}{(y')^2} \ll 1.
\]
Proposition 3.20. (The integral operator $K_f$ commutes with the Laplacian) For $z = x + iy \in \mathfrak{h}$ let
\[
\Delta_z := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)
\]
denote the Laplacian on $L^2(\Gamma \setminus \mathfrak{h})$. Then $\Delta_z K_f = K_f \Delta_z$.

Proof. First note (by a brute force verification) that $\Delta_z K_f(z, z') = \Delta_{z'} K_f(z, z')$.

We compute, using integration by parts (Green’s theorem), that
\[
(\Delta_z K_f \phi)(z) = \int_{\Gamma \setminus \mathfrak{h}} \left( \Delta_{z'} K_f(z, z') \right) \cdot \phi(z') \frac{dx'dy'}{(y')^2}
\]
\[
= \int_{\Gamma \setminus \mathfrak{h}} \left( - \left( \frac{\partial^2}{\partial (x')^2} + \frac{\partial^2}{\partial (y')^2} \right) K_f(z, z') \right) \cdot \phi(z') \, dx'dy'
\]
\[
= \int_{\Gamma \setminus \mathfrak{h}} K_f(z, z') \cdot \left( - \left( \frac{\partial^2}{\partial (x')^2} + \frac{\partial^2}{\partial (y')^2} \right) \phi(z') \right) \, dx'dy'
\]
\[
= \int_{\Gamma \setminus \mathfrak{h}} K_f(z, z') \cdot \left( \Delta_{z'} \phi(z') \right) \frac{dx'dy'}{(y')^2}
\]
\[
= (K_f \Delta_z \phi)(z).
\]

□

Let $\lambda \in \mathbb{C}$ and assume the eigenspace $\ker(\Delta - \lambda)$ is one dimensional. Then since $\Delta$ and $K_f$ commute, it would follow that if $\Delta \phi = \lambda \cdot \phi$ for some $\phi \in L^2(\Gamma \setminus \mathfrak{h})$ and $\lambda \in \mathbb{C}$, then for each $f$ there would exist a function $h_f(\lambda) \in \mathbb{C}$ such that
\[
K_f \phi = h_f(\lambda) \cdot \phi.
\]

We will now show the existence and uniqueness of $h_f(\lambda)$ using differential equations.

Proposition 3.21. (Eigenfunctions of $\Delta$ are eigenfunctions of $K_f$) Assume that $\Delta \phi(z) = \lambda \cdot \phi(z)$ for some $\phi \in L^2(\Gamma \setminus \mathfrak{h})$. Then for every $f : \mathbb{R}^+ \to \mathbb{C}$ (satisfying $f(t) \ll (t + 2)^{-1-\epsilon}$) there exists a unique function $h_f : \mathbb{C} \to \mathbb{C}$ such that
\[
(K_f \phi)(z) = h_f(\lambda) \cdot \phi(z).
\]

Proof. We follow [Hej76]. We want to prove $(K_f \phi)(z) = h_f(\lambda) \cdot \phi(z)$ which is equivalent to:
\[
\begin{align*}
(3.22) \quad \int_{\Gamma} \sum_{\gamma \in \Gamma} f \left( \frac{|z - \gamma z'|^2}{y \cdot \Im(\gamma z')} \right) \phi(z') \frac{dx' dy'}{(y')^2} = \int_{\mathfrak{h}} f \left( \frac{|z - z'|^2}{y \cdot y'} \right) \phi(z') \frac{dx' dy'}{(y')^2} \\
= h_f(\lambda) \cdot \phi(z).
\end{align*}
\]

The Cayley transformation

\[c : \mathfrak{h} \to \mathfrak{U} := \{ u \in \mathbb{C} \mid |u| \leq 1 \}\]

which maps the upper half plane to the unit disk (taking \(i\) to 0) is given by

\[c(z') := \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} z' = \frac{z' - i}{z' + i}, \quad (\forall z' \in \mathfrak{h}).\]

It is easy to see that under this transformation

\[\frac{|dz'|}{\Im(z')} = \frac{2|dc|}{1 - |c|^2}.\]

For fixed \(z = x + iy \in \mathfrak{h}\) we define the modified Cayley transformation

\[w(z') := \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} \begin{pmatrix} y^{-\frac{1}{2}} & -xy^{-\frac{1}{2}} \\ 0 & y^{\frac{1}{2}} \end{pmatrix} z' = 1 - \frac{2y}{i(x - x') + y + y'}\]

which maps the upper half plane \(\mathfrak{h}\) to the unit disk \(\mathfrak{U}\) (taking \(z\) to 0).

Furthermore

\[\frac{|z' - z|^2}{yy'} = \frac{4|w|^2}{1 - |w|^2}.\]

Let us now make the change of coordinates \(z' \to w(z')\) in the integral 3.22 and define \(\phi(z') := \Psi(w)\). It follows that

\[\int_{\mathfrak{U}} f \left( \frac{4|w|^2}{1 - |w|^2} \right) \Psi(w) \, d_A(w) = h_f(\lambda) \cdot \Psi(0)\]

where \(d_A(w)\) denotes the area differential on \(\mathfrak{U}\). We must prove that \(h_f(\lambda)\) depends solely on \(f\) and \(\lambda\).

If we convert to polar coordinates on \(\mathfrak{U}\) the above can be rewritten as

\[\int_{r=0}^{1} f \left( \frac{4r^2}{1 - r^2} \right) \left( \int_{0}^{2\pi} \Psi(re^{i\theta}) \, d\theta \right) \frac{r \, dr}{(1 - r^2)^2} = h_f(\lambda) \cdot \Psi(0).\]
This can be rewritten as
\[
\int_{r=0}^{1} f \left( \frac{4r^2}{1-r^2} \right) \Psi^*(r) \frac{r dr}{(1-r^2)^2} = h_f(\lambda) \cdot \Psi^*(0)
\]
where
\[
\Psi^*(w) := \int_{0}^{2\pi} \Psi (w \cdot e^{i\theta}) \, d\theta.
\]
Note that
\[
\int_{0}^{2\pi} \Psi (r e^{i\theta}) \, d\theta = \int_{0}^{2\pi} \Psi (w \cdot e^{i\theta}) \, d\theta
\]
for all \(w \in \mathcal{W}\) satisfying \(|w| = r\) from which it follows that \(\Psi^*(w)\) is a radially symmetric function which depends only on \(r = |w|\) and, hence, satisfies \(\frac{\partial}{\partial \theta} \Psi^*(w) = 0\).

Furthermore, \(\Psi^*\) is also a radially symmetric eigenfunction of the Laplacian and, therefore, satisfies the differential equation

\[(3.23) \quad \frac{d^2 \Psi^*}{dr^2} + \frac{1}{r} \frac{d\Psi^*}{dr} + \frac{4\lambda}{(1-r^2)^2} \Psi^* = 0.\]

Let \(\lambda \in \mathbb{C}\) be fixed. A possible solution (up to a constant factor) to (3.23) must be of the form
\[
\Psi^*(r) = r^c (1 + a_1 r + a_2 r^2 + \cdots).
\]
We calculate
\[
\frac{1}{r} \Psi^*'(r) = cr^{c-2} + a_1(c+1)r^{c-1} + \cdots
\]
\[
\Psi^*''(r) = c(c-1)r^{c-2} + a_1(c+1)cr^{c-1} + \cdots
\]
Now substitute into the differential equation and we get
\[
0 = \Psi^*''(r) + \frac{1}{r} \Psi^*'(r) + \frac{4\lambda}{(1-r^2)^2} \Psi^*(r) = c^2r^{c-2} + a_1(c+1)^2r^{c-1} + \cdots
\]
It follows that \(c^2 = 0\) and higher order terms are defined recursively from the differential equation. There will be a second solution but it will be of the form \(\log r (1 + b_1 r + \cdots)\) which is singular at the origin. Since a regular solution \(\Psi^*\) exists and is uniquely determined it follows that \(h_f(\lambda)\) exists and is also uniquely determined. \(\Box\)

Remark: The above proof shows that a solution to the equations
\[
\Delta \phi = \lambda \phi, \quad (K_f \phi)(z) = h_f(\lambda)\phi(z)
\]
can be made radially symmetric around \( z \) in the non euclidean sense.
Then \( \Delta \phi = \lambda \phi \) becomes an ordinary differential equation with a 1-
dimensional space of solutions regular near \( z \). This yields the constant
\( h_f(\lambda) \) as in the case of a one-dimensional eigenspace \( \ker(\Delta - \lambda) \).

**Definition 3.24. (The Abel Transform)** Let \( f : \mathbb{R}^+ \to \mathbb{C} \) be a
smooth function such that \( f \) and \( f' \) are integrable on \( \mathbb{R}^+ \). For \( w \geq 0 \)
we define the Abel transform
\[
Q(w) := \int_{-\infty}^{\infty} f(w + \xi^2) \, d\xi = \int_{w}^{\infty} \frac{f(t)}{\sqrt{t - w}} \, dt,
\]
provided the integral converges absolutely.

**Proposition 3.25. (Inverse Abel Transform)** Let \( Q(w) \) be the Abel transform of \( f \). Then
\[
f(t) = \frac{-1}{\pi} \int_{-\infty}^{\infty} Q'(t + w^2) \, dw.
\]

**Proof.** We have
\[
\frac{-1}{\pi} \int_{-\infty}^{\infty} Q'(t + w^2) \, dw = \frac{-1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f'(t + w^2 + \xi^2) \, dw \, d\xi
\]
\[
= \frac{-1}{\pi} \int_{0}^{2\pi} \int_{0}^{\infty} f'(t + r^2) \, rdrd\theta
\]
\[
= -\int_{0}^{\infty} f'(t + u) \, du = f(t).
\]

**Definition 3.26. (Selberg Transform)\( h_f \)** Assume \( \Delta \phi = (\frac{1}{4} + r^2) \cdot \phi \)
for some \( \phi \in L^2(\mathbb{SL}(2,\mathbb{Z})\backslash \mathfrak{h}). \) Let \( f : \mathbb{R}^+ \to \mathbb{C} \) be a smooth function
satisfying \( f(t) \ll_{\epsilon} (t + 2)^{-1-\epsilon} \) for \( t \geq 0 \). Then the Selberg transform is
defined as the unique function \( h_f : \mathbb{C} \to \mathbb{C} \) for which \( K_f \phi = h_f(r) \cdot \phi \)
as in proposition 3.21.

**Proposition 3.27. (Evaluation of the Selberg transform)** Let \( Q \)
be the Abel transform of \( f \) as in definition 3.24. Then we have
\[
h_f(r) = \int_{-\infty}^{\infty} Q(e^u - 2 + e^{-u}) \, e^{iru} \, du.
\]
Proof. Choose \( \phi(z) = y^{\frac{1}{2} + ir} \) in proposition 3.21. Now

\[
\Delta y^{\frac{1}{2} + ir} = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) y^{\frac{1}{2} + ir} = \left( \frac{1}{4} + r^2 \right) \cdot y^{\frac{1}{2} + ir}.
\]

Although \( y^{\frac{1}{2} + ir} \) is not in \( \mathcal{L}^2(SL(2,\mathbb{Z}) \backslash \mathfrak{h}) \), the proof of proposition 3.21 still goes through. It follows that

\[
h_f(r) \cdot y^{\frac{1}{2} + ir} = \int_0^\infty \int_{-\infty}^{\infty} \left( \frac{(x-x')^2 + (y-y')^2}{yy'} \right) (y')^{\frac{1}{2} + ir} \frac{dx' dy'}{y'^2}
\]

Make the transformation

\[
\xi = \frac{x-x'}{\sqrt{yy'}}, \quad d\xi = \frac{dx'}{\sqrt{yy'}}.
\]

Then

\[
h_f(r) \cdot y^{\frac{1}{2} + ir} = \int_0^\infty \int_{-\infty}^{\infty} f \left( \frac{(y-y')^2}{yy'} + \xi^2 \right) (y')^{\frac{1}{2} + ir} \sqrt{yy'} \, d\xi \frac{dy'}{y'^2}
\]

\[
= \int_0^\infty Q \left( \frac{(y-y')^2}{yy'} \right) (y')^{\frac{1}{2} + ir} \sqrt{yy'} \, dy' \frac{dy'}{y'^2}
\]

\[
= y^{\frac{1}{2}} \int_0^\infty Q \left( \frac{y}{y'} - 2 + \frac{y'}{y} \right) (y')^{ir} \, dy' \frac{dy'}{y'}
\]

\[
= y^{\frac{1}{2} + ir} \int_0^\infty Q \left( \frac{1}{y'} - 2 + y' \right) (y')^{ir} \, dy' \frac{dy'}{y'}
\]

The result follows upon making the transformation \( y' = e^u \). \( \square \)

Definition 3.28. (The Selberg transform Functions) Start with a smooth function \( f : \mathbb{R}^+ \to \mathbb{C} \) satisfying \( f(t) \ll \epsilon (t + 2)^{-1-\epsilon} \). Selberg defines the following functions:

\[
Q(w) = \int_{-\infty}^\infty f(w + \xi^2) \, d\xi, \quad \text{(Abel transform of } F),
\]

\[
g(u) = Q(e^u - 2 + e^{-u}), \quad (u \in \mathbb{R}),
\]

\[
h_f(r) := h(r) = \int_{-\infty}^\infty g(u)e^{iru} \, du, \quad \text{(Fourier transform of } g).}
\]
Explicit Example of the Selberg Transform

\[ f(t) = (t + 2)^{-\frac{3}{2}}, \quad Q(w) = \left(1 + \frac{w}{2}\right)^{-1}; \]

\[ g(u) = \text{sech}(u), \quad h(r) = \pi \cdot \text{sech}\left(\frac{\pi r}{2}\right). \]

Since \( \text{sech}(u) = \frac{2}{e^{u} + e^{-u}} \) we see that \( g, h \) have exponential decay.

**Proposition 3.29. Growth of the Selberg Transform Functions**

Let \( \epsilon > 0 \). Then

\[ f(t) \ll \epsilon (t + 2)^{-1-\epsilon}, \quad Q(w) \ll \epsilon w^{-\frac{1}{2}-\epsilon}, \]

\[ g(u) \ll \epsilon e^{-(\frac{1}{2}-\epsilon)|u|}, \quad h(r) \ll \epsilon e^{-(\frac{1}{2}-\epsilon)\pi r}. \]

**Proof.** Exercise for the reader. □

For \( z \in \mathfrak{h} \) let \( \phi_0(z) := \sqrt{\frac{n}{\pi}} \) be the constant function of Petersson norm one, and let \( \phi_1, \phi_2, \ldots \) denote an orthonormal basis of Maass forms consisting of eigenfunctions of the Laplacian where \( \Delta \phi_i = \left(\frac{1}{4} + r_i^2\right) \phi_i \) for \( i = 1, 2 \ldots \)

**Theorem 3.30. (Selberg Spectral Decomposition)** Let \( \phi_0 \) be the constant function of Petersson norm one, and let \( \phi_1, \phi_2, \phi_3, \ldots \) denote an orthonormal basis of Maass forms consisting of eigenfunctions of the Laplacian. Let \( F \in L^2\left(SL(2, \mathbb{Z}) \setminus \mathfrak{h}\right) \). Then for \( z \in \mathfrak{h} \)

\[ F(z) = \sum_{i=0}^{\infty} \langle F, \phi_i \rangle \phi_i(z) + \frac{1}{4\pi} \int_{-\infty}^{\infty} \langle F, E(\ast, 1/2 + ir) \rangle E(z, 1/2 + ir) \, dr. \]

**Proof.** See [Gol06]. □

**Theorem 3.31. (Spectral Decomposition of \( K_f \))** The integral operator \( K_f : L^2\left(SL(2, \mathbb{Z}) \setminus \mathfrak{h}\right) \rightarrow L^2\left(SL(2, \mathbb{Z}) \setminus \mathfrak{h}\right) \) defined in 3.12 has the following spectral decomposition.

\[ K_f(z, z') = \]

\[ = \sum_{i=0}^{\infty} h_f(r_i) \phi_i(z) \overline{\phi_i(z')} + \frac{1}{4\pi} \int_{-\infty}^{\infty} h_f(r) E(z, 1/2 + ir) \overline{E(z', 1/2 + ir)} \, dr \]

for \( z, z' \in \mathfrak{h} \).

Heuristically we expect the trace of \( K_f \) to be given by

\[ \text{Tr}(K_f) = \int_{SL(2, \mathbb{Z}) \setminus \mathfrak{h}} K(z, z) \frac{dxdy}{y^2}. \]
The above integral does not converge! If we restrict the trace (which we denote by $\text{Tr}_{\text{cusp}}$) to the space of cusp forms, $L^2_{\text{cusp}}$ with orthonormal basis $\phi_1, \phi_2, \phi_3, \ldots$ as described above then

$$\text{Tr}_{\text{cusp}}(K_f) = \int_{SL(2,\mathbb{Z})\backslash \mathcal{H}} \sum_{i=1}^{\infty} h_f(r_i) \overline{\phi_i(z)} \phi_i(z) \frac{dxdy}{y^2}$$

$$= \sum_{i=1}^{\infty} h_f(r_i).$$

Note that the above sum over eigenvalues converges absolutely because of the exponential decay properties of the Selberg transform $h_f$. The above expression will turn out to be the spectral side of the Selberg trace formula for $SL(2,\mathbb{R})$.

References

