COMBINATORIAL NUMBER THEORY

PROBLEM SET # 2 (due March 23, 2006)

Problem 1: Determine which of the following numbers are quadratic integers in some field. Hint: Use theorem 238 in Hardy and Wright.

\[9 + \sqrt{7}, \quad 10, \quad \frac{1 + \sqrt{3}}{2}, \quad \frac{9 + 15\sqrt{-7}}{2}, \quad \frac{10 + \sqrt{-108}}{4}, \quad \frac{3 + 2\sqrt{6}}{1 - \sqrt{6}}.\]

Problem 2: Prove that if the product of two integers in a quadratic field is a unit, then each is a unit.

Problem 3: Factor the number \(33 + 11\sqrt{-7}\) into primes in \(\mathbb{Q}(\sqrt{-7})\).

Problem 4: Show that there is no integer in \(\mathbb{Q}(\sqrt{7})\) of norm 3. Show that in spite of this, 3 is not a prime in \(\mathbb{Q}(\sqrt{7})\).

Problem 5: Show that 21 has two essentially different factorizations into primes in \(\mathbb{Q}(\sqrt{-5})\).

Problem 6: Find the greatest common divisor of \(-25 + 47\sqrt{-1}\) and \(34 + 32\sqrt{-1}\) in the Gaussian integers.

Problem 7: Let \(\omega(n)\) denote the number of distinct primes dividing \(n\). Show that \(\omega\) is an additive function, i.e., show that \(\omega(m \cdot n) = \omega(m) + \omega(n)\) if \((m, n) = 1\).

Problem 8: Let \(\sigma(n) = \sum_{d|n} d\) be the sum of the divisors of \(n\). Show that if \(\sigma(n)\) is odd, then \(n\) is a square or the double of a square.

Problem 9: Let \(\mu(n)\) be the Moebius function. Prove that: \(\sum_{d^2|n} \mu(d) = |\mu(n)|\).

Problem 10: Let \(d_2(n) = \sum_{d|n} d^2\). Find the average order of \(d(n)\) for odd integers \(n\). Hint: Evaluate \(\lim_{N \to \infty} \frac{1}{N^{3/2}} \sum_{n \leq N}^{\text{n odd}} d_2(n)\).