Problem 1: Show that for $\Re(s) > 1$, we have

$$\log(\zeta(s)) = \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}},$$

where the first sum goes over primes.

Problem 2: Show that

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{m=1}^{\infty} \frac{\lambda(m)}{m^s}$$

where $\lambda(m) = (-1)^r$ and $r$ denotes the number of prime factors of $m$, with a factor of multiplicity $\ell$ counted $\ell$ times.

Problem 3: (Uniqueness of Dirichlet series) For $n = 1, 2, 3, \ldots$, let $a_n$, $b_n$ be complex numbers with absolute values at most one. Assume that

$$\sum_{n=1}^{\infty} a_n \frac{1}{n^s} = \sum_{n=1}^{\infty} b_n \frac{1}{n^s}$$

for all complex values of $s$ with $\Re(s) > 1$. Prove that we must have $a_n = b_n$ for all $n = 1, 2, 3, \ldots$

Problem 4: Show that

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx.$$ 

For which values of $s$ is the above representation valid? Hint: Use the expansion for $(1 - e^{-x})^{-1}$.

Problem 5: Calculate the mean value

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\zeta(\sigma + it)|^2 dt$$

provided $\sigma > 1$. 

Problem 6: Let \( a_1, a_2, a_3 \ldots \) be a sequence of complex numbers with the property that

\[
S(N) = \sum_{n \leq N} |a_n| \leq \sqrt{N},
\]

for every positive integer \( N \). Show that the Dirichlet series

\[
\sum_{n=1}^{\infty} \frac{a_n}{n^s}
\]

converges absolutely for \( \Re(s) > \frac{1}{2} \).

Problem 7: Let \( f \) be in the Schwartz space with Fourier transform defined by

\[
\hat{f}(x) = \int_{-\infty}^{\infty} f(t) e^{-2\pi itx} \, dt.
\]

Prove that

\[
\hat{f}'(x) = 2\pi ix \hat{f}(x).
\]

Problem 8: Let \( f, g \) be any two functions in the Schwartz space. Define the convolution \( f \circ g \) by the formula

\[
f \circ g(x) = \int_{-\infty}^{\infty} f(x-t)g(t) \, dt.
\]

Show that

\[
\hat{f} \circ \hat{g} = \hat{f} \cdot \hat{g}.
\]

Problem 9: Fix \( u > 0 \). The Fourier transform of \( \frac{u}{\pi(x^2 + u^2)} \) is \( e^{-2\pi|x|u} \). Plug this into the Poisson summation formula to deduce that

\[
\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + u^2} = \frac{\pi}{u} \sum_{n=-\infty}^{\infty} e^{-2\pi|n|u} = \frac{\pi}{u} \left( \frac{1 + e^{-2\pi u}}{1 - e^{-2\pi u}} \right)
\]

By letting \( u \to 0 \), conclude that \( \zeta(2) = \frac{\pi^2}{6} \).