Notes on Pollard’s $p - 1$ Attack on RSA

Let $n = pq$ be an RSA key where $p, q$ are primes. Pollard’s attack is a method to find the factorization of $n$. The method will work (with a choice of a large integer $B$) provided:

- $p - 1$ divides $B!$
- $q - 1$ has a prime factor $> B$.

**Step 1:** Let $a = 2$ and compute $b = 2^B! \pmod{n}$.

**Step 2** Calculate $\text{GCD}(b - 1, n)$. This should give the prime factor $p$.

**Step 3** If $p$ is not found in step 2 repeat steps 1,2 with $a = 3$. If $a = 3$ fails keep trying other values for $a$.

**Why does this work?** Pollard’s attack works because of Fermat’s Little Theorem.

In fact, since $(p-1)|B!$ this implies that $B! = k \cdot (p-1)$ for some integer $k$. It follows that

$$b \equiv 2^B! \pmod{p} \equiv 2^{k \cdot (p-1)} \equiv 1 \pmod{p}.$$ 

Hence $p$ divides $b - 1$.

The method will work as long as $b > 1$, which will certainly be the case if 2 is a primitive root $(\pmod{p})$. If $a = 2$ doesn’t work then try $a = 3$, etc, until a good choice of $a$ is obtained. Eventually the factorization of $n$ will be found.

**EXAMPLE:** $n = 31 \cdot 83 = 2573$ and $B = 5$.

**Step 1:** $B! = 5! = 120 = 64 + 32 + 16 + 8 = 2^6 + 2^5 + 2^4 + 2^3$.

$$2^{2^1} \pmod{2573} \equiv 4$$

$$2^{2^2} \pmod{2573} \equiv 16$$

$$2^{2^3} \pmod{2573} \equiv 256$$

$$2^{2^4} \pmod{2573} \equiv 1211$$

$$2^{2^5} \pmod{2573} \equiv 2484$$

$$2^{2^6} \pmod{2573} \equiv 202$$

$$b \equiv 2^{120} \pmod{2573} \equiv 2^{2^6} \cdot 2^{2^5} \cdot 2^{2^4} \cdot 2^{2^3} \pmod{2573} \equiv 202 \cdot 2484 \cdot 1211 \cdot 256 \pmod{2573} \equiv 280$$

**Step 2:** $\text{GCD}(279, 2573) = 31$. 

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