PARAMETRIZATION OF MODULAR ELLIPTIC CURVES BY POINCARÉ SERIES

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§1. Introduction

Let $E : y^2 = 4x^3 - g_2x - g_3$ be an elliptic curve in Weierstrass normal form which is defined over $\mathbb{Q}$. Fix a base point $p \in E(\mathbb{C})$, and consider the group

$$\Gamma_o(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z},\, ad - bc = 1,\, c \equiv 0 \pmod{N} \right\}$$

which acts on $\mathfrak{h}^* = \mathfrak{h} \cup \mathbb{Q} \cup \{i\infty\}$ where $\mathfrak{h}$ is the upper half plane. Set $X_o(N) = \Gamma_o(N)\backslash\mathfrak{h}^*$. The curve E is said to be modular for $\Gamma_o(N)$ if there exists a non-constant holomorphic map

$$\phi_p : X_o(N) \to E$$

normalized so that $\phi_p(i\infty) = p$ and, in addition, the pullback of the canonical differential $\frac{dx}{y}$ on E, with respect to the local uniformizing parameter $q = e^{2\pi iz}$ at the cusp $i\infty$, is $2\pi i\kappa f(z)dz$ where

$$f(z) = \sum_{n=1}^{\infty} c(n) e^{2\pi inz} \tag{1.1}$$

is a Hecke newform of weight two normalized so that $c(1) = 1$. Here κ is a constant (Manin's constant, see **[M]**) which depends on E. Mazur and Swinnerton-Dyer introduced the concept of strong modular elliptic curves for $\Gamma_o(N)$ in **[Ma-Sw]**. They defined an elliptic curve E' to be strong if it is a modular elliptic curve for $\Gamma_o(N)$, $\mathbb{Q}$-isogenous to E, and the degree of the mapping $X_o(N) \to E'$ is minimal. Manin **[M]** conjectured that $\kappa = \pm 1$ for strong modular elliptic curves. It is clear that if we consider the isogenous curve $[m]E$ given by multiplication by an integer m, then κ also changes by a factor of m.

Let

$$\phi_p^{-1}(O_E) = (\zeta_1, \zeta_2, \ldots, \zeta_d),$$

and let $e(\zeta_i)$ denote the ramification index of ζ_i with respect to the map ϕ_p. The Hurwitz genus formula says that

$$2g - 2 = \sum_{\zeta \in X_o(N)} (e(\zeta) - 1) \tag{1.2}$$

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where g is the genus of $X_o(N)$. If the ζ_i are all distinct, then clearly $e(\zeta_i) = 1$ for $1 \le i \le d$, and d is the degree of the map ϕ_p. In this case we say that ϕ_p is unramified at O_E.

The map $\phi_p : X_o(N) \to E$ can, by definition, be explicitly written in the form

$$\phi_p(z) = \big(\alpha(z), \beta(z)\big)$$

where α, β are meromorphic modular forms of weight zero for $\Gamma_o(N)$ that satisfy the Weierstrass equation

$$\beta(z)^2 = 4\alpha(z)^3 - g_2\alpha(z) - g_3 \tag{1.3}$$

for E. Let us define

$$F(z) = -2\pi i \int_z^{i\infty} f(\tau)\, d\tau, \tag{1.4}$$

so that

$$\kappa F(\sigma z) = \kappa F(z) \pmod{\Lambda}$$

for $\sigma \in \Gamma_o(N)$, and where Λ is the period lattice for the elliptic curve E. Furthermore, the pullback $\phi_p^*\Big(\frac{dx}{y}\Big)$ of the canonical differential on E is

$$\phi_p^*\Big(\frac{dx}{y}\Big) = \frac{d\alpha(z)}{\beta(z)} = \kappa dF(z), \tag{1.5}$$

where d denotes the exterior derivative.

The curve E is parametrized by $\wp(z), \wp'(z)$, the Weierstrass elliptic functions associated to the period lattice Λ. It follows (see **[Go]**) that we may take

$$\alpha(z) = \wp(\kappa F(z) + u)$$

$$\beta(z) = \wp'(\kappa F(z) + u)$$

where our fixed base point $p \in E(\mathbb{C})$ is taken to be $p = (\wp(u), \wp'(u))$ for some $u \in \mathbb{C}/\Lambda$. Consequently, $\phi_p^{-1}(O_E)$ induces a map $\tilde{\phi}_p : X_o(N) \to \mathbb{C}/\Lambda$ given by

$$\tilde{\phi}_p(z) \equiv \kappa F(z) + u \pmod{\Lambda}.$$

Therefore, $\phi_p^{-1}(O_E)$ consists of precisely those points $\zeta \in X_o(N)$ which satisfy the congruence

$$\kappa F(\zeta) \equiv -u \pmod{\Lambda},$$

and these points must be algebraic if p is algebraic.

Under the assumption that E is modular, there must exist meromorphic modular forms $\alpha(z), \beta(z)$ of weight zero for $\Gamma_o(N)$. One would, therefore, like to explicitly construct $\alpha(z), \beta(z)$ in a manner similar to the construction of elliptic functions. This is the principle aim of this paper. Now, the only modular forms one knows how to construct are Poincaré series and theta functions. It is, therefore, natural to try to parametrize a modular elliptic curve by Poincaré series; this goal is materialized in theorem(1.1). Our approach is based on the Petersson- Neunhöffer theory of Poincaré series which we now briefly describe.

In sections 2 and 3 we construct for each positive integer n and complex variables $z, \zeta \in \mathfrak{h}^*$ a Poincaré series $P_n(z,\zeta)$ which is automorphic and harmonic in z except when $z = \zeta$ where it has a pole of order n. The case $\zeta \in \mathfrak{h}$ (inner point) is considered in §2 where it is shown that

$$P_n(z,\zeta) = \frac{1}{(z-\zeta)^n} \ + \ O(1)$$

for $z \to \zeta$.

If ζ is a cusp, then the construction of $P_n(z,\zeta)$ is different and is considered in §3. In the case $\zeta = i\infty$ we construct a Poincaré series $P_n(z,\zeta)$ which is automorphic and harmonic in z except when $z = \zeta$ where it satisfies

$$P_n(z,\zeta) = e^{-2\pi i n z} \ + \ O(1).$$

On the other hand, if ζ is a finite cusp then there exists an element $\sigma \in SL(2,\mathbb{R})$ for which $\sigma(i\infty) = \zeta$ and $\sigma^{-1}\Gamma_\zeta \sigma = \Gamma_{i\infty}$ where Γ_ζ denotes the stability group of ζ. The Poincaré series is then defined by

$$P_n(z,\zeta) = P_n(\sigma z, i\infty).$$

Fourier expansions of $P_n(z,\zeta)$ for the cases $Im(z) > Im(\zeta)$ and $Im(z) < Im(\zeta)$ are developed in §5.

Our main theorem can now be formulated:

Theorem 1.1 *Let $p \in E(\mathbb{C})$. Assume that the map $\phi_p : X_o(N) \to E$ is unramified at O_E and the cusps of $\Gamma_o(N)$ are not contained in $\phi_p^{-1}(O_E) = (\zeta_1, \ldots, \zeta_d)$. Then the map $\phi_p(z) = \big(\alpha(z), \beta(z)\big)$, $\frac{d\alpha(z)}{\beta(z)} = \kappa dF(z)$, with $F(z)$ given by (1.4), $f(z)$ given by (1.1), can be explicitly written*

$$\alpha(z) = \sum_{j=1}^{d} \frac{\kappa^{-2}}{f(\zeta_j)^2}\left[P_2(z,\zeta_j) - \frac{f'(\zeta_j)}{f(\zeta_j)}P_1(z,\zeta_j)\right] \ + \ A$$
$$\beta(z) = \sum_{j=1}^{d} \frac{-\kappa^{-3}}{f(\zeta_j)^3}\left[2P_3(z,\zeta_j) - 3\frac{f'(\zeta_j)}{f(\zeta_j)}P_2(z,\zeta_j) + \left(\frac{3f'(\zeta_j)^2}{f(\zeta_j)^2} - \frac{f''(\zeta_j)}{f(\zeta_j)}\right)P_1(z,\zeta_j)\right] \ + \ B.$$

where A, B are constants and the Poincaré series $P_n(z,\zeta)$ are given in definition (2.4).

The constants A, B can be explicitly computed in terms of the constant terms of the Poincaré series. A similar but more complicated version of theorem (1.1) can also be obtained if the map ϕ_p is ramified at O_E. If the cusps of $\Gamma_o(N)$ are contained in $\phi_p^{-1}(O_E)$ then each cusp ζ will contribute

$$\frac{1}{\kappa^2}\left[P_2(z,\zeta) - c(2)P_1(z,\zeta)\right]$$

to $\alpha(z)$ and

$$\frac{1}{\kappa^3}\Big[-2P_3(z,\zeta)+3c(2)P_2(z,\zeta)+\big(2c(3)-3c(2)^2\big)P_1(z,\zeta)\Big]$$

to $\beta(z)$ where $c(n)$ denotes the n^{th} Fourier coefficient of $f(z)$ in the expansion about ζ and $P_n(z,\zeta)$ is given in definition (3.1).

Since $P_n(z,\zeta)$ is harmonic in z for $z \neq \zeta$, its ℓ^{th} Fourier coefficient must take the form

$$\int_0^1 P_n(z,\zeta)e^{-2\pi i\ell x}\,dx = \rho_n(\ell,\zeta)\,e^{-2\pi\ell y} \;+\; \rho_n^*(\ell,\zeta)e^{2\pi\ell y}.$$

A method for computing $\rho_n(\ell,\zeta)$ and $\rho_n^*(\ell,\zeta)$ is sketched in §5. The following corollary is proved in §4.

Corollary 1.2 *Let $E : y^2 = 4x^3 - g_2x - g_3$ be modular for $\Gamma_o(N)$. Let $p = \big(\wp(u),\wp'(u)\big)$ with $p \in E(\mathbb{C})$ chosen so that ϕ_p is unramified at O_E and the cusps of $\Gamma_o(N)$ are not contained in $\phi_p^{-1}(O_E)$. Set $\phi_p^{-1}(O_E) = (\zeta_1,\dots,\zeta_d)$. Then*

$$g_2 = \frac{2}{\kappa}c(2)\wp(u) + 12\wp(u)^2 - \frac{16}{\kappa^5}\sum_{j=1}^{d}\frac{1}{f(\zeta_j)^2}\left[\rho_2(2,\zeta_j) - \frac{f'(\zeta_j)}{f(\zeta_j)}\rho_1(2,\zeta_j)\right]$$

where $c(2)$ is the second Fourier coefficient of $f(z)$ at the parabolic cusp $i\infty$ as in (1.1).

Remarks: A similar but more complicated formula can also be given for g_3 and any of the other well known invariants of an elliptic curve such as the discriminant, the $j-$invariant, etc. This provides a new connection between the arithmetic theory of elliptic curves and the spectral theory of congruence subgroups of $SL(2,\mathbb{Z})$. Since the Fourier coefficient $\rho_n(\ell,\zeta)$ can be expressed as an infinite sum of Kloosterman sums, these formulae generalize the well known theorems of Rademacher and Zuckerman **[R], [R-Z], [Z]**, on the asymptotics of Fourier coefficients of modular forms of non-positive integral weight. Our approach, however, has followed Petersson **[P1], [P2], [P3], [P4]**, who was the first to give a constructive theory of meromorphic Poincaré series. The convergence problems that arise in weight zero were first overcome by Neunhöffer **[N]** and somewhat later by Niebur **[Ni]**, and the Poincaré series that occur in theorem (1.4) are of Neunhöffer type. Alternatively, one may also develop this theory by the methods of Green's functions as in Hejhal (**[H]** Appendix D). See also Fay **[F]**.

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§2. Poincaré series for inner points

For any $\zeta \in \mathfrak{h}$, we map the upper half-plane $\mathfrak{h} \to U$, the unit disc by the Cayley transform

$$z \mapsto\; w = \frac{z-\zeta}{z-\bar{\zeta}}.$$

Set

$$T_\zeta = \begin{pmatrix} 1 & -\zeta \\ 1 & -\bar{\zeta} \end{pmatrix}$$

and define $w = w(z) = T_\zeta z$.

Let $w \in U$ be written in polar coordinates $w = \rho e^{i\phi}$. Then since $z = {T_\zeta}^{-1} w$, it follows that every automorphic function $f(z)$ has a Laurent expansion in U of the form

$$f(z) = \sum_{m \in \mathbb{Z}} b_m(\rho) e^{im\phi},$$

where $b_m(\rho)$ depend at most on ρ. Clearly, if f is meromorphic at ζ with a pole of order u then the above expansion takes the form

$$f(z) = \sum_{m=-u}^{\infty} b_m \left(\frac{z-\zeta}{z-\bar{\zeta}} \right)^m.$$

We shall also utilize the Gaussian hypergeometric function $F(a, b, c; z)$ which is defined for $|z| < 1$, and c unequal to a negative integer by the convergent series

$$F(a, b, c; z) = 1 + \frac{ab}{c \cdot 1} z + \frac{a(a+1)b(b+1)}{c(c+1) \cdot 1 \cdot 2} z^2 + \cdots$$

With these preliminaries in place, we now define the Neunhöffer Poincaré series for the inner point $\zeta \in \mathfrak{h}$.

Definition 2.1 *Let $n \in \mathbb{Z}$, $\zeta \in \mathfrak{h}$, $s \in \mathbb{C}$, and $\mathrm{Re}(s) > 1$. Then we define for $z \in \mathfrak{h}$ and $z \neq \zeta$, the Neunhöffer Poincaré series $P_n(z, \zeta, s)$ by the convergent series*

$$P_n(z, \zeta, s) = \sum_{M \in T_\zeta \Gamma_o(N) {T_\zeta}^{-1}} (Mw)^n \left(1 - |Mw|^2\right)^s F\left(s+n, s, 2s; 1 - |Mw|^2\right)$$

if $n \geq 0$ and

$$P_n(z, \zeta, s) = \sum_{M \in T_\zeta \Gamma_o(N) {T_\zeta}^{-1}} (\overline{Mw})^{-n} \left(1 - |Mw|^2\right)^s F\left(s-n, s, 2s; 1 - |Mw|^2\right)$$

if $n \leq 0$.

The absolute convergence of the above series for $Re(s) > 1$ is a consequence of the following two identities. For $M = T_\zeta \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) {T_\zeta}^{-1}$ the identities are

$$1 - |M(w)|^2 = \frac{\left(\frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d}\right)(\bar{\zeta}-\zeta)}{\left(\frac{az+b}{cz+d} - \bar{\zeta}\right)\left(\frac{az+b}{cz+d} - \zeta\right)}$$

and

$$M(w) = \frac{\frac{az+b}{cz+d} - \zeta}{\frac{az+b}{cz+d} - \bar{\zeta}}.$$

The absolute convergence of $P_n(z, \zeta, s)$ for $Re(s) > 1$ now easily follows from the absolute convergence of the Eisenstein series

$$E(z, s) = \sum_{\sigma \in \Gamma_\infty \backslash \Gamma_o(N)} \Big(Im(\sigma z)\Big)^s.$$

Proposition 2.2 (Neunhöffer) *For $z \neq \zeta$, $P_n(z,\zeta,s)$ has a meromorphic continuation to all $s \in \mathbb{C}$ and satisfies*

$$\nabla P_n(z,\zeta,s) = s(1-s)P_n(z,\zeta,s)$$

where $\nabla = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ is the non-Euclidean Laplacian. Moreover, for $Re(s) > 1$,

$$\lim_{z\to\zeta}\left(\frac{\overline{z}-\overline{\zeta}}{\overline{z}-\zeta}\right)^n P_n(z,\zeta,s) = 2\frac{\Gamma(n)\Gamma(2s)}{\Gamma(s+n)\Gamma(s)}$$

for $n > 0$, while

$$\lim_{z\to\zeta}\left(\frac{z-\zeta}{z-\overline{\zeta}}\right)^{-n} P_n(z,\zeta,s) = 2\frac{\Gamma(-n)\Gamma(2s)}{\Gamma(s-n)\Gamma(s)}$$

for $n < 0$. Finally

$$P_o(z,\zeta,s) = -4\frac{\Gamma(2s)}{\Gamma(s)^2}\,\log\left(\frac{z-\zeta}{z-\overline{\zeta}}\right) \;+\; O(1)$$

for $z \to \zeta$.

The proof of the above proposition follows from the observations that

$$\nabla = \frac{\left(1-\rho^2\right)^2}{4}\left(\frac{\partial^2}{\partial\rho^2} \;+\; \frac{1}{\rho}\frac{\partial}{\partial\rho} \;+\; \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)$$

when written in polar coordinates on the unit disk, and that the differential equation

$$\nabla g(\rho)e^{in\phi} = s(1-s)g(\rho)e^{in\phi}$$

has exactly two independent solutions

$$g(\rho) = \begin{cases} \rho^{|n|}(1-\rho^2)^s F(s+|n|, s, 1+|n|; \rho^2) \\ \rho^{|n|}(1-\rho^2)^s F(s+|n|, s, 2s; 1-\rho^2). \end{cases}$$

The second eigenfunction $g(\rho)e^{in\phi}$ may be summed for $z \neq \zeta$ over all translates in the group $T_\zeta\Gamma_o(N){T_\zeta}^{-1}$, and as shown earlier, the sum converges absolutely for $Re(s) > 1$. Clearly, the sum must be an eigenfunction of ∇ with eigenvalue $s(1-s)$ since each term has that property.

Finally, the limiting values $z \to \zeta$ in the proposition are obtained from the limiting values

$$\lim_{\rho\to 0}\,\rho^{2|n|}F(s+|n|, s, 2s; 1-\rho^2) = \frac{\Gamma(|n|)\Gamma(2s)}{\Gamma(s+|n|)\Gamma(s)}$$

and

$$F(s,s,2s;1-\rho^2) = \frac{-2\Gamma(2s)\log(\rho)}{\Gamma(s)^2} \;+\; O(1), \qquad\qquad \text{for } \rho \to 0,$$

of the hypergeometric function.

In order to construct Poincaré series which are harmonic in z and have a prescribed singularity at $z = \zeta$, we consider

$$\lim_{s\to 1} P_n(z,\zeta,s) = P_n(z,\zeta,1). \tag{2.1}$$

Since $\nabla P_n(z,\zeta,1) = 0$, this function is clearly harmonic. For our purposes, it is only necessary to consider the case when $n < 0$. To study the function $P_n(z,\zeta,s)$ we let

$$P_n(z,\zeta,s) = \sum_{\sigma\,\in\,\Gamma_\infty\backslash\Gamma_o(N)} Q_n(\sigma z,\zeta,s) \tag{2.2}$$

where

$$Q_n(z,\zeta,s) = 2\sum_{\ell\in\mathbb{Z}} \left(\frac{\overline{z}+\ell-\overline{\zeta}}{\overline{z}+\ell-\zeta}\right)^{|n|} \left(1-\left|\frac{z+\ell-\zeta}{z+\ell-\overline{\zeta}}\right|^2\right)^s F\left(s+|n|,s,2s;1-\left|\frac{z+\ell-\zeta}{z+\ell-\overline{\zeta}}\right|^2\right).$$

(The factor 2 appears because both $\left(\begin{smallmatrix}1&\ell\\0&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}-1&\ell\\0&-1\end{smallmatrix}\right)$ are in Γ_∞.) Applying the formula

$$F(1+n,1,2;z) = \frac{(1-z)^{-n}\, -\, 1}{nz},$$

we obtain

$$Q_n(z,\zeta,1) = \frac{1}{|n|}\sum_{\ell\,\in\,\mathbb{Z}}\left[\left(\frac{z+\ell-\overline{\zeta}}{z+\ell-\zeta}\right)^{|n|} - \left(\frac{\overline{z}+\ell-\overline{\zeta}}{\overline{z}+\ell-\zeta}\right)^{|n|}\right]. \tag{2.3}$$

Setting $z = x+iy$, and $\zeta = \nu+i\eta$, it follows from the Poisson summation formula that

$$Q_n(z,\zeta,1) = \frac{1}{|n|}\sum_{\ell\,\in\,\mathbb{Z}} e^{-2\pi i\ell(x+\nu)}\, I_n(y,\eta,\ell) \tag{2.4}$$

where

$$I_n(y,\eta,\ell) = \int_{-\infty}^{\infty}\left[\left(1+\frac{2\eta i}{x-i(-y+\eta)}\right)^{|n|} - \left(1+\frac{2\eta i}{x-i(y+\eta)}\right)^{|n|}\right]\cdot e^{-2\pi i\ell x}\,dx.$$

Now $I_n = I_n(y,\eta,\ell)$ may be computed by the calculus of residues. Firstly, if $\ell > 0,\ y > \eta$ we have

$$I_n = e^{2\pi\ell(-y+\eta)}\sum_{j=1}^{n}\frac{(4\pi\eta\ell)^j\binom{n}{j}}{\ell(j-1)!}. \tag{2.5}$$

Secondly, if $\ell > 0,\ y < \eta$, then $I_n = 0$. On the other hand if $\ell < 0,\ y > \eta$ then

$$I_n = e^{2\pi\ell(y+\eta)}\sum_{j=1}^{n}\frac{(4\pi\eta\ell)^j\binom{n}{j}}{\ell(j-1)!}. \tag{2.6}$$

Finally, in the remaining case $\ell < 0,\ y < \eta$

$$I_n = \left(e^{2\pi\ell y} - e^{-2\pi\ell y}\right) e^{2\pi\ell\eta} \sum_{j=1}^{n} \frac{(4\pi\eta\ell)^j \binom{n}{j}}{\ell(j-1)!}. \tag{2.7}$$

Now, if $n \neq 0,\ P_n(z,\zeta,s)$ is nonsingular for $z \neq \zeta$ at the point $s = 1$. This is proved in §5, and by another method in **[N]**. It then follows from (2.1), (2.2) that

$$P_n(z,\zeta,1) \;=\; Q_n(z,\zeta,1) \;+\; \lim_{s\to 1} \sum_{\substack{\sigma\in\Gamma_\infty\backslash\Gamma_o(N) \\ \sigma\not\equiv\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\ (\mathrm{mod}\ \Gamma_\infty)}} Q_n(\sigma z,\zeta,s). \tag{2.8}$$

where the second term above converges for $Re(s) > 1$ to a nonsingular function, and hence, by analytic continuation must also be non singular for all pairs $z,\,\zeta \in X_o(N)$ when $s = 1$. On the other hand, (2.3) gives

$$\begin{aligned} Q_n(z,\zeta,1) \;&=\; \frac{1}{|n|}\left(\frac{z-\overline{\zeta}}{z-\zeta}\right)^{|n|} \;+\; O(1) \\ &=\; \frac{1}{|n|}\left(1+\frac{2i\eta}{z-\zeta}\right)^{|n|} \;+\; O(1) \\ &=\; \frac{1}{|n|}\sum_{j=0}^{|n|}\binom{n}{j}\frac{(2i\eta)^j}{(z-\zeta)^j} \;+\; O(1). \end{aligned} \tag{2.9}$$

Finally, we obtain from (2.8) and (2.9) that

$$\begin{aligned} P_{-3}(z,\zeta,1) \;&=\; \frac{1}{3}\left[\frac{-8i\eta^3}{(z-\zeta)^3} - \frac{12\eta^2}{(z-\zeta)^2} + \frac{6i\eta}{z-\zeta}\right] \;+\; O(1) \\ P_{-2}(z,\zeta,1) \;&=\; \left[\frac{-2\eta^2}{(z-\zeta)^2} + \frac{2i\eta}{z-\zeta}\right] \;+\; O(1) \\ P_{-1}(z,\zeta,1) \;&=\; \frac{2i\eta}{z-\zeta} \;+\; O(1). \end{aligned}$$

It immediately follows that

$$\frac{3i}{8\eta^3}\left[P_{-3}(z,\zeta,1) - 2P_{-2}(z,\zeta,1) + 3P_{-1}(z,\zeta,1)\right] \;=\; \frac{1}{(z-\zeta)^3} \;+\; O(1) \tag{2.10}$$

$$\frac{-1}{2\eta^2}\left[P_{-2}(z,\zeta,1) + P_{-1}(z,\zeta,1)\right] \;=\; \frac{1}{(z-\zeta)^2} \;+O(1) \tag{2.11}$$

$$\frac{-i}{2\eta}P_{-1}(z,\zeta,1) \;=\; \frac{1}{z-\zeta} \;+\; O(1), \tag{2.12}$$

and this completes the required construction of Poincaré series with specified singularity type. It is, therefore, natural to define the following functions.

Definition 2.4 *Let*

$$\begin{aligned} P_1(z,\zeta) &= \frac{-i}{2\eta} P_{-1}(z,\zeta,1) \\ P_2(z,\zeta) &= \frac{-1}{2\eta^2}\big[P_{-2}(z,\zeta,1)+P_{-1}(z,\zeta,1)\big] \\ P_3(z,\zeta) &= \frac{3i}{8\eta^3}\big[P_{-3}(z,\zeta,1)-2P_{-2}(z,\zeta,1)+3P_{-1}(z,\zeta,1)\big], \end{aligned}$$

where $\eta = Im(\zeta)$ and $P_n(z,\zeta,1)$ is given by definition (2.1).

Definition (2.4) together with equations (2.10), (2.11) and (2.12) immediately yield

$$P_n(z,\zeta) = \frac{1}{(z-\zeta)^n} + O(1) \tag{2.13}$$

for $1 \le n \le 3$.

§3. Poincaré series for cusps

We now consider Poincaré series for the cusp $i\infty$. Let

$$I_s(y) = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(s+k+1)} \left(\frac{y}{2}\right)^{s+2k}$$

denote the I-Bessel function. Then

$$\nabla\left(y^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi y)\, e^{2\pi i x}\right) = s(1-s)\, y^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi y)\, e^{2\pi i x}$$

where $\nabla = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$. By summing translates of the above eigenfunction of the non-Euclidean Laplacian ∇ we obtain a Poincaré series which is also an eigenfunction of ∇.

Definition 3.1 *Let $n \in \mathbb{Z},\ z \in X_o(N),\ Re(s) > 1$. we define the Poincaré series $P_n(z, i\infty, s)$ by the absolutely convergent series*

$$P_n(z,i\infty,s) = \sum_{M\in\Gamma_\infty\backslash\Gamma_o(N)} \Big(Im(Mz)\Big)^{\frac{1}{2}} I_{s-\frac{1}{2}}\big(2\pi|n|Im(Mz)\big)\, e^{2\pi i n\, Re(Mz)}.$$

This function was first introduced by Neunhöffer **[N]** who found the functional equation

$$P_n(z,i\infty,s) - P_n(z,i\infty,1-s) = \frac{a_{-n}(s)}{2s-1}\, E(z,1-s)$$

where $E(z,s) = \sum_{M\in\Gamma_\infty\backslash\Gamma_o(N)} (ImMz)^s$ is the Eisenstein series and

$$a_n(s) = \frac{2\pi^s |n|^{s-\frac{1}{2}}}{\Gamma(s)} \sum_{c>0} S(m,0;c)c^{-2s}.$$

Here $S(m,n;c)$ denotes the Kloosterman sum

$$S(m,n;c) = \sum_{\substack{0\le d<|c| \\ \begin{pmatrix} a & * \\ c & d\end{pmatrix}\in\Gamma_o(N)}} e^{2\pi i\frac{(ma+nd)}{c}}.$$

Somewhat later Niebur **[Ni]** computed the Fourier expansion

$$P_n(z,i\infty,s) = e^{2\pi inx}\, y^{\frac12}\, I_{s-\frac12}(2\pi|n|y) \;+\; \sum_{m=-\infty}^{\infty} b_m(y,s;n)e^{2\pi imx} \tag{3.1}$$

where for $m\neq 0$

$$b_m(y,s;n) = 2\sum_{c>0} S(m,n;c)\, c^{-1}\, M_{2s-1}(4\pi(mn)^{\frac12}\, c^{-1})\;\cdot y^{\frac12} K_{s-\frac12}(2\pi|n|y)$$

with

$$M_{2s-1}\left((mn)^{\frac12}y\right) = \begin{cases} J_{2s-1}(|mn|^{\frac12}y), & \text{if } mn>0 \\ I_{2s-1}(|mn|^{\frac12}y), & \text{if } mn<0,\end{cases}$$

while

$$b_o(y,s;n) = 2\sum_{c>0} S(0,n;c)\, c^{-1} M_{2s-1}(4\pi(mn)^{\frac12}\, c^{-1})\;\cdot y^{1-s}.$$

Let $P_n(z,i\infty) = \lim_{s\to1} P_n(z,i\infty,s)$. It immediately follows that $\nabla P_n(z,i\infty) = 0$ which implies that $P_n(z,i\infty)$ is harmonic. Furthermore, the above Fourier expansion gives

$$P_n(z,i\infty) = e^{2\pi inx}e^{-2\pi|n|y} \;+\; O(1).$$

§4. Proof of theorem 1.1

Recall the relations (1.3), (1.5)

$$\beta(z)^2 = 4\alpha(z)^3 - g_2\alpha(z) - g_3 \tag{4.1}$$

$$\frac{d\alpha(z)}{\beta(z)} = \kappa dF(z) \tag{4.2.}$$

Since we are assuming ϕ_p is unramified at O_E, this implies that $p\neq O_E$, $f(\zeta)\neq 0$ for $\zeta\in\phi_p^{-1}(O_E)$. Therefore, ζ cannot be a cusp, or an elliptic fixed point, $dF(z) = f(z)dz$, and we have Laurent expansions of the form

$$f(z) = \sum_{n=0}^{\infty} c(n)(z-\zeta)^n \tag{4.3}$$

$$\alpha(z) = \sum_{n=-2}^{\infty} a(n)(z-\zeta)^n \tag{4.4}$$

$$\beta(z) = \sum_{n=-3}^{\infty} b(n)(z-\zeta)^n. \tag{4.5}$$

Here $c(n) = \frac{f^n(\zeta)}{n!}$ depends on ζ and $a(n), b(n)$ also depend on ζ.

Substituting the power series (4.3), (4.4) and (4.5) into equations (4.1) and (4.2) we find

$$\sum_{\ell=-3}^{n+3} b(\ell)b(n-\ell) = 4 \sum_{\substack{\ell_1+\ell_2+\ell_3=n \\ \ell_i \geq -2}} a(\ell_1)a(\ell_2)a(\ell_3) \; - \; g_2 a(n) \; - \; g_3 \delta_{n,0}$$

for $n \geq -6$, and

$$(n+1)a(n+1) = \kappa \sum_{\ell=0}^{n+3} c(\ell)b(n-\ell)$$

for $n \geq -3$. These equations may be recursively solved, and we obtain

$$a(-2) = \frac{1}{\kappa^2 c(0)^2}$$

$$b(-3) = \frac{-2}{\kappa^3 c(0)^3}$$

$$a(-1) = \frac{-c(1)}{\kappa^2 c(0)^3}$$

$$b(-2) = \frac{3c(1)}{\kappa^3 c(0)^4}$$

$$a(0) = \frac{1}{\kappa^2}\left[\tfrac{3}{4}\frac{c(1)^2}{c(0)^4} \; - \; \tfrac{2}{3}\frac{c(2)}{c(0)^3}\right]$$

$$b(-1) = \frac{1}{\kappa^3}\left[-3\frac{c(1)^2}{c(0)^5} \; + \; 2\frac{c(2)}{c(0)^4}\right]$$

$$a(1) = \frac{1}{\kappa^2}\left[-\tfrac{1}{2}\frac{c(3)}{c(0)^3} \; - \; \tfrac{1}{2}\frac{c(1)^3}{c(0)^5} \; + \; \frac{c(1)c(2)}{c(0)^4}\right]$$

$$b(0) = \frac{1}{\kappa^3}\left[\tfrac{5}{2}\frac{c(1)^3}{c(0)^6} \; - \; 4\frac{c(1)c(2)}{c(0)^5} \; + \; \tfrac{3}{2}\frac{c(3)}{c(0)^4}\right]$$

etc. In general, $\kappa^2 a(n)$, $\kappa^3 b(n)$ can be expressed as polynomials over $\mathbb{Q}$ in $c(0)^{-1}, c(n)$ (for $n \geq 0$), and g_2 and g_3. The first occurrence of g_2 is in

$$a(2) = \frac{1}{\kappa^2}\left[\tfrac{5}{16}\frac{c(1)^4}{c(0)^6} - \frac{c(1)^2c(2)}{c(0)^5} + \frac{\frac{1}{3}c(2)^3 + \frac{3}{4}c(1)c(3)}{c(0)^4} - \tfrac{2}{5}\frac{c(4)}{c(0)^3}\right] + \frac{\kappa^2}{20}c(0)^2 g_2$$

while the first occurrence of g_3 is in $b(3)$. The proof of theorem (1.1) immediately follows from these calculations since

$$\alpha(z) - \sum_{\zeta\in\phi_p^{-1}(O_E)} \Big(a(-2)P_2(z,\zeta) + a(-1)P_1(z,\zeta)\Big)$$

and

$$\beta(z) - \sum_{\zeta\in\phi_p^{-1}(O_E)} \Big(b(-3)P_3(z,\zeta) + b(-2)P_2(z,\zeta) + b(-1)P_1(z,\zeta)\Big)$$

are nonsingular harmonic functions on $X_o(N)$ which must, therefore, be constants.

If the cusps of $\Gamma_(N)$ are contained in $\phi_p^{-1}(O_E)$, then each cusp must be unramified since the constant term in the Fourier expansion at any cusp must be 1. The expansions are exactly the same as equations (4.3), (4.4) and (4.5), except that $(z-\zeta)^n$ must be replaced by q^n where $q = e^{2\pi iz}$ is the uniformizing parameter at the cusp $i\infty$. The rest of the calculation is straightforward and left to the reader.

To prove corollary (1.2) we redo the above calculations at the cusp $\zeta = i\infty$. Since $i\infty \notin \phi_p^{-1}(O_E)$, it follows that $\alpha(z)$ and $\beta(z)$ must be regular at $z = \zeta$. The expansions (4.3), (4.4), and (4.5) now take the form

$$f(z) = \sum_{n=1}^{\infty} c(n)e^{2\pi inz} \tag{4.6}$$

$$\alpha(z) = \sum_{n=0}^{\infty} a(n)e^{2\pi inz} \tag{4.7}$$

$$\beta(z) = \sum_{n=0}^{\infty} b(n)e^{2\pi inz}, \tag{4.8}$$

where $c(1) = 1$, $a(0) = \wp(u)$, $b(0) = \wp'(u)$ and $p = \big(\wp(u), \wp'(u)\big)$ is our fixed base point. The relation (4.2) now becomes

$$\frac{\alpha'(z)}{\beta(z)} = 2\pi i\kappa f(z). \tag{4.9}$$

Substituting (4.6), (4.7), (4.8) into (4.1) and (4.9) and recursively solving as before, we obtain

$$a(1) = \kappa\wp'(u)$$
$$b(1) = 6\wp(u)^2\kappa - \frac{\kappa}{2}g_2$$

$$a(2) = \frac{\kappa}{2}c(2)\wp(u) + 3\kappa^2\wp(u)^2 - \frac{\kappa^2}{4}g_2$$
$$b(2) = 3\kappa\wp(u)^2c(2) + 6\kappa^2\wp(u)\wp'(u) - \frac{\kappa}{4}c(2)g_2$$

etc.

Since $a(2)$ must be equal to the second Fourier coefficient (at the parabolic cusp $\zeta = i\infty$) of the sum of Poincaré series on the right hand side of theorem (1.1), a simple computation completes the proof of corollary (1.2).

§5. Fourier expansions of Poincaré series for inner points

The Fourier expansion of the Poincaré series $P_n(z,\zeta,s)$ (see definition (2.1)), at the parabolic cusp $i\infty$, was computed by Neunhöffer **[N]** in the special case where $n=0$ and $Im(z) < Im(\zeta)$. Since the Poincaré series has a singularity at $z=\zeta$, the Fourier expansion takes a different form when $Im(z) > Im(\zeta)$, and does not seem to be given in the literature. We now sketch a general method for finding all such Fourier expansions. Details are given only for $n \le 0$, since by symmetry the case $n \ge 0$ is entirely similar.

Recall equation (2.2),

$$P_n(z,\zeta,s) = \sum_{\sigma\,\in\,\Gamma_\infty\backslash\Gamma_o(N)} Q_n(\sigma z,\zeta,s) \tag{5.1}$$

where

$$Q_n(z,\zeta,s) = 2\sum_{\ell\in\mathbb{Z}}\left(\frac{\overline{z}+\ell-\overline{\zeta}}{\overline{z}+\ell-\zeta}\right)^{|n|}\left(1-\left|\frac{z+\ell-\zeta}{z+\ell-\overline{\zeta}}\right|^2\right)^s F\Big(s+|n|,s,2s;1-\left|\frac{z+\ell-\zeta}{z+\ell-\overline{\zeta}}\right|^2\Big).$$

Applying the Poisson summation formula, we obtain

$$\begin{aligned} Q_n(z,\zeta,s) = 2\sum_{\ell\in\mathbb{Z}} e^{2\pi i\ell(x-\nu)}\int_{-\infty}^{\infty}\left(\frac{x_1-i(y-\eta)}{x_1-i(y+\eta)}\right)^{-n}\left(\frac{4y\eta}{x_1^2+(y+\eta)^2}\right)^s\cdot \\ \cdot F\Big(s+|n|,s,2s;\frac{4y\eta}{x_1^2+(y+\eta)^2}\Big)\,e^{-2\pi i\ell x_1}\,dx_1. \end{aligned} \tag{5.2}$$

Substituting the integral formula for the hypergeometric function,

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt,$$

into (5.2), it follows that

$$\begin{aligned} Q_n(z,\zeta,s) = 2\frac{\Gamma(2s)(4y\eta)^s}{\Gamma(s)^2}\sum_{\ell\in\mathbb{Z}} e^{2\pi i\ell(x-\nu)}\int_0^1 t^{s-1}(1-t)^{s-1}\cdot \\ \cdot\int_{-\infty}^{\infty}\frac{\Big((x_1-i(y-\eta))(x_1+i(y+\eta))\Big)^{|n|}}{\Big(x_1^2+(y+\eta)^2-4y\eta t\Big)^{s+|n|}}\,e^{-2\pi i\ell x_1}\,dx_1\,dt. \end{aligned} \tag{5.3}$$

Now,

$$\nabla\!\cdot\int_0^1 Q_n(z,\zeta,s)e^{-2\pi i\ell x}\,dx = s(1-s)\int_0^1 Q_n(z,\zeta,s)e^{2\pi i\ell x}\,dx,$$

and by (5.3) for $Re(s) > 1$,

$$\lim_{y\to 0} |y^{-s} Q_n(z,\zeta,s)| \ll 1$$

which implies

$$\int_0^1 Q_n(z,\zeta,s) e^{2\pi i \ell x}\, dx = \begin{cases} B_\ell(\zeta,s)\sqrt{y}\, I_{s-\frac{1}{2}}(2\pi|\ell|y) & \text{if } \ell \neq 0 \\ B_o(\zeta,s) y^s & \text{if } \ell = 0 \end{cases} \tag{5.4}$$

for some function $B_\ell(\zeta,s)$. Clearly

$$B_\ell(\zeta,s) = \begin{cases} \lim_{y\to 0} \frac{\int_0^1 Q_n(z,\zeta,s) e^{-2\pi i \ell x}\, dx}{\sqrt{y}\, I_{s-\frac{1}{2}}(2\pi|\ell|y)} & \text{if } \ell \neq 0 \\ \lim_{y\to 0} \frac{\int_0^1 Q_n(z,\zeta,s)\, dx}{y^s} & \text{if } \ell = 0. \end{cases} \tag{5.5}$$

But for $\ell \neq 0$,

$$\lim_{y\to 0} \frac{y^{s-\frac{1}{2}}}{I_{s-\frac{1}{2}}(2\pi|\ell|y)} \;=\; \frac{\Gamma(s+\frac{1}{2})}{(\pi|\ell|)^{s-\frac{1}{2}}},$$

and it, therefore, follows from (5.3) and (5.5) that

$$\begin{aligned} B_\ell(\zeta,s) = \frac{2\Gamma(s+\frac{1}{2})\Gamma(2s)(4\eta)^s e^{-2\pi i \ell \nu}}{(\pi|\ell|)^{s-\frac{1}{2}}\Gamma(s)^2} & \int_0^1 t^{s-1}(1-t)^{s-1} \cdot \\ & \cdot \int_{-\infty}^{\infty} (x+i\eta)^{2|n|} \frac{e^{-2\pi i \ell x}}{(x^2+\eta^2)^{s+|n|}}\, dx\, dt. \end{aligned}$$

On the other hand, we have

$$\left(\frac{\partial}{\partial w}\right)^m \int_{-\infty}^{\infty} \frac{e^{-2\pi i \ell x}}{\big((x-w)(x-\overline{w})\big)^s}\, dx = g_m(s) \int_{-\infty}^{\infty} \frac{(x-\overline{w})^m\, e^{-2\pi i \ell x}}{\big((x-w)(x-\overline{w})\big)^{s+m}}\, dx$$

where

$$g_m(s) = \begin{cases} 1 & \text{if } m = 0 \\ (-1)^m s(s+1)\cdots(s+m-1) & \text{if } m = 1,2,3,\ldots \end{cases}$$

Let $w = \nu + i\eta$, the integral formula

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i \ell x}}{\big((x-w)(x-\overline{w})\big)^s}\, dx \;=\; \frac{2\sqrt{\pi}(\pi|\ell|)^{s-\frac{1}{2}}}{\Gamma(s)\eta^{s-\frac{1}{2}}}\, K_{s-\frac{1}{2}}(2\pi|\ell|\eta)\, e^{-2\pi i \ell \nu}$$

for the K-Bessel function, together with the beta function integral

$$\int_0^1 t^{s-1}(1-t)^{s-1}\, dt = \frac{\Gamma(s)^2}{\Gamma(2s)}$$

then yield

$$B_\ell(\zeta,s) = G_n(s) \frac{\eta^s}{(\pi|\ell|)^{|n|}} \left(\frac{\partial}{\partial w}\right)^{2|n|} \left(\eta^{\frac{1}{2}+|n|-s}\, K_{s-|n|-\frac{1}{2}}(2\pi|\ell|\eta) e^{-2\pi i \ell \nu}\right) \tag{5.6}$$

where

$$G_n(s) = \frac{\sqrt{\pi}4^{s+1}\Gamma(s+\frac{1}{2})}{g_{2|n|}(s-|n|)\,\Gamma(s-|n|)}$$

and

$$\frac{\partial}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial \nu} - i\frac{\partial}{\partial \eta}\right).$$

Returning to the case $\ell = 0$, we have for $w = \nu + i\eta$

$$\begin{aligned} B_o(\zeta, s) &= 2(4\eta)^s \int_{-\infty}^{\infty} \frac{(x+i\eta)^{2|n|}}{(x^2+\eta^2)^{s+|n|}}\,dx \\ &= \frac{2(4\eta)^s}{g_{2|n|}(s-|n|)}\left(\frac{\partial}{\partial w}\right)^{2|n|}\int_{-\infty}^{\infty}\frac{dx}{\big((x-w)(x-\overline{w})\big)^{s-|n|}} \\ &= \frac{2(4\eta)^s}{g_{2|n|}(s-|n|)}\left(\frac{-i}{2}\frac{\partial}{\partial \eta}\right)^{2|n|}\left(\eta^{1-2s+2|n|}\cdot\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)^{s-|n|}}\right), \end{aligned}$$

since the integral

$$\int_{-\infty}^{\infty}\frac{dx}{\big((x-w)(x-\overline{w})\big)^{s-|n|}}$$

is independent of ν. It follows that

$$B_o(\zeta, s) = \frac{\sqrt{\pi}\,4^{2s+1-|n|}\,\eta^s\,\Gamma(s-|n|-\frac{1}{2})}{g_{2|n|}(s-|n|)\Gamma(s-|n|)}\left(\frac{-i}{2}\frac{\partial}{\partial \eta}\right)^{2|n|}\left(\eta^{1-2s+2|n|}\right).$$

It is an immediate consequence of the above formula that $B_o(\zeta, 1) = 0$ for $n = -1, -2, \ldots$

The combination of equations (5.1), (5.2), (5.4), (5.6), and definition (3.1) now give for $n = -1, -2, \ldots$

$$P_n(z,\zeta,1) = Q_n(z,\zeta,1) + G_n(1)\sum_{\ell\in\mathbb{Z}}\frac{k_\ell(\nu,\eta,n)}{(\pi|\ell|)^{|n|}}\left[P_\ell(z,i\infty,1) - \sqrt{y}I_{\frac{1}{2}}(2\pi|\ell|y)e^{2\pi i\ell x}\right]$$

where

$$k_\ell(\nu,\eta,n) = \eta\cdot\left(\frac{\partial}{\partial w}\right)^{2|n|}\left(\eta^{|n|-\frac{1}{2}}K_{|n|-\frac{1}{2}}(2\pi|\ell|\eta)\,e^{-2\pi i\ell\nu}\right).$$

Furthermore, the Fourier expansion of the function $Q_n(z,\zeta,1)$ is given by equations (2.4), (2.5), (2.6),and (2.7), while the Fourier expansion of $P_\ell(z,i\infty,1)$ is given in (3.1). Although somewhat complicated, these Fourier expansions can then be used (see corollary (1.2)) to express the invariants g_2, g_3 of an elliptic curve as infinite sums of Kloosterman sums.

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