Application of functional transcendence to bounding the number of points on curves

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Let $g \geq 0$ and $d \geq 1$ be two integers. Let $C$ be an irreducible smooth projective curve of genus $g$, defined over a number field $K$ of degree $d$. In 1983, Faltings proved the Mordell Conjecture.

**Theorem (Faltings 1983)**

*When $g \geq 2$, the set $C(K)$ is finite.*

However Faltings’s 1983 proof does not give a good upper bound on $\#C(K)$. 
Faltings’s Theorem

In 1988, Vojta gave a second proof to Faltings’s Theorem. The proof was simplified and generalized by Faltings in 1989, and further simplified by Bombieri in 1990. This new proof (BFV) gives an upper bound

\[ \#C(K) \leq c(g, d, h_\Theta(J))^{1 + \text{rk}_Z J(K)} \]

where \( J = \text{Jacobian of } C \), and \( h_\Theta(J) \) is the Theta height of \( J \).
Mazur asked the following question in 1986.

**Conjecture (Mazur 1986)**

Let $P_0 \in C(\overline{\mathbb{Q}})$, $J =$Jacobian of $C$ and $j_{P_0} : C \to J$ be the Abel-Jacobi embedding via $P_0$. Let $\Gamma$ be a finite rank subgroup of $J(\overline{\mathbb{Q}})$. Then

$$\#(\Gamma \cap j_{P_0}(C)(\overline{\mathbb{Q}})) \leq c(g)^{1+\text{rk} \Gamma}.$$  

(1)

Our main result is

**Theorem (Dimitrov-G’-Habegger, forthcoming)**

For any $g \geq 2$, there exists a number $\delta(g) > 0$ such that the following property holds: If $h_\Theta(J) > \delta(g)$, then (1) holds.
Two upshots

Theorem (Dimitrov-G’-Habegger, forthcoming)

For any $g \geq 2$, there exists a number $\delta(g) > 0$ such that the following property holds: If $h_\Theta(J) > \delta(g)$, then

$$\#(\Gamma \cap j_{P_0}(C)(\overline{Q})) \leq c(g)^{1+\text{rk}\Gamma}.$$ 

Two particular cases of this theorem.

- If we take $P_0 \in C(K)$ and $\Gamma = J(K)$, then the upper bound gives
  $$\#C(K) \leq c(g, d)^{1+\text{rk}J(K)}.$$ This is a weaker form of Mazur’s conjecture.

- If we take $\Gamma = J(\overline{Q})_{\text{tor}}$, then the upper bound becomes a desired bound on the size of torsion packets on $C$, which by abuse of notation we denote by $C(\overline{Q})_{\text{tor}}$ (towards uniform Manin-Mumford).
Known results on Mazur’s conjecture

- For **rational points**, here are some known results towards the upper bound $\#C(K) \leq c(g, d)^{1+\text{rk}J(K)}$.
  - David–Philippon 2007: when $J \subset E^n$.
  - Katz–Rabinoff–Zureick-Brown 2016: when $\text{rk}J(K) \leq g - 3$.
  - Alpoge 2018: average number of $\#C(K)$ is finite when $g = 2$.

- For **algebraic torsion points**, here are some known results towards the upper bound $\#C(\mathbb{Q}) \leq c(g, d)$.
  - DeMarco–Holly–Ye 2018: $g = 2$ bi-elliptic independent of $d$. 
Review of the BFV method

(For simplicity assume $\Gamma$ is finitely generated.)

On $J(\overline{Q})$, there is a function $\hat{h}_L : J(\overline{Q}) \to \mathbb{R}_{\geq 0}$ vanishing precisely on $J(\overline{Q})_{\text{tor}}$.

$\Rightarrow \hat{h}_L : \Gamma \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}_{\geq 0}$.

$\Rightarrow$ “Normed Euclidean space” $(\Gamma \otimes_{\mathbb{Z}} \mathbb{R}, \hat{h}_L)$, and $\Gamma$ becomes a lattice in it.

Theorem (Vojta, Faltings, Bombieri, David–Philippon, Rémond)

$\#\text{large points} \leq c(g)7^{\text{rk}\Gamma}$.

$\Rightarrow$ to prove Mazur’s Conj, it suffices to establish

Algebraic points are “far” from each other, i.e.

$\hat{h}_L(P - Q) \geq c'(g)h_{\Theta}(J)$ for all $P \neq Q$ in $J(\overline{Q})$.

In practice, need to add a calibration term to this inequality.
A height inequality

Theorem (G’–Habegger, Ann. 2019)

Let $S$ be a curve over $\overline{\mathbb{Q}}$, let $\pi : A \to S$ be an abelian scheme, and let $L$ be a relatively ample line bundle on $A$. Let $X$ be an irreducible subvariety of $A$ dominant to $S$.

➢ If $X$ is non-degenerate (defined later), then there exist $c = c(X) > 0$ and a Zariski open dense $U \subseteq X$ such that

$$c(\hat{h}_L(x) + 1) \geq h_\Theta(s)$$

for all $s \in S(\overline{\mathbb{Q}})$ and $x \in (U \cap A_s)(\overline{\mathbb{Q}})$.

➢ $X$ is degenerate $\iff X = \text{abelian subscheme} + \text{torsion section} + \text{“constant part”}$.
Let $\mathcal{C} \to S$ be a 1-parameter of curves of genus $g \geq 3$. Assume it is non-isotrivial. We have a relative Jacobian $\mathcal{J} \to S$.
- The morphism $\mathcal{C} \times_S \mathcal{C} \to \mathcal{J}$, fiberwise defined by $(P, Q) \mapsto [P - Q]$, defines a subvariety of $\mathcal{J}$, which we call $\mathcal{C} - \mathcal{C}$.
- Apply the height inequality to $A = \mathcal{J}$ and $X = \mathcal{C} - \mathcal{C}$. It is not hard to show that $X$ is non-degenerate by part (2) of the theorem. We get

$$c(\hat{h}_L(P - Q) + 1) \geq h_\Theta(\mathcal{J}_s)$$

for all $s \in S(\overline{Q})$, $P, Q \in \mathcal{C}_s(\overline{Q})$ such that $P - Q \in U$.

$\hat{c}h_L(x) + c' \geq h_\Theta(s)$.

- $c'$ (partly) comes from the Height Machine.
- Let us normalize $L$ so that it is trivial along the zero section.
- The height function $h_\Theta$ on $S(\mathcal{Q})$ is defined by an ample line bundle, say $\mathcal{M}$.
- Then the height inequality becomes the comparison of two line bundles $L$ and $\pi^* \mathcal{M}$ when restricted to $X$. We wish that $L^{\otimes N}|_X$ is “bigger” than $\pi^* \mathcal{M}|_X$ for some $N \gg 0$.
- To achieve this, it suffices to prove that $L|_X$ is “big”.
- So the definition of $X$ non-degenerate should reflect the fact that $L|_X$ is big.

This is not vigorous as $S$, $\mathcal{A}$, $X$ are not projective.
General case: preparation

To prove the result for general case, we need to investigate the universal curve $\mathcal{C}_g \to M_g$ of genus $g$. Here $M_g$ is the moduli space with some level structure.

As before, $\mathcal{C}_g - \mathcal{C}_g$ is a well-defined subvariety of $\text{Jac}(\mathcal{C}_g/M_g)$.

One step further, let us put everything in the universal abelian variety $\pi: \mathcal{A}_g \to A_g$,

\[
\begin{array}{ccc}
\mathcal{X} := \mathcal{C}_g - \mathcal{C}_g & \to & \text{Jac}(\mathcal{C}_g/M_g) & \to & \mathcal{A}_g \\
\downarrow & & \downarrow & & \downarrow \\
M_g & \to & \mathcal{A}_g \\
\end{array}
\]

so that now, we wish to study subvarieties of $\mathcal{A}_g$. 
Non-degeneracy: geometric definition

On $\mathcal{A}_g \to \mathcal{A}_g$, there is a canonical symmetric relatively ample line bundle $\mathcal{L}_g$, which is trivial along the zero section. Consider the $(1, 1)$-form $c_1(\mathcal{L}_g)$ (representative of the first Chern class). It is a non-negative form as $\mathcal{L}_g$ is nef.

**Definition**

A subvariety $X$ of $\mathcal{A}_g$ is said to be non-degenerate if

$$c_1(\mathcal{L}_g)|_X^{\dim X} \not\equiv 0.$$  

Vaguely speaking, this means that $\mathcal{L}_g|_X$ is big.
Let $\mathcal{H}_g$ be the Siegel upper half space. Then we have a uniformization $\mathcal{H}_g \to \mathbb{A}_g$ in the category of complex spaces. Construct a similar uniformization $\mathcal{X}_{2g}$ of the universal abelian variety $\mathbb{A}_g$ as follows:

- real-algebraic: $\mathcal{X}_{2g} = \mathbb{R}^{2g} \times \mathcal{H}_g$;
- complex structure: $\mathbb{R}^{2g} \times \mathcal{H}_g \tilde\to \mathbb{C}^g \times \mathcal{H}_g$, $(a, b, Z) \mapsto (a + Zb, Z)$.

Then the 2-form $da \wedge db$ descends to a $(1, 1)$-form on $\mathbb{A}_g$, which is a representative of $c_1(\mathcal{L}_g)$ (N. Mok).

Upshot: the kernel of the skew-symmetric bilinear form defined by $c_1(\mathcal{L}_g)$ is precisely $\{ \{ r \} \times \mathcal{H}_g : r \in \mathbb{R}^{2g} \}$. 

Non-degeneracy in Diophantine way: Betti map

\( u : \mathcal{X}_{2g} \to \mathcal{A}_g. \)

**Definition (Bertrand, Zannier, Masser, Corvaja, André)**

The Betti map is defined to be the natural projection \( b : \mathcal{X}_{2g} \to \mathbb{R}^{2g}. \) It is a real-analytic map with complex fibers.

By the discussion in the previous slide, we have the following definition for non-degeneracy.

**Definition**

A subvariety \( X \) of \( \mathcal{A}_g \) is said to be non-degenerate if

\[
\text{rk}_{\mathbb{R}} db|_{\tilde{X}} = 2 \dim X.
\]

Here \( \tilde{X} \) is an irreducible component of \( u^{-1}(X) \).

In particular, \( X \) is always degenerate if \( \dim X > g \).
Question ((weak) Ax-Schanuel)

Let $q: \Omega \to S$ be a surjective holomorphic morphism between algebraic varieties. Let $Z \subseteq \Omega$ be complex analytic. Then

$$\dim \overline{Z}^{\text{Zar}} + \dim \overline{q(Z)}^{\text{Zar}} \geq \dim Z + \dim \overline{q(Z)}^{\text{biZar}}.$$

Here $\overline{q(Z)}^{\text{biZar}}$ means the smallest bi-algebraic subvariety of $S$ containing $q(Z)$, where bi-algebraic means “both algebraic in $\Omega$ and in $S$”.

Theorem (Ax, 1971, 1973)

Ax-Schanuel holds for semi-abelian varieties.
Functional Transcendence: (weak) Ax-Schanuel

**Theorem (Mok–Pila–Tsimerman, 2019 Ann.)**

\[ \text{Ax-Schanuel holds for } \mathcal{H}_g \rightarrow \mathbb{A}_g. \]

Extensions of Mok–Pila–Tsimerman in two directions.

**Theorem (Bakker–Tsimerman, 2019 Inv.)**

\[ \text{Ax-Schanuel holds for variations of pure Hodge structures.} \]

**Theorem (G’, 2018 preprint)**

\[ \text{Ax-Schanuel holds for } u: \mathcal{X}_{2g} \rightarrow \mathcal{A}_g \text{ (mixed Shimura variety).} \]

We will use this mixed version to study the Betti map.
From non-degeneracy to unlikely intersection

For the notation

\[ X_{2g} = \mathbb{R}^{2g} \times \mathcal{H}_g \xrightarrow{u} \mathcal{A}_g \]

\[ \xrightarrow{\pi} \mathcal{H}_g \xrightarrow{} \mathbb{A}_g \]

with \( X \subseteq \mathcal{A}_g \) and \( \tilde{X} \) a component of \( u^{-1}(X) \), we have

\[ X \text{ is degenerate} \iff \tilde{X} = \bigcup_{r \in \mathbb{R}^{2g}, \tilde{C} \text{ curve in } \mathcal{H}_g} (\{r\} \times \tilde{C}) \]

\[ \iff X = \bigcup_{r \in \mathbb{R}^{2g}, \tilde{C} \text{ curve in } \mathcal{H}_g} u(\{r\} \times \tilde{C}) \]
From non-degeneracy to unlikely intersection

So $X$ is degenerate $\Rightarrow X = \bigcup_{r \in \mathbb{R}^{2g}} \tilde{C}_{\text{curve in } \mathcal{H}_g} \cup \{r\} \times \tilde{C}$. Now let us study

$$Y := u(\{r\} \times \tilde{C})$$

Apply mixed Ax-Schanuel to $Z = \{r\} \times \tilde{C}$ (version of G’). We get

$$\dim \tilde{Z} + \dim u(\tilde{Z}) \geq \dim \tilde{Z} + \dim u(\tilde{Z})$$

It then becomes

$$\dim(\{r\} \times \tilde{C}) + \dim Y > \dim Y$$

As $\dim \tilde{C} \leq \dim \pi(Y)$, we have

$$\dim Y > \dim Y - \dim \pi(Y)$$
So we have

\[ X \text{ is degenerate} \iff X = \bigcup_{\dim Y > \dim \overline{Y}^{\text{biZar}} - \dim \overline{\pi(Y)}^{\text{biZar}}} Y. \]

The previous slide showed $\Rightarrow$ using mixed Ax-Schanuel. The direction $\Leftarrow$ follows directly from the description of bi-algebraic subvarieties in $\mathcal{A}_g$.

We will show that the union on the right hand side is actually a finite union, and get a criterion to degeneracy from this!
Finiteness of the union

**Theorem (Bogomolov, ’70)**

Let $A$ be an abelian variety and let $X$ be a subvariety. There are only finitely many abelian subvarieties $B$ of $A$ with $\dim B > 0$ satisfying:

1. $a + B \subseteq X$ for some $a \in A$;
2. $B$ is maximal for the property described in (1).

- Ullmo (2014) proved the corresponding result for pure Shimura varieties, for the purpose of studying the André-Oort conjecture.
- Habegger–Pila (2016) introduced the notion of *weakly optimal* subvarieties when studying the more general Zilber-Pink conjecture. They also proved the corresponding finiteness result for the case $Y(1)^N$.
In a preprint (2018), G’ extended Daw–Ren’s result to $\mathcal{A}_g$ (mixed Shimura varieties). Applying this extension to the union in earlier slides (and some other results), we get the following criterion for degeneracy. To ease notation let us denote by $S = \pi(X) \subseteq \mathcal{A}_g$, and $\mathcal{A} := \mathcal{A}_g \times \mathcal{A}_g \, S$. We also assume that $X$ is not contained in a proper subgroup scheme of $\mathcal{A} \to S$.

**Theorem (G’, 2018 preprint)**

* $X$ is degenerate if and only if there exists an abelian subscheme $\mathcal{B}$ of $\mathcal{A} \to S$ such that $\dim \varphi(X) < \dim X - (g - g')$. 

$$
\begin{align*}
\varphi: \mathcal{A} &\to \mathcal{A}/\mathcal{B} &\to \mathcal{A}_{g'} \\
S &\to S &\to \mathcal{A}_{g'}
\end{align*}
$$
Back to our original question (study $X = \mathcal{C}_g - \mathcal{C}_g$). It is clear, by dimension reasons, that $X$ is degenerate. But applying the criterion in the previous slide, one can easily show that $X^m := X \times_{\mathcal{M}_g} \ldots \times_{\mathcal{M}_g} X$ ($m$-copies) is non-degenerate when $m \geq 3g - 3$, if $g \geq 3$.

**Upshot:** Around each algebraic point in $J(\overline{\mathbb{Q}})$, there are at most $3g - 3$ points which are NOT far from it.

This gives the desired bound up to some packing argument.

**In fact,** it is better to consider the Faltings–Zhang map

$$D_m : \mathcal{U}_g^{m+1} \rightarrow \mathcal{U}_g^m$$

fiberwise defined by $(P_0, P_1, \ldots, P_m) \leftrightarrow (P_1 - P_0, \ldots, P_m - P_0)$. Then $D_m(\mathcal{C}_g^{m+1})$ is non-degenerate for $m \geq 3g - 2$, if $g \geq 2$. 