Application of functional transcendence to bounding the number of points on curves

Ziyang Gao

CNRS, IMJ-PRG

July 22, 2019

Faltings's Theorem

Let $g \geq 0$ and $d \geq 1$ be two integers. Let C be an irreducible smooth projective curve of genus g, defined over a number field K of degree d. In 1983, Faltings proved the Mordell Conjecture.

Theorem (Faltings 1983)

When $g \geq 2$, the set $C(K)$ is finite.

However Faltings's 1983 proof does not give a good upper bound on $\#C(K)$.

Faltings's Theorem

In 1988, Vojta gave a second proof to Faltings's Theorem. The proof was simplified and generalized by Faltings in 1989, and further simplified by Bombieri in 1990. This new proof (BFV) gives an upper bound

$$\#C(K) \leq c(g, d, h_{\Theta}(J))^{1+\mathrm{rk}_{\mathbb{Z}} J(K)}$$

where J =Jacobian of C, and $h_{\Theta}(J)$ is the *Theta height* of J.

Upper bound: Mazur's conjecture

Mazur asked the following question in 1986.

Conjecture (Mazur 1986)

Let $P_0 \in C(\overline{\mathbb{Q}})$, $J =$Jacobian of C and $j_{P_0} : C \to J$ be the Abel-Jacobi embedding via P_0. Let Γ be a finite rank subgroup of $J(\overline{\mathbb{Q}})$. Then

$$\#(\Gamma \cap j_{P_0}(C)(\overline{\mathbb{Q}})) \leq c(g)^{1+\mathrm{rk}\Gamma}. \quad (1)$$

Our main result is

Theorem (Dimitrov-G'-Habegger, forthcoming)

For any $g \geq 2$, there exists a number $\delta(g) > 0$ such that the following property holds: If $h_{\Theta}(J) > \delta(g)$, then (1) holds.

Two upshots

Theorem (Dimitrov-G'-Habegger, forthcoming)

For any $g \geq 2$, there exists a number $\delta(g) > 0$ such that the following property holds: If $h_\Theta(J) > \delta(g)$, then

$$\#(\Gamma \cap j_{P_0}(C)(\overline{\mathbb{Q}})) \leq c(g)^{1+\mathrm{rk}\Gamma}.$$

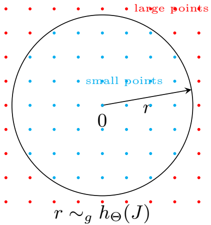
Two particular cases of this theorem.

- If we take $P_0 \in C(K)$ and $\Gamma = J(K)$, then the upper bound gives $\#C(K) \leq c(g,d)^{1+\mathrm{rk}J(K)}$. This is a weaker form of Mazur's conjecture.
- If we take $\Gamma = J(\overline{\mathbb{Q}})_{\mathrm{tor}}$, then the upper bound becomes a desired bound on the size of torsion packets on C, which by abuse of notation we denote by $C(\overline{\mathbb{Q}})_{\mathrm{tor}}$ (towards uniform Manin-Mumford).

Known results on Mazur's conjecture

- For rational points, here are some known results towards the upper bound $\#C(K) \leq c(g, d)^{1+\mathrm{rk} J(K)}$.
  - David–Philippon 2007: when $J \subset E^n$.
  - David–Nakamaye–Philippon 2007: for some particular families of curves.
  - Katz–Rabinoff–Zureick-Brown 2016: when $\mathrm{rk} J(K) \leq g-3$.
  - Alpoge 2018: average number of $\#C(K)$ is finite when $g=2$.
- For algebraic torsion points, here are some known results towards the upper bound $\#C(\overline{\mathbb{Q}}) \leq c(g, d)$.
  - Katz–Rabinoff–Zureick-Brown 2016: assuming some good reduction behavior.
  - DeMarco–Holly–Ye 2018: $g=2$ bi-elliptic independent of d.

Review of the BFV method



(For simplicity assume Γ is finitely generated.)
On $J(\overline{\mathbb{Q}})$, there is a function $\hat{h}_L \colon J(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$ vanishing precisely on $J(\overline{\mathbb{Q}})_{\mathrm{tor}}$.
$\rightsquigarrow \hat{h}_L \colon \Gamma \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}_{\geq 0}$.
$\rightsquigarrow$ "Normed Euclidean space" $(\Gamma \otimes_{\mathbb{Z}} \mathbb{R}, \hat{h}_L)$, and Γ becomes a lattice in it.

Theorem (Vojta, Faltings, Bombieri, David–Philippon, Rémond)

$\#$large points $\leq c(g)7^{\mathrm{rk}\Gamma}$.

$\Rightarrow$ to prove Mazur's Conj, it suffices to establish

Algebraic points are "far" from each other, *i.e.*
$\hat{h}_L(P-Q) \geq c'(g)h_\Theta(J)$ for all $P \neq Q$ in $J(\overline{\mathbb{Q}})$.

In practice, need to add a calibration term to this inequality.

A height inequality

Theorem (G'–Habegger, Ann. 2019)

Let S be a curve over $\overline{\mathbb{Q}}$, let $\pi\colon \mathcal{A} \to S$ be an abelian scheme, and let $\mathcal{L}$ be a relatively ample line bundle on $\mathcal{A}$. Let X be an irreducible subvariety of $\mathcal{A}$ dominant to S.

- *If X is non-degenerate (defined later), then there exist $c = c(X) > 0$ and a Zariski open dense $U \subsetneq X$ such that*

$$c(\hat{h}_{\mathcal{L}}(x) + 1) \geq h_{\Theta}(s)$$

for all $s \in S(\overline{\mathbb{Q}})$ and $x \in (U \cap \mathcal{A}_s)(\overline{\mathbb{Q}})$.

- *X is degenerate $\Leftrightarrow$ $X =$ abelian subscheme + torsion section + "constant part".*

Mazur's conjecture for 1-parameter families

Let $\mathfrak{C} \to S$ be a 1-parameter of curves of genus $g \geq 3$. Assume it is non-isotrivial. We have a relative Jacobian $\mathfrak{J} \to S$.
- The morphism $\mathfrak{C} \times_S \mathfrak{C} \to \mathfrak{J}$, fiberwise defined by $(P, Q) \mapsto [P - Q]$, defines a subvariety of $\mathfrak{J}$, which we call $\mathfrak{C} - \mathfrak{C}$.
- Apply the height inequality to $\mathcal{A} = \mathfrak{J}$ and $X = \mathfrak{C} - \mathfrak{C}$. It is not hard to show that X is non-degenerate by part (2) of the theorem. We get

$$c(\hat{h}_{\mathcal{L}}(P - Q) + 1) \geq h_{\Theta}(\mathfrak{J}_s)$$

for all $s \in S(\overline{\mathbb{Q}})$, $P, Q \in \mathfrak{C}_s(\overline{\mathbb{Q}})$ such that $P - Q \in U$.
⇝ [Dimitrov–G'–Habegger, preprint 2019] With some extra details (packing argument), we prove the main theorem for 1-parameter families.

Digest of non-degeneracy

$$c\hat{h}_{\mathcal{L}}(x) + c' \geq h_{\Theta}(s).$$

- c' (partly) comes from the Height Machine.
- Let us normalize $\mathcal{L}$ so that it is trivial along the zero section.
- The height function h_{Θ} on $S(\overline{\mathbb{Q}})$ is defined by an ample line bundle, say $\mathcal{M}$.
- Then the height inequality becomes the comparison of two line bundles $\mathcal{L}$ and $\pi^*\mathcal{M}$ when restricted to X. We wish that $\mathcal{L}^{\otimes N}|_X$ is "bigger" than $\pi^*\mathcal{M}|_X$ for some $N \gg 0$.
- To achieve this, it suffices to prove that $\mathcal{L}|_X$ is "big".
- So the definition of X non-degenerate should reflect the fact that $\mathcal{L}|_X$ is big.

This is not vigorous as S, $\mathcal{A}$, X are not projective.

General case: preparation

To prove the result for general case, we need to investigate the universal curve $\mathfrak{C}_g \to M_g$ of genus g. Here M_g is the moduli space with some level structure.

As before, $\mathfrak{C}_g - \mathfrak{C}_g$ is a well-defined subvariety of $\mathrm{Jac}(\mathfrak{C}_g/M_g)$.

One step further, let us put everything in the universal abelian variety $\pi\colon \mathfrak{A}_g \to \mathbb{A}_g$,

$$\begin{array}{ccccc} X := \mathfrak{C}_g - \mathfrak{C}_g & \longrightarrow & \mathrm{Jac}(\mathfrak{C}_g/M_g) & \longrightarrow & \mathfrak{A}_g \\ & \searrow & \swarrow & & \downarrow \pi \\ & & M_g & \longrightarrow & \mathbb{A}_g \end{array}$$

so that now, we wish to study subvarieties of $\mathfrak{A}_g$.

Non-degeneracy: geometric definition

On $\mathfrak{A}_g \to \mathbb{A}_g$, there is a canonial symmetric relatively ample line bundle $\mathfrak{L}_g$, which is trivial along the zero section. Consider the $(1,1)$-form $c_1(\mathfrak{L}_g)$ (representative of the first Chern class). It is a non-negative form as $\mathfrak{L}_g$ is nef.

Definition

A subvariety X of $\mathfrak{A}_g$ is said to be non-degenerate if

$$c_1(\mathfrak{L}_g)|_X^{\wedge \dim X} \not\equiv 0.$$

Vaguely speaking, this means that $\mathfrak{L}_g|_X$ is big.

Non-degeneracy: a more Diophantine definition

Let $\mathcal{H}_g$ be the Siegel upper half space. Then we have a uniformization $\mathcal{H}_g \to \mathbb{A}_g$ in the category of complex spaces.
Construct a similar uniformization $\mathcal{X}_{2g}$ of the universal abelian variety $\mathfrak{A}_g$ as follows:

- real-algebraic: $\mathcal{X}_{2g} = \mathbb{R}^{2g} \times \mathcal{H}_g$;
- complex structure: $\mathbb{R}^{2g} \times \mathcal{H}_g \xrightarrow{\sim} \mathbb{C}^g \times \mathcal{H}_g$, $(a, b, Z) \mapsto (a + Zb, Z)$.

Then the 2-form $da \wedge db$ descends to a $(1,1)$-form on $\mathfrak{A}_g$, which is a representative of $c_1(\mathfrak{L}_g)$ (N. Mok).
⇝ Upshot: the kernel of the skew-symmetric bilinear form defined by $c_1(\mathfrak{L}_g)$ is precisely $\{\{r\} \times \mathcal{H}_g : r \in \mathbb{R}^{2g}\}$.

Non-degeneracy in Diophantine way: Betti map

$u\colon \mathcal{X}_{2g} \to \mathfrak{A}_g$.

Definition (Bertrand, Zannier, Masser, Corvaja, André)

The Betti map is defined to be the natural projection $b\colon \mathcal{X}_{2g} \to \mathbb{R}^{2g}$. It is a real-analytic map with complex fibers.

By the discussion in the previous slide, we have the following definition for non-degeneracy.

Definition

A subvariety X of $\mathfrak{A}_g$ is said to be non-degenerate if

$$\mathrm{rk}_{\mathbb{R}} db|_{\tilde{X}} = 2\dim X.$$

Here $\widetilde{X}$ is an irreducible component of $u^{-1}(X)$.

In particular, X is always degenerate if $\dim X > g$.

Functional Transcendence: (weak) Ax-Schanuel

Question ((weak) Ax-Schanuel)

Let $q\colon \Omega \to S$ be a surjective holomorphic morphism between algebraic varieties. Let $Z \subseteq \Omega$ be complex analytic. Then

$$\dim \overline{Z}^{\mathrm{Zar}} + \dim \overline{q(Z)}^{\mathrm{Zar}} \geq \dim Z + \dim \overline{q(Z)}^{\mathrm{biZar}}.$$

Here $\overline{q(Z)}^{\mathrm{biZar}}$ means the smallest bi-algebraic subvariety of S containing $q(Z)$, where bi-algebraic means "both algebraic in Ω and in S".

Theorem (Ax, 1971, 1973)

Ax-Schanuel holds for semi-abelian varieties.

Functional Transcendence: (weak) Ax-Schanuel

Theorem (Mok–Pila–Tsimerman, 2019 Ann.)

Ax-Schanuel holds for $\mathcal{H}_g \to \mathbb{A}_g$.

Extensions of Mok–Pila–Tsimerman in two directions.

Theorem (Bakker–Tsimerman, 2019 Inv.)

Ax-Schanuel holds for variations of pure Hodge structures.

Theorem (G', 2018 preprint)

Ax-Schanuel holds for $u\colon \mathcal{X}_{2g} \to \mathfrak{A}_g$ *(mixed Shimura variety).*

We will use this mixed version to study the Betti map.

From non-degeneracy to unlikely intersection

For the notation

$$\begin{array}{ccc} \mathcal{X}_{2g} = \mathbb{R}^{2g} \times \mathcal{H}_g & \xrightarrow{u} & \mathfrak{A}_g \\ \downarrow & & \downarrow \pi \\ \mathcal{H}_g & \longrightarrow & \mathbb{A}_g \end{array}$$

with $X \subseteq \mathfrak{A}_g$ and $\widetilde{X}$ a component of $u^{-1}(X)$, we have

$$\begin{aligned} & X \text{ is degenerate} \\ \Leftrightarrow\ & \widetilde{X} = \bigcup_{r \in \mathbb{R}^{2g},\, \widetilde{C} \text{ curve in } \mathcal{H}_g} (\{r\} \times \widetilde{C}) \\ \Leftrightarrow\ & X = \bigcup_{r \in \mathbb{R}^{2g},\, \widetilde{C} \text{ curve in } \mathcal{H}_g} u(\{r\} \times \widetilde{C}) \end{aligned}$$

From non-degeneracy to unlikely intersection

So X is degenerate $\Rightarrow X = \bigcup_{r\in\mathbb{R}^{2g}, \widetilde{C} \text{ curve in } \mathcal{H}_g} \overline{u(\{r\}\times\widetilde{C})}^{\mathrm{Zar}}$.
Now let us study

$$Y := \overline{u(\{r\}\times\widetilde{C})}^{\mathrm{Zar}}$$

Apply mixed Ax-Schanuel to $Z = \{r\}\times\widetilde{C}$ (version of G'). We get

$$\dim \overline{Z}^{\mathrm{Zar}} + \dim \overline{u(Z)}^{\mathrm{Zar}} \geq \dim Z + \dim \overline{u(Z)}^{\mathrm{biZar}}.$$

It then becomes

$$\dim(\{r\}\times\widetilde{C}^{\mathrm{Zar}}) + \dim Y > \dim \overline{Y}^{\mathrm{biZar}}.$$

As $\dim \widetilde{C}^{\mathrm{Zar}} \leq \dim \overline{\pi(Y)}^{\mathrm{biZar}}$, we have

$$\dim Y > \dim \overline{Y}^{\mathrm{biZar}} - \dim \overline{\pi(Y)}^{\mathrm{biZar}}.$$

From non-degeneracy to unlikely intersection

So we have

$$X \text{ is degenerate} \Leftrightarrow X = \bigcup_{\dim Y > \dim \overline{Y}^{\mathrm{biZar}} - \dim \overline{\pi(Y)}^{\mathrm{biZar}}} Y.$$

The previous slide showed $\Rightarrow$ using mixed Ax-Schanuel. The direction $\Leftarrow$ follows directly from the description of bi-algebraic subvarieties in $\mathfrak{A}_g$.

We will show that the union on the right hand side is actually a finite union, and get a criterion to degeneracy from this!

Finiteness of the union

Theorem (Bogomolov, '70)

Let A be an abelian variety and let X be a subvariety. There are only finitely many abelian subvarieties B of A with $\dim B > 0$ satisfying:
(1) $a + B \subseteq X$ for some $a \in A$;
(2) B is maximal for the property described in (1).

- Ullmo (2014) proved the corresponding result for pure Shimura varieties, for the purpose of studying the André-Oort conjecture.
- Habegger–Pila (2016) introduced the notion of *weakly optimal* subvarieties when studying the more general Zilber-Pink conjecture. They also proved the corresponding finiteness result for the case $Y(1)^N$.
- Daw–Ren (2018) proved the finiteness result for pure Shimura varieties.

Criterion to degeneracy

In a preprint (2018), G' extended Daw–Ren's result to $\mathfrak{A}_g$ (mixed Shimura varieties).
Applying this extension to the union in earlier slides (and some other results), we get the following criterion for degeneracy. To ease notation let us denote by $S = \pi(X) \subseteq \mathbb{A}_g$, and $\mathcal{A} := \mathfrak{A}_g \times_{\mathbb{A}_g} S$. We also assume that X is not contained in a proper subgroup scheme of $\mathcal{A} \to S$.

Theorem (G', 2018 preprint)

X is degenerate if and only if there exists an abelian subscheme $\mathcal{B}$ of $\mathcal{A} \to S$ such that $\dim \varphi(X) < \dim X - (g - g')$.

$$\begin{array}{ccccc} \varphi : \mathcal{A} & \longrightarrow & \mathcal{A}/\mathcal{B} & \longrightarrow & \mathfrak{A}_{g'} \\ \downarrow & & \downarrow \lrcorner & & \downarrow \\ S & \xrightarrow{=} & S & \longrightarrow & \mathbb{A}_{g'} \end{array}$$

Back to Mazur's Conjecture

Back to our original question (study $X = \mathfrak{C}_g - \mathfrak{C}_g$). It is clear, by dimension reasons, that X is degenerate. But applying the criterion in the previous slide, one can easily show that $X^m := X \times_{\mathbb{M}_g} \ldots \times_{\mathbb{M}_g} X$ (m-copies) is non-degenerate when $m \geq 3g-3$, if $g \geq 3$.
⇝ Upshot: Around each algebraic point in $J(\overline{\mathbb{Q}})$, there are at most $3g-3$ points which are NOT far from it.
⇝ This gives the desired bound up to some packing argument.
In fact, it is better to consider the Faltings–Zhang map

$$D_m \colon \mathfrak{A}_g^{m+1} \to \mathfrak{A}_g^m$$

fiberwise defined by $(P_0, P_1, \ldots, P_m) \mapsto (P_1 - P_0, \ldots, P_m - P_0)$. Then $D_m(\mathfrak{C}_g^{m+1})$ is non-degenerate for $m \geq 3g-2$, if $g \geq 2$.