p-Converse Theorem (CM Case)

Ye Tian

Morningside Center of Mathematics

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(Joint work with Burungale and Skinner)

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   They fit into an short exact sequence:

$$0 \to E(F) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}_{p^\infty}(E/F) \to \mathrm{III}(E/F)[p^\infty] \to 0$$

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Moreover, under these conditions, the formula holds:

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The p-part of BSD formula for E means that both sides (conjecturally to be rational numbers) have the same p-valuation.

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$$E \in \{ny^2 = x^3 + ax + b, \ n \in F^\times/F^{\times 2}\}$$

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Goldfeld conjecture for the above family will follows if one can show the $p = 2$-converse ($r = 0, 1$) for them.

Results on p-Converse

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$$\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E/F) = 1 \quad \Longrightarrow \quad \mathrm{ord}_{s=1}L(E/F,s) = 1,$$

provided that $p \nmid 6D_F N_E h^-_{MF/F}$. Here N_E is the conductor of E, M is the CM field of E and h^- is the relative class number.

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The Theorem of Gross-Zagier implies that

$$\mathrm{ord}_{s=1} L(E_K, s) = 1 \iff y_K \neq 0 \text{ in } E(K) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

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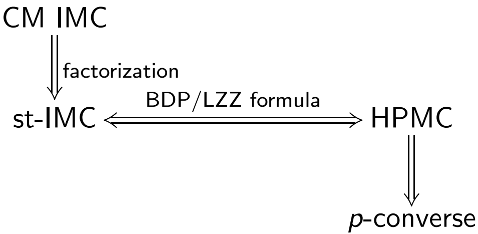

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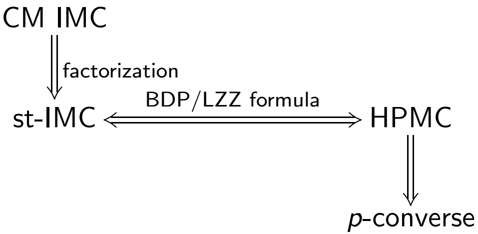

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Theorem (Hsieh)

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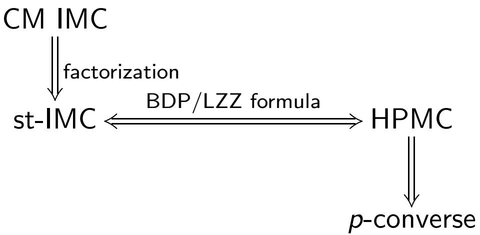

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\begin{array}{ccc}
\text{CM IMC} & & \\
\Big\Downarrow \text{factorization} & & \\
\text{st-IMC} & \xLeftrightarrow{\text{BDP/LZZ formula}} & \text{HPMC} \\
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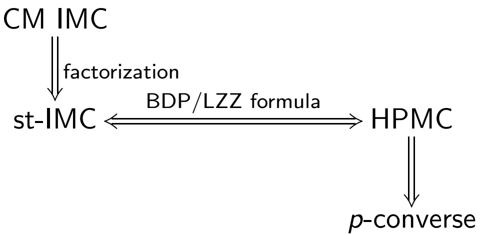

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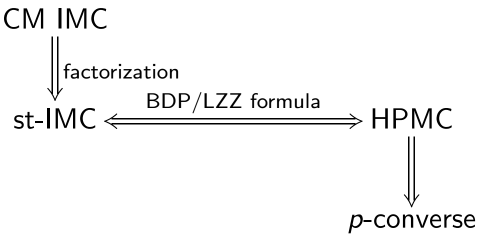

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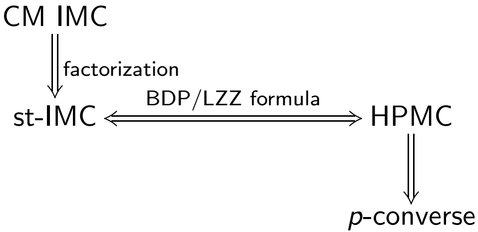

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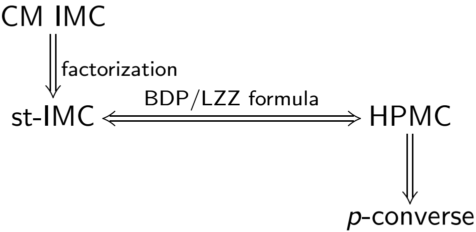

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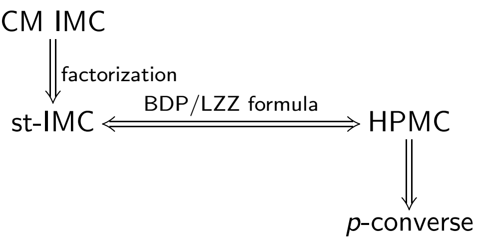

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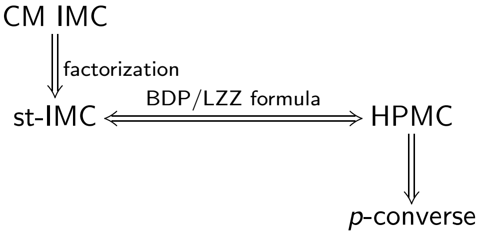

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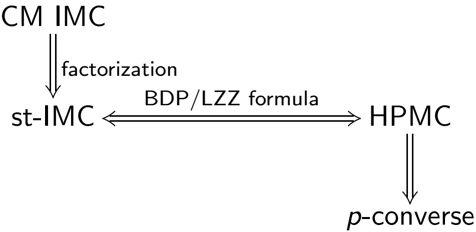

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Thanks

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