Bounds for $GL(3) \times GL(2)$ Rankin-Selberg L-functions

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Automorphic L-function $L(s, \pi)$

Let π be an automorphic form for $GL_d(\mathbb{A}_\mathbb{Q})$. Then $\pi = \otimes_p \pi_p$, and we have the associated L-function

$$L(s, \pi) = \prod_{p \text{ prime}} L_p(s, \pi) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{n^s}$$

with

$$L_p(s, \pi) = L(s, \pi_p) = \prod_{i=1}^{d} (1 - \alpha_i(p) p^{-s})^{-1}.$$

The local factor at infinity is given by

$$L(s, \pi_\infty) = \prod_{i=1}^{d} \Gamma_\mathbb{R}(s - \mu_i)$$

where

$$\Gamma_\mathbb{R}(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right).$$

- Converge absolutely for $\mathrm{Re}(s) = \sigma > \sigma_0$ and extend to a meromorphic function on $\mathbb{C}$.
- Functional equation

$$\Lambda(s, \pi) := q_\pi^{s/2} L(s, \pi_\infty) L(s, \pi) = \omega_\pi \Lambda(1 - s, \tilde{\pi})$$

for some $q_\pi \in \mathbb{N}$, $|\omega_\pi| = 1$ (Godement-Jacquet).
- Note: The centre is at $s = 1/2$.

Main Conjectures

Grand Riemann Hypothesis

All non-trivial zeros of $L(s, \pi)$ are on the central line

$$\frac{1}{2} + it$$

Generalised Lindelöf Hypothesis

Analytic conductor:

$$C(\pi, t) = q_\pi \prod_{i=1}^{d} (1 + |\mu_i + it|) \asymp \begin{cases} q_\pi & \text{level} \\ t^d & t \\ \Lambda_\pi = \prod \mu_i & \text{spectral} \end{cases}$$

GRH $\implies$ For any $\varepsilon > 0$ one has

$$|L(\tfrac{1}{2} + it, \pi)| \leq c(\varepsilon) C(\pi, t)^\varepsilon$$

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GRH $\implies$ For any $\varepsilon > 0$ one has

$$L(\tfrac{1}{2} + it, \pi) \ll C(\pi,t)^{\varepsilon}.$$

The Subconvexity Problem

Convexity bound

Functional equation + Phragmen-Lindelöf principle $\implies$

$$L(\tfrac{1}{2} + it, \pi) \ll C(\pi, t)^{\frac{1}{4}+\varepsilon}.$$

The Subconvexity Problem

To establish a bound of the form (with $\delta > 0$)

$$L(\tfrac{1}{2} + it, \pi) \ll C(\pi, t)^{\frac{1}{4}-\delta}.$$

We can also formulate the problem w.r.t. each parameter separately (level aspect q_π, t-aspect, depth aspect μ_i)

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$$L(\tfrac{1}{2}+it,\pi) \ll t^{\frac{d}{4}-\delta}.$$

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Convexity bound

Functional equation + Phragmen-Lindelöf principle $\Longrightarrow$

$$L(\tfrac{1}{2}+it,\pi) \ll C(\pi,t)^{\frac{1}{4}+\varepsilon}.$$

The Subconvexity Problem

To establish a bound of the form (with $\delta > 0$)

$$L(\tfrac{1}{2}+it,\pi) \ll \Lambda_\pi^{\frac{1}{4}-\delta}.$$

We can also formulate the problem w.r.t. each parameters separately (t-aspect, level aspect q_π, depth aspect μ_i)

Examples of subconvex bounds: Degree 1

L-function	Subconvex bounds
$\zeta(s)$	$\zeta(1/2+it) \ll t^{\frac{1}{4}-\frac{1}{12}+\varepsilon} = t^{\frac{1}{6}+\varepsilon}$ Weyl-Hardy-Littlewood (1920)
	$\zeta(1/2+it) \ll t^{\frac{1}{6}-\frac{1}{84}+\varepsilon}$ (Bombieri-Iwaniec) Bourgain (2017)
$L(s,\chi)$ χ mod M	$L(1/2,\chi) \ll M^{\frac{1}{4}-\frac{1}{16}+\varepsilon}$ Burgess (1962)
$L(s,\chi)$, $\chi^2 = 1$	$L(1/2,\chi) \ll M^{\frac{1}{4}-\frac{1}{12}+\varepsilon}$ Conrey-Iwaniec (2000) ($\chi^2 \neq 1$: Petrow-Young (preprint 2018))

Degree 2

L-function	Subconvex bounds
$L(s, f)$ f modular or Maass	$L(1/2 + it, f) \ll t^{\frac{1}{2}-\frac{1}{6}+\varepsilon} = t^{\frac{1}{3}+\varepsilon}$ Good (1980) Meurman
$L(s, f)$	$L(1/2, f) \ll q_f^{\frac{1}{4}-\delta}$ Duke-Friedlander-Iwaniec (1994, 2001, 2002) Blomer-Harcos-Michel (2007) $\delta = 1/1889$
	$L(1/2, f) \ll \Lambda_\pi^{\frac{1}{6}+\varepsilon}$ Iwaniec (1992) (conditional), Ivic (2001)
$L(s, f \otimes \chi)$ χ mod M	$L(1/2, f \otimes \chi) \ll M^{\frac{1}{2}-\frac{1}{6}+\varepsilon}$ DFI (1992), Blomer-Harcos (2008)

Degree 2×2

L-function	Subconvex bounds
$L(s, f_1 \otimes f_2)$ f_i modular wt k_i	$L(1/2, f_1 \otimes f_2) \ll_{f_2} k_1^{1-\frac{7}{165}+\varepsilon}$ Sarnak (2001)
f_i Maass/modular	$L(1/2, f_1 \otimes f_2) \ll_{f_2} k_1^{1-\frac{1}{3}+\varepsilon}$ Lau-Liu-Ye (2006)
levels q_i	$L(1/2, f_1 \otimes f_2) \ll_{f_2} q_1^{\frac{1}{2}-\delta}$ Kowalski-Michel-Vanderkam (2002) $\delta = 1/80$ Michel (2004), Harcos-Michel (2006)
$(q_1, q_2) = 1$	$L(1/2, f_1 \otimes f_2) \ll (q_1 q_2)^{\frac{1}{2}-\delta}$ Holowinsky-M. (2013), Raju (preprint 2018)

Degree 3

$L(s, \mathrm{Sym}^2 f)$ f modular/Maass	$L(1/2+it, \mathrm{Sym}^2 f) \ll t^{\frac{3}{4}-\frac{1}{16}+\varepsilon}$ X. Li (2011)
π Hecke-Maass for $SL_3(\mathbb{Z})$	$L(1/2+it, \pi) \ll t^{\frac{11}{16}+\varepsilon}$ M. (2015) Aggarwal (preprint 2019) $3/4 - 3/40 + \varepsilon$
χ mod M generalization	$L(1/2, \pi \otimes \chi) \ll_\pi M^{\frac{3}{4}-\delta}$ M. (2015, arxiv 2016) Holowinsky-Nelson (2018) $\delta = 1/36$ Kowalski-Lin-Michel-Sawin (preprint 2019)
generic spec. par.	$L(1/2, \pi) \ll \Lambda_\pi^{\frac{1}{4}-\delta}$ Blomer-Buttcane (2019)

Higher degree

L-function	Subconvex bounds
$L(s, f_1 \otimes f_2 \otimes f_3)$ f_i Maass forms spectral para. t_i	$L(1/2, f_1 \otimes f_2 \otimes f_3) \ll_{f_1,f_2} t_3^{2-\frac{1}{3}+\varepsilon}$ Bernstein-Reznikov (2005)
f_3 of level p	$L(1/2, f_1 \otimes f_2 \otimes f_3) \ll_{f_1,f_2} p^{1-\frac{1}{13}+\varepsilon}$ Venkatesh (2010) Woodbury (thesis 2011) Hu (2017)
g modular/Maass wt./spec. par. k	$L(1/2, \mathrm{Sym}^2 f \otimes g) \ll_f k^{\frac{3}{2}-\frac{1}{8}+\varepsilon}$ X. Li (2011)

General results

- Diaconu-Garrett (2010): π $GL(2)$ automorphic form over a number field K of degree d,

$$L(1/2+it,\pi) \ll_\pi t^{\frac{d}{2}-\delta}$$

- Michel-Venkatesh (2010): π_i $GL(2)$ automorphic forms

$$L(1/2,\pi_1\otimes\pi_2) \ll_{\pi_1} C(\pi_1\otimes\pi_2)^{\frac{1}{4}-\delta}$$

- Soundararajan (2010), Soundararajan-Thorner (2019) π $GL(d)$ automorphic form

$$L(1/2,\pi) \ll \frac{C(\pi)^{1/4}}{(\log C(\pi))^{\delta}}$$

- Bernstein-Reznikov (2005), Venkatesh (2010)

New result

Theorem

Let f be a $SL_2(\mathbb{Z})$ modular/Maass form, and π be a $SL_3(\mathbb{Z})$ Maass form, then

$$L(\tfrac{1}{2}+it, \pi \otimes f) \ll t^{\frac{3}{2}-\frac{1}{42}+\varepsilon}.$$

(As usual we will take $t > 2$.)

- The proof is based on separation of oscillation via circle method
- It is not sensitive to f or π being cuspidal or not. Hence as a corollary we obtain:

$$L(\tfrac{1}{2}+it, \chi) \ll_{\chi} t^{\frac{1}{4}-\frac{1}{252}+\varepsilon}$$

$$L(\tfrac{1}{2}+it, f) \ll_{f} t^{\frac{1}{2}-\frac{1}{126}+\varepsilon}.$$

$$L(\tfrac{1}{2}+it, \pi) \ll_{\pi} t^{\frac{3}{4}-\frac{1}{84}+\varepsilon}.$$

Related results

Theorem (Sharma arxiv 2019)

Let f be a $SL_2(\mathbb{Z})$ modular/Maass form, and π be a $SL_3(\mathbb{Z})$ Maass form, and $\chi \bmod p$ then

$$L(\tfrac{1}{2}, \pi \otimes f \otimes \chi) \ll p^{\frac{3}{2} - \frac{1}{32} + \varepsilon}.$$

- As before taking Eisenstein series in place of f or π or both, we get subconvex bounds for $L(1/2, \pi \otimes \chi)$, $L(1/2, f \otimes \chi)$ and $L(1/2, \chi)$

Theorem (Kumar upcoming)

Let f be a $SL_2(\mathbb{Z})$ modular/Maass form with weight/spectral parameter k, and π be a $SL_3(\mathbb{Z})$ Maass form, then

$$L(\tfrac{1}{2}, \pi \otimes f) \ll k^{\frac{3}{2} - \delta}.$$

- This extends the result of X. Li to non self-dual.

Related results 2

Theorem (Kumar-Mallesham-Singh upcoming)

Let f be a $SL_2(\mathbb{Z})$ modular/Maass form, and π be a $SL_3(\mathbb{Z})$ Maass form with spectral parameters in generic position. Suppose

$$\sum_{L<\ell\leq 2L} |\lambda_\pi(1,\ell)|^2 \gg L^{1/2+\eta}$$

with $\eta > 0$. Then

$$L(\tfrac{1}{2}, \pi \otimes f) \ll \Lambda_\pi^{\frac{1}{4}-\delta}.$$

- Not clear how to get rid off the irritating condition!
- Without the condition the result would give a generalization of the work of Blomer-Buttcane (2019).

Sketch of proof

- Approximate functional equation:

$$L(s, \pi \otimes f) = \sum_{n,r=1}^{\infty}\sum \frac{\lambda_\pi(n, r)\lambda_f(n)}{(nr^2)^s}.$$

AFE implies that

$$L(\tfrac{1}{2} + it, \pi \otimes f) \ll t^\varepsilon \sup_{N \ll t^{3+\varepsilon}} \frac{|S(N)|}{\sqrt{N}} + t^{-2019}$$

with

$$S(N) = \sum_{n,r=1}^{\infty}\sum \lambda_\pi(n, r)\lambda_f(n)(nr^2)^{it} V\left(\frac{nr^2}{N}\right).$$

- Trivial estimation of $S(N)$ yields convexity.

Sketch of proof

- We will establish non-trivial cancellation in

$$S(N) = \sum_{n \sim N} \lambda_\pi(n,1)\lambda_f(n)n^{it}$$

  for $N = t^3$.
- We write

$$S(N) = \sum\sum_{n,m \sim N} \lambda_\pi(n,1)\lambda_f(m)m^{it}\delta(n,m)$$
$$= \frac{1}{K}\int W\left(\frac{v}{K}\right)\sum\sum_{n,m \sim N} \lambda_\pi(n,1)n^{iv}\lambda_f(m)m^{it-iv}\delta(n,m)$$

- δ-method: We use a Fourier expansion of δ

$$\delta(n,m) = \frac{1}{Q}\sum_{q \ll Q} \frac{1}{q}\sum_{a \bmod q}^{\star} e\left(\frac{a(n-m)}{q}\right) h\left(\frac{q}{Q}, \frac{n-m}{qQ}\right)$$

  where h is a reasonably nice function.

Sketch of proof

- We pick
$$Q = \sqrt{N/K}.$$
- Roughly speaking we have
$$S(N) = \frac{1}{KQ^2} \int_{v \sim K} \sum_{q \sim Q} \sideset{}{^\star}\sum_{a \bmod q} \sum_{n \sim N} \lambda_\pi(n,1) n^{iv} e\left(\frac{an}{q}\right) \times \sum_{m \sim N} \lambda_f(m) m^{it-iv} e\left(-\frac{am}{q}\right)$$
- Trivial estimation at this point yields $S(N) \ll N^2$.

Sketch of proof

- Next we apply Voronoi summation formulae.
- $GL(3)$ Voronoi transfers

$$\sum_{n\sim N} \lambda_\pi(n,1) n^{iv} e\left(\frac{an}{q}\right) \mapsto \sum_{n\sim K^{3/2}N^{1/2}} \lambda_\pi(1,n) S(-\bar{a},n;q) \int (\dots),$$

  and gives a saving of $N^{1/4}/K^{3/4}$.
- $GL(2)$ Voronoi transfers

$$\sum_{m\sim N} \lambda_f(m) m^{it-iv} e\left(-\frac{am}{q}\right) \mapsto \sum_{m\sim t^2/K} \lambda_f(m) e\left(\frac{\bar{a}m}{q}\right) \int (\dots),$$

  and gives a saving of $(NK)^{1/2}/t$.

Sketch of proof

- Next we see that the character sum

$$\sum_{a \bmod q}^{\star} S(-\bar{a}, n; q) e\left(\frac{\bar{a}m}{q}\right) \approx qe\left(-\frac{\bar{m}n}{q}\right),$$

  and we save $\sqrt{Q}$.
- We also save $\sqrt{K}$ in the integral over v.
- It remains to save t (plus extra) in the sum

$$\sum_{q\sim Q} \sum_{n\sim K^{3/2}N^{1/2}} \lambda_\pi(1,n) \sum_{m\sim t^2/K} \lambda_f(m)\, e\left(-\frac{\bar{m}n}{q}\right) \mathcal{I}$$

Sketch of proof

- Now we apply Cauchy inequality, and our job reduces to saving t^2 (plus extra) in

$$\sum_{n\sim K^{3/2}N^{1/2}} \left| \sum_{q\sim Q} \sum_{m\sim t^2/K} \lambda_f(m)\, e\left(-\frac{\bar{m}n}{q}\right)\, \mathcal{I} \right|^2$$

- Open absolute square and apply Poisson summation formula. In the diagonal (zero frequency) we save Qt^2/K which is sufficient if $K < t$.
- In the off-diagonal we save $K^{3/2}N^{1/2}/K^{1/2}$ which is sufficient if $K > t^{1/2}$.
- Notice the structural advantage. If one had a generic character sum in place of $e(-\bar{m}n/q)$ then one would save $K^{3/2}N^{1/2}/QK^{1/2}$ which would be sufficient if $K > t^{4/3}$!

Limit of the method

- Suppose $P(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ irreducible of degree d. Consider the counting problem

$$N(B) = \#\{(x_1, \ldots, x_n) \in \mathbb{Z}^n \cap [-B, B]^n : P(x_1, \ldots, x_n) = 0\}$$

- δ-method (circle method w/o minor arc) can give asymptotic for $N(B)$ (for n sufficiently large) for $d \leq 3$. But fails for $d \geq 4$.
- However δ-method can be modified to count points on degree $d = 4$ varieties arising as complete intersection of two quadrics, or $d = 6$ varieties given by a cubic and a quadratic.
- To go beyond degree $d = 3$ one needs non-trivial analysis of minor arc. Is higher rank δ method (trace formula) the right way to introduce minor arcs in the automorphic setting?

Thank You!