The RM values of the Dedekind-Rademacher cocycle

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Complex multiplication and real quadratic fields

The values of modular forms at CM points, indexed by ideal classes of imaginary quadratic orders, are of great arithmetic interest.

They generate explicit class fields of imaginary quadratic fields, and give several basic instances of "Euler systems": Elliptic units, and Heegner points. (Cf. the lectures of Ye Tian and Chris Skinner on the first day.)

Question: Is there a similar theory when quadratic imaginary fields are replaced by real quadratic fields?

Hope: The theory of complex multiplication *does* extend to this setting, after replacing classical modular forms by less familiar avatars: so-called *rigid analytic* or *meromorphic cocycles*.

Origins: p-adic uniformisation and Ceřednik-Drinfeld

$\mathbb{H} := \mathbb{Q}[1, i, j, k]$, the skew field of rational Hamilton quaternions;

$R := \mathbb{Z}[1, i, j, \frac{1+i+j+k}{2}]$ Hurwitz's maximal order.

Let $\Gamma := (R[1/p])_1^\times$.

Jacquet-Langlands: The first cohomology $H^1(\Gamma, \mathbb{Q})$ is (non-canonically) isomorphic to $S_2(\Gamma_0(2p))^{\mathrm{new}}$ as a Hecke module.

Modularity: If E is an elliptic curve over $\mathbb{Q}$ of conductor $2p$, there is a Hecke eigenclass $\varphi_E \in H^1(\Gamma, \mathbb{Z})$ satisfying

$$T_\ell(\varphi_E) = a_\ell(E) \cdot \varphi_E, \quad \text{for all } \ell \neq 2, p,$$

where $a_\ell(E) = \ell + 1 - \#E(\mathbb{Z}/\ell\mathbb{Z})$.

Rigid analytic functions

The cocycle φ_E encodes the periods of a *rigid analytic function*.

Let $\mathcal{H}_p := \mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p)$ be Drinfeld's p-adic upper half-plane.

For $z := (z_1 \colon z_2) \in \mathbb{P}_1(\mathbb{C}_p)$ with $z_1, z_2 \in (\mathcal{O}_{\mathbb{C}_p}^2)'$,

$$\mathcal{H}_p^{\geq\epsilon} := \left\{ z \text{ with } |az_1 - bz_2| \geq \epsilon, \text{ for all } (a,b) \in (\mathbb{Z}_p^2)' \right\},$$

$$\mathcal{H}_p = \bigcup_{\epsilon>0} \mathcal{H}_p^{\geq\epsilon}.$$

Definition. A function on $\mathcal{H}_p$ is *rigid analytic* if its restriction to all $\mathcal{H}_p^{\geq\epsilon}$ is a uniform limit of rational functions with poles outside $\mathcal{H}_p^{\geq\epsilon}$.

When p is an odd prime, fixing $\mathbb{H} \otimes \mathbb{Q}_p = M_2(\mathbb{Q}_p)$, the group Γ acts discretely on $\mathcal{H}_p$ by Möbius transformations.

Trivialising the one-cocycle φ_E

$\mathcal{A}^\times :=$ group of nowhere vanishing rigid analytic functions on $\mathcal{H}_p$.

Theorem. (Mumford-Tate, Cerednik-Drinfeld). Let q_E be the Tate period of $E_{/\mathbb{Q}_p}$. Then there is a rigid analytic function $J_E \in \mathcal{A}^\times$ satisfying

$$J_E(\gamma z)/J_E(z) = q_E^{\varphi_E(\gamma)}, \quad \text{for all } \gamma \in \Gamma.$$

The function $J_E \in H^0(\Gamma, \mathcal{A}^\times/q_E^{\mathbb{Z}})$ gives a *modular parametrisation* from the Shimura curve of level $2p$ to E:

$$J_E : \Gamma \backslash \mathcal{H}_p \longrightarrow \mathbb{C}_p^\times/q_E^{\mathbb{Z}} = E(\mathbb{C}_p).$$

Heegner points arising from Shimura curves

Definition. A CM point of $\mathcal{H}_p$ is a point τ satisfying $\mathrm{Stab}_{\mathbb{H}^\times}(\tau) = K^\times$, where K is a quadratic subfield of $\mathbb{H}$. (Necessarily, K is *imaginary*, and 2 and p are non-split.)

Theorem. If Δ is a degree 0 divisor on $\mathcal{H}_p$ supported on *CM* points attached to K, then

$$J_E(\Delta) \in E(\mathbb{C}_p)$$

is algebraic, and defined over a ring class field of K.

The numerical calculation of $J_E(\Delta)$ can be carried out efficiently on a computer, thanks to ideas of Pollack-Stevens and Matthew Greenberg.

p-adic uniformisation via $\mathbf{SL}_2(\mathbb{Z}[1/p])$

Let $\Gamma = \mathbf{SL}_2(\mathbb{Z}[1/p])$, acting on $\mathcal{H}_p$ by Mobius transformations.

It does not act discretely, and $H^1(\Gamma, \mathbb{Q}) = 0$.

The interesting cohomology occurs in degree 2.

Because $\Gamma = \mathbf{SL}_2(\mathbb{Z}) *_{\Gamma_0(p)} \mathbf{SL}_2(\mathbb{Z})$,

$$H^2(\Gamma, \mathbb{Q}) = H^1(\Gamma_0(p), \mathbb{Q}) \text{ as a Hecke module.}$$

Modularity: If E is an elliptic curve over $\mathbb{Q}$ of conductor p, there is a Hecke eigenclass $\varphi_E \in H^2(\Gamma, \mathbb{Z})$ satisfying

$$T_\ell(\varphi_E) = a_\ell(E) \cdot \varphi_E, \qquad \text{for all} \qquad \ell \neq p.$$

Trivialising the two-cocyle φ_E

Theorem. (D, 2000). There is a one-cochain $J_E : \Gamma \longrightarrow \mathcal{A}^\times$ satisfying, for all $\gamma_1, \gamma_2 \in \Gamma$:

$$J_E(\gamma_1) \times J_E(\gamma_1\gamma_2)^{-1} \times \gamma_1(J_E(\gamma_2)) = q_E^{\varphi_E(\gamma_1,\gamma_2)}.$$

This theorem is a *formal consequence* of a 1986 conjecture of Mazur, Tate and Teitelbaum, proved by Greenberg-Stevens in 1990. Thus, it was proved 10 years before it was stated!

Key ingredient in Greenberg-Stevens: p-adic deformations of modular forms and Galois representations, and Galois cohomology.

This proof *differs markedly* from that of Cerednik-Drinfeld, which relies on more geometric ideas.

Rigid analytic cocycles

Definition. The class of J_E in $H^1(\Gamma, \mathcal{A}^\times/q_E^{\mathbb{Z}})$ is called the *rigid analytic cocycle* attached to E.

For $\tau \in \mathcal{H}_p$, there is an evaluation map $\mathrm{ev}_\tau : \mathcal{A}^\times \longrightarrow \mathbb{C}_p^\times$,

$$\mathrm{ev}_\tau : H^1(\Gamma, \mathcal{A}^\times/q_E^{\mathbb{Z}}) \longrightarrow H^1(\Gamma_\tau, \mathbb{C}_p^\times/q_E^{\mathbb{Z}}).$$

If $\Gamma_\tau = 1$, the target is trivial. It is non-trivial when $\Gamma_\tau \simeq \mathbb{Z}$, which occurs precisely when $\mathbb{Q}(\tau)$ is real quadratic.

$$J_E[\tau] := J_E(\gamma_\tau)(\tau), \qquad \langle \gamma_\tau \rangle = \mathrm{Stab}_\Gamma(\tau).$$

This quantity is called the *value* of J_E at the RM point τ.

Stark-Heegner points

Conjecture (D, 2000)

If τ is an RM point and $F = \mathbb{Q}(\tau)$, then $J_E[\tau]$ is a global point on E, defined over a ring class field of F.

The RM values $J_E[\tau]$ are called *Stark-Heegner points*: they are to Heegner points what Stark units are to elliptic units.

The "Stark-Heegner point conjecture" touches on the basic mystery of constructing algebraic points on elliptic curves, and remains completely open.

The Dedekind-Rademacher cocycle

Samit Dasgupta's thesis (2004): The group $H^2(\Gamma, \mathbb{Z}) = H^1(\Gamma_0(p), \mathbb{Z})$ contains a class φ_{DR} that is *Eisenstein*, the Dedekind Rademacher homomorphism encoding the periods of the modular unit $\Delta(pz)/\Delta(z)$.

Theorem. (Samit Dasgupta, D, 2003; Jan Vonk, D, 2018). There is a one-cochain $J_{\mathrm{DR}} : \Gamma \longrightarrow \mathcal{A}^\times$ satisfying, for all $\gamma_1, \gamma_2 \in \Gamma$:

$$J_{\mathrm{DR}}(\gamma_1) \times J_{\mathrm{DR}}(\gamma_1\gamma_2)^{-1} \times \gamma_1(J_{\mathrm{DR}}(\gamma_2)) = p^{\varphi_{\mathrm{DR}}(\gamma_1,\gamma_2)}.$$

Its image in $H^1(\Gamma, \mathcal{A}^\times/p^{\mathbb{Z}})$ is the *Dedekind-Rademacher cocycle*.

The RM values of the Dedekind-Rademacher cocycle

Conjecture (Dasgupta, D, 2003)

If τ is an RM point and $F = \mathbb{Q}(\tau)$, then $J_{\mathrm{DR}}[\tau]$ is a global p-unit in a ring class field of F.

This conjecture is much more tractable than the conjecture on Stark-Heegner points!

There is now substantial theoretical evidence for it.

Gross-Stark units

The conjectural p-units $J_{\mathrm{DR}}[\tau]$ can be related to Gross-Stark units.

$F = \mathbb{Q}(\tau)$, $\quad \psi : \mathrm{Cl}^{+}(F) \longrightarrow \mathbb{C}$ the indicator function of $[1, \tau]$.

$L(F, \psi, s) = \sum_{I \triangleleft \mathcal{O}_F} \psi(I) \mathrm{Nm}(I)^{-s}$,

$L_p(F, \psi, s) =$ the associated Deligne-Ribet p-adic L-function.

$J^{+}_{\mathrm{DR}}[\tau] := J_{\mathrm{DR}}[\tau] \times J_{\mathrm{DR}}[\tau'] = \mathrm{Norm}_{\mathbb{Q}_p(\tau)/\mathbb{Q}_p} J_{\mathrm{DR}}[\tau]$.

Theorem (Dasgupta, D, 2003)

The quantity $\log_p(J^{+}_{\mathrm{DR}}[\tau])$ *is equal to* $L'_p(F, \psi, 0)$*, and hence its algebraicity follows from the p-adic Gross-Stark conjecture.*

Dasgupta's thesis

Theorem (Dasgupta, D, 2003)

The quantity $\log_p(J^+_{\mathrm{DR}}[\tau])$ *is equal to* $L'_p(F,\psi,0)$*, and hence its algebraicity follows from the p-adic Gross-Stark conjecture.*

The original proof in Dasgupta's thesis relies on a direct comparison between $J^+_{\mathrm{DR}}[\tau]$ and $L'_p(F,\psi,0)$, expressing $L_p(F,\psi,1-k)$ in terms of the p-adic Mellin transform of a measure, à la Cassou-Noguès.

It has been extended, notably by Pierre Charollois and Samit Dasgupta, to arbitrary totally real F.

I will now describe an alternate approach inspired by Siegel's proof of the rationality of $L(F,\psi,1-k)$

The winding cocycle

Let $[a, b; c, d] := \frac{(a-c)(b-d)}{(a-d)(b-c)}$ be the cross-ratio.

For $z_1, z_2 \in \mathcal{H}^*$, $[z_1, z_2] :=$ hyperbolic segment from z_1 to z_2.

$[z_1, z_2] \cdot [z_3, z_4] \in \{-1, 0, 1\} :=$ intersection product.

Fix base points $\xi \in \mathcal{H}$ and $\eta \in \mathcal{H}_p$.

Proposition (Pozzi, Vonk, D)

The infinite product

$$J_w(\gamma)(z) := \prod_{\gamma \in \Gamma} [z, \eta; \alpha 0, \alpha\infty]^{[\alpha 0, \alpha \infty] \cdot [\xi, \gamma\xi]}$$

converges to a rigid analytic function on $\mathcal{H}_p$ and satisfies

$$J_w(\gamma_1\gamma_2) = J_w(\gamma_1) \times \gamma_1 \cdot J_w(\gamma_2) \pmod{\mathbb{C}_p^\times}.$$

Spectral expansion of the winding cocycle

Unlike the Dedekind-Rademacher cocycle, the winding cocycle is not a Hecke eigenclass.

Proposition (Pozzi, Vonk, D)

The winding cocycle admits an expansion

$$J_w = J_{\mathrm{DR}} + \sum_f \frac{L(f,1)}{\Omega_f^+} J_f,$$

where the sum ranges over a basis of newforms in $S_2(\Gamma_0(p))$.

Corollary. *The generating series*

$$C + \sum_{n=1}^{\infty} \log_p(J_w[T_n\tau])q^n, \qquad \text{with } C = \frac{(1-p)}{12}\log_p J_{\mathrm{DR}}[\tau],$$

is a modular form of weight two on $\Gamma_0(p)$.

Hilbert modular Eisenstein series

Hilbert Modular Eisenstein series of parallel weight k on $\mathbf{SL}_2(\mathcal{O}_F)$:

$$E_k(1,\psi) := c_k + 4\sum_{\nu\in\mathfrak{d}_+^{-1}} \sigma_{k-1,\psi}(\nu\mathfrak{d})\exp(2\pi i(\nu_1 z_1 + \nu_2 z_2)),$$

$$c_k = L(F,\psi,1-k), \qquad \sigma_{k-1,\psi}(\alpha) := \sum_{I|(\alpha)} \psi(I)\mathrm{Nm}(I)^{k-1}.$$

p-stabilised Eisenstein series on $\Gamma_0(p\mathcal{O}_F)$:

$$E_k^{(p)}(1,\psi) := c_k^{(p)} + 4\sum_{\nu\in\mathfrak{d}_+^{-1}} \sigma_{k-1,\psi}^{(p)}(\nu\mathfrak{d})\exp(2\pi i(\nu_1 z_1 + \nu_2 z_2)),$$

$$c_k^{(p)} = L_p(F,\psi,1-k), \qquad \sigma_{k-1,\psi}^{(p)}(\alpha) := \sum_{p\nmid I|(\alpha)} \psi(I)\mathrm{Nm}(I)^{k-1}.$$

The fourier coefficients of $E_k^{(p)}(1,\psi)$ extend to p-adic analytic functions of the variable k.

Diagonal restrictions

Let $G_k(\psi) :=$ the diagonal restriction of $E_k^{(p)}(1, \psi)$.

$G_k(\psi) \in M_{2k}(\Gamma_0(p))$.

Theorem (Alice Pozzi, Jan Vonk, D)

When p is inert in F, the modular form $G_1(\psi)$ vanishes identically. The derivative

$$G_1'(\psi) := \frac{d}{dk}(G_k(\psi))_{k=1} = L_p'(F, \psi, 0) + \sum_{n=1}^{\infty} a_n q^n$$

is a p-adic modular form satisfying, for all n with $(p, n) = 1$,

$$\lim_{t \to \infty} a(np^t) = \log_p(J_w^+[T_n \tau]).$$

Generating series and the winding cocycle

Let $e_{\mathrm{ord}} := \lim U_p^{n!}$ be the ordinary projection.

The modular form $e_{\mathrm{ord}} G_1'(\psi)$ has fourier expansion given by

$$e_{\mathrm{ord}} G_1'(\psi) = L_p'(F, \psi, 0) + \sum_{n=1}^{\infty} \log_p(T_n J_w^+[\tau]) q^n \quad \in \quad M_2(\Gamma_0(p)).$$

On the other hand, we had already shown that

$$\frac{(1-p)}{12} \log_p J_{\mathrm{DR}}^+[\tau] + \sum_{n=1}^{\infty} \log_p(T_n J_w^+[\tau]) q^n \text{ is in } M_2(\Gamma_0(p)).$$

It follows that $L_p'(F, \psi, 0) = \frac{(1-p)}{12} \log_p J_{\mathrm{DR}}^+[\tau]$.

This approach is reminiscent of the "analytic proof" of the theorem of Gross-Zagier on factorisations of singular moduli.

The Gross-Stark conjecture

The algebraicity of $J_{\rm DR}[\tau]$ now follows from Gross's p-adic analogue of the Stark conjecture.

This conjecture turns out to be *far more tractable* than its archimedean counterpart.

Theorem (Dasgupta, Rob Pollack, D, (2007))

The p-adic Gross-Stark conjecture is true. In particular, $J_{\rm DR}^{+}[\tau]$ is, up to torsion, a p-unit in the narrow Hilbert class field of F.

The proof is based on a careful analysis of the p-adic *cuspidal* deformations of $E_1(1,\psi)$ in the parallel weight direction, and of their associated p-adic Galois representations.

Removing the norm

Problem: The algebraicity of $J_{\mathrm{DR}}^{+}[\tau]$ implies that $J_{\mathrm{DR}}[\tau]$ is also algebraic, but only up to multiplication by elements of $\mathbb{Q}_{p^2}^{\times}$ of norm one.

It would be desirable to do away with this ambiguity, which is built into the Gross-Stark conjecture.

Recent work of Samit Dasgupta and Mahesh Kakde achieves remarkable progress in this direction.

The work of Samit Dasgupta and Mahesh Kakde

Theorem (Dasgupta, Mahesh Kakde (2019))

The algebraicity of $J_{\mathrm{DR}}[\tau]$ up to torsion follows from Gross's "tame refinement" of the Gross-Stark conjecture.

The argument employed in the proof is reminiscent of the "tame-patching technique" of Taylor-Wiles.

Theorem (Dasgupta, Mahesh Kakde (2019))

Gross's tame refinement in true.

The proof is based on a careful analysis of the *tame* deformations of $E_1(1, \psi)$, and of their associated mod p^n Galois representations.

An alternate approach to Dasgupta-Kakde

The study of the diagonal restriction of the p-adic family of Hilbert modular Eisenstein series can be refined, to consider the *antiparallel deformations* of the Eisenstein series of weight one.

This leads to an alternate proof of the theorem of Dasgupta-Kakde, which is "purely p-adic" and does not require tame deformations.

This is the content of a joint work in progress with Alice Pozzi and Jan Vonk.

p-adic deformations of Hilbert Eisenstein series

Theorem (Alice Pozzi, Jan Vonk, D, (2019))

There is an infinitesimal p-adic deformation $E_1'(\psi)$ *of* $E_1(1,\psi)$ *in the* antiparallel weight direction *satisfying:*

- $a_0(E_1'(\psi)) = \log_p(u_\tau)$ *where* $u_\tau :=$ *Gross-Stark unit*;

- $$e_{\mathrm{ord}}(E_1'(\psi)|_{\mathrm{Diag}}) = \log_p(J_{\mathrm{DR}}[\tau])E_2^{(p)}(z) + \sum_f \lambda_f \cdot f,$$

where the sum runs over a newform basis of $S_2(\Gamma_0(p))$ *and* $\lambda_f = \log_p(J_f[\tau])L(f,1)$.

Comparing constant terms, $J_{\mathrm{DR}}[\tau]^{\frac{(p-1)}{12}} = u_\tau \pmod{\mathrm{tors}}$.

The key ingredient, again

The approaches of Dasgupta-Pollack-D + Dasgupta-Kakde, and of Pozzi-Vonk-D differ in several key respects,

However, they all rest crucially on p-adic deformations of modular forms and Galois representations, Galois cohomology, and global class field theory.

This is just like the work of Greenberg and Stevens, and indeed, much of our theoretical understanding of the emerging theory of "real multiplication" is based on these notions.

A more geometric approach would be of great interest!

Beyond L-functions?

As Chris remarked in his lecture, we seem to be better at constructing unramified rather than ramified invariants.

Philosophical justification. Elliptic units are related to special values of L-functions, via the Kronecker limit formula.

On the other hand, *singular moduli* – the values of meromorphic modular functions, like $j(z)$, at CM points – are algebraic numbers with highly non-trivial factorisations.

Our belief that the values of rigid analytic cocycles, leading to Stark-Heegner points and Gross-Stark units, have a geometric underpinning would be *reinforced* by a convincing counterpart of singular moduli in this setting.

Rigid meromorphic cocycles

Definition (Jan Vonk, D, 2018) A *rigid meromorphic cocycle* is a class in $H^1(\Gamma, \mathcal{M}^\times)$, where $\mathcal{M}^\times$ is the multiplicative group of rigid meromorphic functions on $\mathcal{H}_p$.

Whereas $H^1(\Gamma, \mathcal{A}^\times/\mathbb{C}_p^\times)$ is a finitely generated $\mathbb{Z}$-module (of rank roughly $\frac{p}{12}$), the group $H^1(\Gamma, \mathcal{M}^\times)$ is infinitely generated.

Hope. If J is a suitable rigid meromorphic cocycle, the RM values $J[\tau]$ lead to real quadratic analogues of singular moduli, with interesting "Gross-Zagier style" factorisations.

The classification of rigid meromorphic cocycles

Let $\Gamma_\infty := \mathrm{Stab}_\Gamma(\infty) \subset \Gamma$, and S the standard matrix of order 2 in $\mathbf{SL}_2(\mathbb{Z})$.

Theorem (Jan Vonk, D, 2018) Let J be a rigid meromorphic cocycle satisfying $J(\Gamma_\infty) = 1$. Then J is completely determined by its value at S, a *finite* product

$$J(S) = \prod_{\tau \in \Gamma \backslash \mathcal{H}_p^{\mathrm{RM}}} \alpha_\tau(z)^{n_\tau},$$

where $\alpha_\tau(z)$ is a rigid meromorphic function having its zeroes (resp. poles) at the negative norm elements of $\Gamma \cdot \tau$ that are positive (resp. negative).

The finite formal sum $\sum_\tau n_\tau \cdot (\tau) \in \mathrm{Div}(\Gamma \backslash \mathcal{H}_p^{\mathrm{RM}})$ is called the *divisor* of J.

A real quadratic Borchers lift

Let $-\Delta$ be a fundamental discriminant which is a square mod p.

Theorem. (Jan Vonk, D). Let $\phi := \sum_{n \gg -\infty} c_\phi(n) q^n \in M^{!!}_{1/2}(4p)$ be a weakly holomorphic modular form with integer fourier coefficients and poles only at ∞. There is a unique rigid meromorphic cocycle $J_{-\Delta,\phi} \in H^1(\Gamma, \mathcal{M}^\times)$ satisfying

$$\mathrm{Divisor}(J_{-\Delta,\phi}) = \sum_{\substack{D \equiv 0,3 \\ \pmod 4}} c_\phi(-D) \left(\sum_{\tau_\mathfrak{a} \in \Gamma \backslash \mathcal{H}_p^{\Delta D}} \chi_{-\Delta,-D}(\mathfrak{a})[\tau_\mathfrak{a}] \right),$$

with the outer sum running over the D satisfying $\left(\frac{-D}{p}\right) \neq 1$.

The rigid meromorphic cocycle $J_{-\Delta,\phi}$ is called the *real quadratic Borcherds lift* attached to $-\Delta$ and ϕ.

Real quadratic singular moduli

Conjecture (Vonk, D, 2019) Let J be a real quadratic Borcherds lift, and τ an *RM* point. The value $J[\tau]$ belongs to the narrow ring class field associated to τ, and admits an *explicit factorisation* depending on $\mathrm{Divisor}(J)$ and τ.

Strategy for proving this conjecture:

1. Show that $J[\tau]$ can be packaged in generating series that are "*p*-adic mock modular forms". (Over $\mathbb{C}$: Gross-Zagier, Kudla-Rapoport-Yang, Jan Bruinier, Bill Duke, Yingkun Li, Stephan Ehlen, Maryna Viazovska, ...)

2. Use the p-adic deformation theory of modular forms and Galois representations to relate the fourier coefficients of these mock modular forms to logarithms of algebraic numbers. (Lauder-Rotger-D, Rotger-Rivero, ...)

Summary

CM Theory	RM Theory
$\mathcal{H}$	$\mathcal{H}_p$
$\mathbf{SL}_2(\mathbb{Z})$	$\mathbf{SL}_2(\mathbb{Z}[1/p])$
Modular functions	Rigid meromorphic cocycles
CM points	RM points
Heegner points	Stark-Heegner points
Elliptic units	Gross-Stark units
Singular moduli	RM values of rigid meromorphic cocycles

Conclusion

The approach to “real multiplication”, based on p-adic deformations of modular forms and Galois representations, leads to a good understanding of the Dedekind-Rademacher cocycle, and of the RM values of rigid meromorphic cocycles.

Yet it is not at all satisfying: the algebraicity of Stark-Heegner points remains wide open.

Question. Is there a geometric interpretation for rigid meromorphic cocycles, making the algebraicity of their RM values apparent?

The factorisation patterns of real quadratic singular moduli reinforce the hope that such an interpretation should exist.

Thank you for your attention!