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Kloosterman Zeta Functions for GL$(n, \mathbf{Z})$

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1. Introduction. Analytic number theory has made considerable strides in the past few years. Let me begin by citing two of the most striking recent results.

Fouvry [**9**] has shown that there exist infinitely many primes p such that $p-1$ has a prime factor larger than $p^{2/3}$. This, in conjunction with a theorem of L. M. Adleman and D. R. Heath-Brown [**1**] (extensions of Sophie Germaine's criterion), enables one to show that Fermat's equation

$$x^p + y^p = z^p \qquad (p \nmid xyz)$$

has no positive integral solutions for infinitely many primes p.

Another major breakthrough has been the work of Deshouillers-Iwaniec [**8**] which has led to many spectacular results. For example, Bombieri-Friedlander-Iwaniec [**2**, **3**] have recently obtained an averaged form of the prime number theorem for arithmetic progressions (of Bombieri-Vinogradov type). In the case of the distribution of primes $\leq x$ with respect to moduli $> \sqrt{x}$, their results go beyond what can be obtained upon assumption of the generalized Riemann hypothesis.

The proofs of the above theorems have a new common ingredient; uniform estimates (of the type first proved by Kuznietsov [**16**]) for the distribution of Kloosterman sums. For other applications of this innovative idea, the excellent survey article of Iwaniec [**13**] is to be commended. Accordingly, our attention is turned to a general theory of Kloosterman sums, hyper-Kloosterman sums, and their zeta functions.

Let M, $N \in \mathbf{Z}$ and $s \in \mathbf{C}$. The Kloosterman zeta function for GL$(2, \mathbf{Z})$ is defined to be

$$Z(M, N; s) = \sum_{c=1}^{\infty} S(M, N; c) c^{-2s} \tag{1.1}$$

where

$$S(M, N; c) = \sum_{\substack{a=1 \\ (a,c)=1}}^{c} e^{2\pi i (aM + \bar{a}N)/c} \qquad (a\bar{a} \equiv 1 \bmod c) \tag{1.2}$$

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is the classical Kloosterman sum. The bound $S(M,N;c)=O(c^{1/2+\varepsilon})$ of A. Weil [**22**] shows that (1.1) converges absolutely for $\mathrm{Re}(s)>\frac{3}{4}$.

The function (1.1) was first introduced by A. Selberg [**19**] who obtained its meromorphic continuation to the whole complex s-plane. In [**12**], by use of bounds for the resolvent operator, Goldfeld and Sarnak have shown that

$$Z(M,N;s)=O(|s|^{1/2+\varepsilon}) \tag{1.3}$$

for $\mathrm{Re}(s)>\frac{1}{2}+\varepsilon$ and $|\mathrm{Im}(s)|>\varepsilon$. The bound (1.3) leads to a simple proof of Kuznietsov's theorem [**16**]

$$\sum_{c\le x}\frac{S(M,N;c)}{c}=O(x^{1/6+\varepsilon}) \tag{1.4}$$

which we discussed before. Other generalizations of (1.4) have been obtained by Deshouillers and Iwaniec [**8**] and Proskurin [**18**]. Also, Bruggeman [**4**] has developed a Kuznietsov trace formula which also leads to (1.4).

We shall now consider Kloosterman zeta functions for higher rank groups, focussing on $\mathrm{GL}(n,\mathbf{Z})$ with $n>2$. Uniform estimates for the distribution of hyper-Kloosterman sums and products of classical Kloosterman sums should be the outcome of this endeaver.

2. Notation. For $n=2,3,\ldots$ let $G=\mathrm{GL}(n,\mathbf{R})$, $\Gamma=\mathrm{GL}(n,\mathbf{Z})$, $X\subset G$ be the set of all upper triangular matrices with ones on the diagonal, and $Y\subset G$ the set of diagonal matrices of type

$$\mathrm{diag}(y_1\cdots y_{n-1},y_1\cdots y_{n-2},\ldots,y_1,1) \tag{2.1}$$

with $y_i>0$. We consider the homogeneous space $H\cong G/\mathrm{O}(n,\mathbf{R})\cdot\mathbf{R}$ where $\mathrm{O}(n,\mathbf{R})$ is the orthogonal group. By the Iwasawa decomposition, every $z\in H$ has a unique decomposition $z\equiv xy \pmod{\mathrm{O}(n,\mathbf{R})\cdot\mathbf{R}}$ with $x\in X$ and $y\in Y$. The discrete group Γ acts on H by left matrix multiplication. Let $\mathcal{L}^2(\Gamma\backslash H)$ denote the Hilbert space with inner product

$$\langle f,g\rangle=\int_{\Gamma\backslash H}f(z)\overline{g(z)}\,d^*z$$

where both f, $g\colon H\to\mathbf{C}$ are left-invariant under Γ and the invariant volume element d^*z satisfies

$$d^*z=\prod_{1<i<j\le n}dx_{i,j}\prod_{i=1}^{n-1}y_i^{-i(n-i)-1}\,dy_i$$

where $x=(x_{ij})\in X$, $y\in Y$ is given by (2.1) and $z\equiv xy$.

Henceforth $M=(M_1,\ldots,M_{n-1})$, $N=(N_1,\ldots,N_{n-1})$ are in $\mathbf{Z}^{n-1}$ and $s=(s_1,\ldots,s_{n-1})$, $u=(u_1,\ldots,u_{n-1})$, $v=(v_1,\ldots,v_{n-1})$, and $k=(k_1,\ldots,k_{n-1})$ are in $\mathbf{C}^{n-1}$. By θ_M, θ_N, we mean characters of X given by

$$\theta_M(x)=e(M_1x_{1,2}+\cdots+M_{n-1}x_{n-1,n})$$

with $x=(x_{ij})\in X$ and $e(\theta)=e^{2\pi i\theta}$.

3. Kloosterman sums associated to double coset decompositions of GL(n, $\mathbf{Z}$). Let W denote the Weyl group of G. If $D \subset G$ is the subgroup of diagonal matrices, then we have the Bruhat decomposition $G = \bigcup_{w \in W} XDwX$. This induces the decomposition $\Gamma = \bigcup_{w \in W} \Gamma_w$ where $\Gamma_w = (XDwX) \cap \Gamma$ are termed Bruhat cells. The cell corresponding to the so called long element

$$w = \begin{pmatrix} & & & \pm 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix} \tag{3.1}$$

is called the big cell.

Consider the minimal parabolic subgroup $P = X \cap \Gamma$ of Γ, and the various subgroups $P_w = ((w^{-1})^t P w) \cap P$ indexed by $w \in W$. Let $(c_1, \ldots, c_{n-1})$ be nonzero integers and set

$$c = \operatorname{diag}(1/c_{n-1}, c_{n-1}/c_{n-2}, \ldots, c_2/c_1, c_1). \tag{3.2}$$

For M, $N \in \mathbf{Z}^{n-1}$, the generalized Kloosterman sum $S_w(M, N; c)$ is defined as

$$S_w(M, N; c) = \sum_{\substack{\gamma \in P \backslash \Gamma_w / P_w \\ \gamma = b_1 c w b_2}} \theta_M(b_1) \theta_N(b_2) \tag{3.3}$$

where θ_M, θ_N are characters of X. This reduces to the classical sum (1.2) for $n = 2$ and $w = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$.

The sum (3.3) was first considered in Bump-Friedberg-Goldfeld [**6**, **7**] for $n = 3$, and somewhat later for $n > 3$ by Friedberg [**10**], Stevens [**21**] and Piatetski-Shapiro. As shown in [**10**], the Kloosterman sums are multiplicative in c, nonzero only if $w \in W$ is of the form

$$(\operatorname{mod} \mathrm{SL}(n, \mathbf{Z}) \cap D), \qquad w = \begin{pmatrix} & & & I_1 \\ & & I_2 & \\ & \cdot^{\cdot^{\cdot}} & & \\ I_l & & & \end{pmatrix}$$

where the I_j are identity matrices; and factor into nondegenerate classical Kloosterman sums of type (1.2) if $c_1, \ldots, c_{n-1}$ are pairwise coprime and w is the long element (3.1). If c is given by (3.2) and c_i are suitable powers of a fixed prime p, then (3.3) will be associated to an algebraic variety over $\mathbf{F}_p$. In the special case

$$c = (p^{1-n}, p, \ldots, p), \qquad w = \begin{pmatrix} & \pm 1 \\ I_{n-1} & \end{pmatrix},$$

[**10**] has shown that $S_w(M, N; c)$ is a power of p times

$$\sum_{x_1 \cdots x_{n-1} \equiv M_1 \cdots M_{n-1} N_{n-1} \pmod p} e((x_1 + \cdots + x_{n-1})/p)$$

and is, therefore, associated to a Kloosterman hypersurface. In general (see [**21**]), the associated varieties are not smooth and their classification is still an open problem.

4. Kloosterman zeta functions. Let $M,\ N \in \mathbf{Z}^{n-1}$,

$$c = \operatorname{diag}(1/c_{n-1}, c_{n-1}/c_{n-2}, \ldots, c_2/c_1, c_1),$$

$w \in W$, and $s \in \mathbf{C}^{n-1}$. The Kloosterman zeta function associated to the Bruhat cell Γ_w is defined to be

$$Z_w(M,N;s) = \sum_{c_1=1}^{\infty} \cdots \sum_{c_{n-1}=1}^{\infty} S_w(M,N;c) c_1^{-ns_1} \cdots c_{n-1}^{-ns_{n-1}}. \tag{4.1}$$

For $n > 2$, little is known about this function at present. We expect, however, that (4.1) has a meromorphic continuation in s, and that the polar divisors of (4.1) may be a subset of the polar divisors of the global zeta function

$$Z(M,N;s) = \sum_{w \in W} Z_w(M,N;s). \tag{4.2}$$

If this were the case for $n = 3$, then by the arguments of [**6**, **7**] the generalized Ramanujan conjecture would follow for $n = 3$, and by the Gelbart-Jacquet lift [**11**], also for $n = 2$.

The meromorphic continuation of (4.2) has been obtained for $n = 3$ in [**6**, **7**] by considering the inner product of two Poincaré series. We indicate an approach to generalizing our results to $n > 3$.

For $v \in \mathbf{C}^{n-1}$, define $I_v\colon H \to \mathbf{C}$ by the formula

$$I_v(z) = \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} y_i^{b_{ij} v_j}, \tag{4.3}$$

$$b_{ij} = \begin{cases} (n-i)j, & 1 \le j \le i, \\ (n-j)i, & i \le j \le n-i, \end{cases}$$

where $z \equiv xy$ with $x \in X$ and $y \in Y$ given by (2.1). Let $\mathcal{D}$ denote the polynomial ring of differential operators defined on $\Gamma\backslash H$. Every $d \in \mathcal{D}$ determines a character $\lambda_v(d)$ given by $dI_v = \lambda_v(d) I_v$.

A Maass form of type $k \in \mathbf{C}^{n-1}$ is a smooth function $\varphi \in \mathcal{L}^2(\Gamma\backslash H)$ satisfying $d\varphi = \lambda_k(d)\varphi$ for all $d \in \mathcal{D}$. If it is "cuspidal" it has a Whittaker expansion [**17**, **20**]

$$\varphi(z) = \sum_{M_1=1}^{\infty} \cdots \sum_{M_{n-1}=1}^{\infty} \sum_{\gamma \in \Lambda_{n-1}} a_M \prod_{i=1}^{n-1} M_i^{(-i(n-i))/2} W_k \left((M) \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} z \right) \tag{4.4}$$

where $M = (M_1, \ldots, M_{n-1})$,

$$(M) = \operatorname{diag}(M_1 \cdots M_{n-1}, M_1 \cdots M_{n-2}, \ldots, M_1, 1),$$

$a_M \in \mathbf{C}$, $\Lambda_{n-1} = U_{n-1}\backslash\mathrm{GL}(n-1,\mathbf{Z})$, $U_{n-1} \subset \mathrm{SL}(n-1,\mathbf{Z})$ is the subgroup of upper triangular matrices with ones on the diagonal, and $W_k(z)$ is the Whittaker function given by

$$W_k(z) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} I_k(w_\circ uz) e\left(-\sum_{j=1}^{n-1} u_{j,j+1}\right) \cdot \prod_{1<i<j\le n} du_{ij}$$

where $w_\circ$ is the long element (3.1) and $u = (u_{ij}) \in X$. Although we have not defined "cuspidal," we may take (4.4) as a definition.

The meromorphic continuation in $s \in \mathbf{C}^{n-1}$ of the Mellin transform of the Whittaker function

$$\mathcal{M}_k(s) = \int_0^{\infty} \cdots \int_0^{\infty} y_1^{s_1} \cdots y_{n-1}^{s_{n-1}} W_k(y) \prod_{i=1}^{n-1} \frac{dy_i}{y_i}$$

is still unknown except for $n = 2, 3$ (see Bump [5]). A heuristic argument of Ka-Lam Kueh suggests that $\mathcal{M}_k(s)$ has its first simple poles at

$$s_i = \left(\sum_{j=1}^{n-1} b_{n-i,j} k_j\right) - i(n-i) \qquad (i = 1, \ldots, n-1) \tag{4.5}$$

with b_{ij} given by (4.3).

Let $N \in \mathbf{Z}^{n-1}$ and θ_N be a character of X. An e-function is a bounded function $e_N\colon H \to \mathbf{C}$ satisfying $e_N(uz) = \theta_N(u)e_N(z)$ for all $u \in X$ and $z \in H$. Let $v \in \mathbf{C}^{n-1}$. We consider the Poincaré series

$$P_N(z; v) = \sum_{\gamma \in P\backslash\Gamma} I_v(\gamma z) e_N(\gamma z).$$

For u, $v \in \mathbf{C}^{n-1}$, M, $N \in \mathbf{Z}^{n-1}$, the inner product of $P_M(z,v)$ and $P_N(z,u)$ satisfies

$$\langle P_M, P_N\rangle = \sum_{w\in W} \sum_{c} \frac{S_w(M,N;c)}{c^{nv}} \cdot \int_{X_w} \int_{y_1=0}^{\infty} \cdots \int_{y_{n-1}=0}^{\infty} I_v(wz) e_M(cwz) \overline{I_u(z)}\, \overline{e_N(z)}\, d^*z \tag{4.6}$$

where $X_w = ((w^{-1})^t X w) \cap X$, $c^{nv} = c_1^{nv_1} \cdots c_{n-1}^{nv_{n-1}}$, and the sum on the right side of (4.6) goes over all $N = (\varepsilon_1 N_1, \ldots, \varepsilon_{n-1}N_{n-1})$ with $\varepsilon_i = \pm 1$ and $\varepsilon_1 \cdots \varepsilon_{n-1} = 1$.

For fixed u, the integrals on the dexter side of (4.6) can be continued as analytic functions of v. The polar set of $Z(M,N;v)$ can then be obtained from the polar set of $\langle P_M, P_N\rangle$. Let φ be a Maass form of type $k \in \mathbf{C}^{n-1}$. The projection of P_M, P_N onto φ yields a contribution $\langle P_M, \varphi\rangle \cdot \overline{\langle P_N, \varphi\rangle}$ to $\langle P_M, P_N\rangle$. Some formal calculations in conjunction with (4.4) give

$$\langle P_M, \varphi\rangle = \bar{a}_M \left[\prod_{i=1}^{n-1} M_i^{\left(i(n-i)/2 - \sum_{j=1}^{n-1} b_{n-i,j} v_j\right)}\right] \mathcal{M}_{\bar{k}}(t)$$

with $t = (t_1, \ldots, t_{n-1})$ and $t_i = \left(\sum_{j=1}^{n-1} b_{n-i,j} v_j\right) - i(n-i)$. Since $P_M(z,v)$ is orthogonal to the residual spectrum of $\mathcal{D}$, we do not obtain residual polar divisors. By (4.5), we are led to expect that $Z(M,N;s)$ has a meromorphic continuation in s with simple polar divisors which contain the hyperplanes

$$\sum_{j=1}^{n-1} b_{n-i,j} s_j = \sum_{j=1}^{n-1} b_{ij} \bar{k}_j \qquad (i = 1, \ldots, n-1).$$

5. Kloosterman decompositions of Selberg's kernel function. Another approach to the distribution of Kloosterman sums is based on a double coset decomposition of Selberg's kernel function. This method was used by Zagier (see Iwaniec [**14**]) to give an alternate proof of Kuznietsov's sum formula, and more recently by Ye [**23**] to give a new proof of quadratic base change. We consider generalizations to $G = \mathrm{GL}(n, \mathbf{R})$ with $n \geq 2$ which lead to new types of trace formulae.

Consider the Cartan decomposition $G = KAK$ where $K = \mathrm{O}(n, \mathbf{R})$ and $A \subset G$ is the subgroup of diagonal matrices with positive entries. Let $\varphi\colon K\backslash G/K \to \mathbf{C}$ be a K-biinvariant function.

Formally, the Selberg kernel function for Γ is

$$K(z, z') = \sum_{\gamma \in \Gamma} \varphi(z^{-1}\gamma z') \tag{5.1}$$

where z, $z' \in H$. For suitably chosen φ the dexter side of (5.1) converges absolutely and uniformly on compact subsets of $H \times H$.

Now (5.1) can be rewritten

$$K(z, z') = \sum_{(m) \in P} \sum_{\gamma \in P\backslash\Gamma} \varphi(((m)z)^{-1}\gamma z') \tag{5.2}$$

with $(m) = (m_{ij})$. For fixed $(m) \in P$, the inner sum on the dexter side of (5.2), denoted $K_{(m)}(z, z')$ is an automorphic form for $\Gamma\backslash H$. It has a Fourier expansion in x with Nth Fourier coefficient (here $N \in \mathbf{Z}^{n-1}$) given by

$$\int_{P\backslash X} K_{(m)}(z, z') \overline{\theta_N(x)}\, dx \tag{5.3}$$

which is itself a Poincaré series in z'. For $M \in \mathbf{Z}^{n-1}$, the Mth Fourier coefficient in x' of (5.3) is

$$\int_{P\backslash X} \int_X \sum_{\gamma \in P\backslash\Gamma} \varphi(z^{-1}\gamma z') \overline{\theta_N(x)}\, \overline{\theta_M(x')}\, dx\, dx',$$

which is just

$$\sum_{w \in W} \sum_c S_w(M, N; c) \int_{X_w} \int_X \varphi(z^{-1}cwz') \overline{\theta_N(x)}\, \overline{\theta_M(x')}\, dx\, dx' \tag{5.4}$$

with the notation of (4.6).

For $x \in \mathbf{C}$, $z \in H$, and $P_\circ$ the maximal parabolic subgroup $\left(\begin{smallmatrix} & * \\ 0 \cdots 0 & 1\end{smallmatrix}\right)$ of SL$(n, \mathbf{Z})$, let $E(z,s) = \sum_{\gamma \in P_\circ \backslash \Gamma} (\det \gamma z)^s$ denote the maximal parabolic Eisenstein series which converges absolutely and uniformly on compact subsets of H for $\mathrm{Re}(s) > 1$. Now, if $K_\circ(z, z')$ is the projection of $K(z,z')$ onto the space of cuspidal Maass forms, we are interested in computing the trace

$$\operatorname*{Res}_{s=1} \int_{\Gamma \backslash H} K_\circ(z,z) E(z,s)\, d^*z. \tag{5.5}$$

After some formal computations (see Jacquet [**15**]), it follows from (5.4) that the essential contribution to the trace (5.5) is given by

$$\operatorname*{Res}_{s=1} \sum_{w \in W} \sum_{c} \sum_{N \neq (0)} S_w(N,N;c) F_w(N,c,s) \tag{5.6}$$

where

$$F_w = \int_{y_1=0}^{\infty} \cdots \int_{y_{n-1}=0}^{\infty} \int_{X_w} \int_X \varphi(y^{-1}x^{-1}cwx'y)\theta_N(x+x')dx\,dx' \cdot \prod_{i=1}^{n-1} y_i^{(s-1)(n-i)-1}\, dy_i.$$

In the special case $n = 2$, (5.6) takes the form

$$\operatorname*{Res}_{s=1} \sum_{N \neq 0} \sum_{c=1}^{\infty} \frac{S(N,N;c)}{c^{s+1}} F\left(\frac{N}{c}, s\right)$$

where

$$F(B,s) = \int_0^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi\left(\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} \begin{pmatrix} y & \\ & y^{-1} \end{pmatrix} \begin{pmatrix} 1 & x' \\ & 1 \end{pmatrix} \right) \cdot e(-By(x+x'))y^s\, dx\, dx'\, dy$$

decays rapidly to zero as $B \to \infty$.

To compute the above residue we use a method of Kuznietsov. Let

$$S(N,N;c) = \sum_{-c/2 \leq l \leq c/2} v(c,l) e\left(\frac{lN}{c}\right)$$

where $v(c,l)$ denotes the number of solutions $a \pmod c$ of $a^2 - al + 1 \equiv 0 \pmod c$. By Poisson summation

$$\sum_N e\left(\frac{lN}{c}\right) F\left(\frac{N}{c}, s\right) = \sum_{h \in \mathbf{Z}} \int_{-\infty}^{\infty} F\left(\frac{\xi}{c}, s\right) e\left(\left(\frac{l}{c} + h\right)\xi\right) d\xi$$

where the integral on the right is bounded by $(hc)^{-m}$ for $h \neq 0$ and bounded by l^{-m} for $h = 0$, $l \neq 0$ (after integrating by parts $m > 2$ times). Consequently, the residue (5.7) is given by

$$\operatorname*{Res}_{s=1} \left[\sum_{c=1}^{\infty} \sum_{l=-\infty}^{\infty} \frac{v(c,l)}{c^s} \right] \cdot \int_{-\infty}^{\infty} F(\xi, s) e(-l\xi)\, d\xi.$$

We have, therefore, expressed the principal cuspidal contribution to the Selberg trace formula in terms of special values of quadratic L-functions.

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