Problem 1: (Uniqueness of Dirichlet series) For \( n = 1, 2, 3, \ldots \), let \( a_n, b_n \) be complex numbers with absolute values at most one. Assume that

\[
\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{n=1}^{\infty} \frac{b_n}{n^s}
\]

for all complex values of \( s \) with \( \Re(s) > 1 \). Prove that we must have \( a_n = b_n \) for all \( n = 1, 2, 3, \ldots \)

Problem 2: Explicitly construct all Dirichlet characters (mod 15). Each such character is a completely multiplicative function \( \chi : \mathbb{Z} \to \mathbb{C} \) satisfying \( \chi(n + 15) = \chi(n) \) for all \( n \in \mathbb{Z} \).

Problem 3: Let \( q \) be an integer which has the property that every Dirichlet character \( \chi \) (mod \( q \)) is real valued (takes on only the values 0, \( \pm 1 \)). Show that \( q \) must divide 24.

Problem 4: Let \( \chi_0 \) be the trivial character (mod \( q \)), and let \( q_1 \) be some factor of \( q \). For any character \( \chi_1 \) (mod \( q_1 \)) there is a character \( \chi \) (mod \( q \)) defined by \( \chi = \chi_0 \chi_1 \). Express \( L(s, \chi) \) in terms of \( L(s, \chi_1) \). Conclude that \( L(1, \chi) \neq 0 \) if and only if \( L(1, \chi_1) \neq 0 \).

Problem 5: Calculate the mean value

\[
\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\zeta(\sigma + it)|^2 \, dt
\]

provided \( \sigma > 1 \).
Problem 6: By the functional equation for
\[ \xi(s) = \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \zeta(s), \]
the function \( s(1-s)\xi(s) \) can be regarded as an entire function of \( s^2-s \); what is the order of this function? Use this to obtain the alternative infinite product
\[ \xi(s) = \frac{\xi(1/2)}{4(s - s^2)} \prod_{\rho} \left( 1 - \left( \frac{s - \frac{1}{2}}{\rho - \frac{1}{2}} \right)^2 \right) \]
the product extending over zeros \( \rho \) of \( \xi(s) \) whose imaginary part is positive. [This symmetrical form eliminates the exponential factors \( e^{A+Bs} \) and \( e^{s/\rho} \) occurring in the usual Hadamard factorization of \( \xi(s) \).]

Problem 7: Use the partial fraction decomposition of \( \zeta'/\zeta \) to obtain the following exact formula:
\[ \psi(x) = \sum_{p^k \leq x} \log p = x - \sum_{\zeta(\rho) = 0} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log \left( 1 - x^{-2} \right). \]
Here \( \sum_{\rho} \) is taken to mean \( \lim_{T \to \infty} \sum_{|\rho| < T} \). If \( x = p^k \) is a prime power, so that \( \psi(x) \) is discontinuous at \( x \), then we interpret \( \psi(x) \) as \( \left( \psi(x - \epsilon) + \psi(x + \epsilon) \right) / 2 \).

Hint: You may view \( \frac{1}{2} \log \left( 1 - x^{-2} \right) \) as the sum \(-x^r/r \) over the trivial zeros \( r = -2, -4, -6, \ldots \).