Analytic Number Theory
Homework #1
(due Thursday, February 9, 2012)

Problem 1: Show that the Gamma function \( \Gamma(s) = \int_0^\infty e^{-u}u^s \frac{du}{u} \) has simple poles at \( s = 0, -1, -2, -3, \ldots \) Determine the residues at these poles.

Problem 2: Let \( f : \mathbb{Z} \to \mathbb{C} \) be a completely multiplicative function, i.e., \( f(mn) = f(m)f(n) \) for all \( m, n \in \mathbb{Z} \). Assume also that \( |f(n)| \leq 1 \) for all \( n \in \mathbb{Z} \) and that \( f(1) = 1 \). Prove that for \( \Re(s) > 1 \), we have
\[
\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left( 1 - \frac{f(p)}{p^s} \right)^{-1}
\]
where the product goes over all primes \( p \).

Problem 3: Evaluate the integral
\[
\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{x^s}{s(s+1)\cdots(s+k)} \, ds
\]
for any \( \sigma > 0 \) and any integer \( k = 1, 2, 3, \ldots \) Give a rigorous proof of your evaluation.

Problem 4: Fix \( u > 0 \). Show that the Fourier transform of \( \frac{u}{\pi(x^2+u^2)} \) is \( e^{-2\pi|x|u} \). Plug this into the Poisson summation formula to deduce that
\[
\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + u^2} = \frac{\pi}{u} \sum_{n=-\infty}^{\infty} e^{-2\pi|n|u}.
\]
By letting \( u \to 0 \) prove that \( \zeta(2) = \pi^2/6 \).