Analytic Number Theory
Homework #1

(due Thursday, February 9, 2012)

Problem 1: Show that the Gamma function $\Gamma(s) = \int_0^\infty e^{-u} u^s \, \frac{du}{u}$ has simple poles at $s = 0, -1, -2, -3, \ldots$ Determine the residues at these poles.

Problem 2: Let $f : \mathbb{Z} \to \mathbb{C}$ be a completely multiplicative function, i.e., $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{Z}$. Assume also that $|f(n)| \leq 1$ for all $n \in \mathbb{Z}$ and that $f(1) = 1$. Prove that for $\Re(s) > 1$, we have

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 - \frac{f(p)}{p^s}\right)^{-1}$$

where the product goes over all primes p.

Problem 3: Evaluate the integral

$$\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{x^s}{s(s+1)\cdots(s+k)} \, ds$$

for any $\sigma > 0$ and any integer $k = 1, 2, 3, \ldots$ Give a rigorous proof of your evaluation.

Problem 4: Fix $u > 0$. Show that the Fourier transform of $\frac{u}{\pi(x^2+u^2)}$ is $e^{-2\pi|x|u}$. Plug this into the Poisson summation formula to deduce that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + u^2} = \frac{\pi}{u} \sum_{n=-\infty}^{\infty} e^{-2\pi|n|u}.$$

By letting $u \to 0$ prove that $\zeta(2) = \pi^2/6$.