ON AUTOMORPHIC FUNCTIONS OF HALF-INTEGRAL WEIGHT WITH APPLICATIONS TO ELLIPTIC CURVES

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1. Introduction

The theory of automorphic forms of 1/2-integral weight has attracted a considerable amount of attention in recent years. The striking difference between the case of integral and 1/2-integral weight is the fact that the Fourier coefficients of 1/2-integral weight forms are expressible in terms of the values of L-functions. In fact, Waldspurger [W] in answering a question of Shimura [Sh] has recently shown that if

$$f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$$

is a holomorphic cusp form (normalized new form of weight k) for a congruence subgroup of $SL_2(\mathbb{Z})$, then there exists a cusp form

$$F(z) = \sum_{n=1}^{\infty} c(n)e^{2\pi inz}$$

of weight $(k+1)/2$ whose D-th Fourier coefficient (where D is a fundamental discriminant of a quadratic field subject to certain congruence conditions) is given by

$$c(|D|)^2 = \Omega|D|^{\frac{k-1}{2}} L_f\left(\frac{k}{2}, \chi\right) .$$

Here, $\chi(n) = (\frac{D}{n})$ is Kronecker's symbol,

$$L_f(s,\chi) = \sum_{n=1}^{\infty} a(n)\chi(n)n^{-s}$$

and Ω is a fixed constant independent of D.

N. Koblitz (ed.), *Number Theory Related to Fermat's Last Theorem*

It seems likely that an analogue of Ramanujan's conjecture on the growth of the Fourier coefficients of cusp forms also holds in the 1/2-integral weight case. We propose the following

Conjecture. *For every* $\varepsilon > 0$, $C(|D|) = O(|D|^{\frac{k-1}{4}+\varepsilon})$ *where the* O-*constant depends only on* ε *and* f.

In view of Waldspurger's results, this conjecture is entirely consistent with the generalized Lindelöf hypothesis which states that

$$\left|L_f\left(\frac{k}{2} + it, \chi\right)\right| = O((1+|t|)^{\varepsilon}|D|^{\varepsilon}) ,$$

for every $\varepsilon > 0$. At present, the best bound we can obtain is

$$C(|D|) = O\left(|D|^{\frac{k}{4}+\varepsilon}\right) .$$

In this paper, we obtain results similar to Waldspurger. The essential difference is that we deal with the continuous spectrum instead of the cuspidal spectrum. Also we work over an algebraic number field k of degree N over $\mathbb{Q}$. In order to simplify the proofs, we assume k is totally imaginary with class number one. Let $L_k(s,\psi) = \sum_{\mathfrak{a}\in k} \psi(\mathfrak{a}) N(\mathfrak{a})^{-s}$ be an arbitrary Hecke L-function for k formed with a Grössencharakter ψ. Our main result states (see Propositions (2.2), (4.3)) that there exists an automorphic form for a congruence subgroup of $SL_2(\mathcal{O}(k))$ ($\mathcal{O}(k)$ denotes the ring of integers of k) lying in the continuous spectrum of the Laplacian whose α-th Fourier coefficient (for $\alpha \in \mathcal{O}(k)$) is given by

$$(\text{Whittaker function}) \times L_k(s,\psi\chi) \quad ,$$

where χ is a primitive quadratic character with conductor dividing α. The Whittaker function is given explicitly in Section 3.

We assume all our Hecke L-series are normalized to have functional equations $s \to 1-s$. An immediate consequence of our main theorem is that for any complex s with $\mathrm{Re}(s) \geq 1/2$, there exist infinitely many quadratic twists by χ where

$$L_k(s,\psi\chi) \neq 0 \quad .$$

In the special case when k is an imaginary quadratic field of class number one and E is an elliptic curve with complex multiplication by k it is known that for suitable ψ, $L_k(s,\psi)$ is the Hasse-Weil

L-function of E over Q. Using the deep theorem of Coates-Wiles [C.W.], Arthaud [A], our results imply that there exist infinitely many quadratic extensions $Q(\sqrt{d})$ where the rank of the Mordell-Weil group

$$\mathrm{rank}(E/Q(\sqrt{d})) = \mathrm{rank}(E/Q) .$$

In Section 5 we consider certain Mellin transforms of our automorphic form. We obtain the analytic continuation of a family of Dirichlet series whose Dirichlet coefficients are given by quadratic twists of $L_k(s,\psi)$. These Dirichlet series can be used to obtain mean value estimates c.f. [G.V.]. It also follows from this that the generalized Lindelöf hypothesis holds on the average.

An important open problem that still remains is to construct an automorphic form whose Fourier coefficients is given by twists of higher order characters. At present, we do not know how to attack this problem.

2. Eisenstein Series

Let k be a totally imaginary field of degree N and define S_∞ to be the set of infinite (complex) places of k. Let

$$H = \prod_{v\in S_\infty} \mathbb{H}^3 ,$$

where $\mathbb{H}^3$ is the hyperbolic 3-space, which we regard as the set of quaternions $\{x+iy+kt;\, t>0, x,y\in\mathbb{R}\}$. Recall that $SL_2(\mathbb{C})$ acts on $\mathbb{H}^3$ by

$$g\cdot w = (aw+b)(cw+d)^{-1}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \qquad w\in\mathbb{H}^3 ,$$

where the multiplication is as quaternions, and where we regard $\mathbb{C}$ as the subfield $\{x+iy;\ x,y\in\mathbb{R}\}$ of the quaternions. This action can be extended to $GL_2(\mathbb{C})$ if we let λI (for $\lambda\in\mathbb{C}^\times$, I = Identity in $SL_2(\mathbb{C})$) act trivially (i.e., $(\lambda I)\cdot w = w$). For each $v\in S_\infty$ let $i_v : k \to \mathbb{C}$ be an embedding. Let

$$G = \prod_{v\in S_\infty} GL_2(\mathbb{C}) .$$

If $\mathcal{O}_k$ denotes the ring of integers of k, we map $SL_2(\mathcal{O}_k)$, $SL_2(k)$, $GL_2(k)$, etc. into G by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} i_v(a) & i_v(b) \\ i_v(c) & i_v(d) \end{pmatrix} .$$

Let $\Gamma = SL_2(\mathcal{O}(k))$; it acts discontinuously on $H = H(k)$. If $\mathfrak{a}$ is an ideal of k, put

$$\Gamma_0(\mathfrak{a}) = \left\{ \gamma \in \Gamma ; \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (\bmod\ \mathfrak{a}) \right\} .$$

For each $v \in S_\infty$, let ρ_v be a representation of SU_2, and set, for

$$g = \begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix} \in \mathfrak{G} , \quad w = (w_v) \in H ,$$

$$j_\rho(g,w) = \bigotimes_{v \in S_\infty} \rho_v((c_v w_v + d_v)/\|c_v w_v + d_v\|) ,$$

$$W_\rho = \bigotimes_{v \in S_\infty} w_v ,$$

where we have identified the group of unit quaternions with SU_2. Here $\|\ \|$ denotes the quaternionic norm. An elementary calculation shows that

$$j_\rho(gh,w) = j_\rho(g,hw) \cdot j_\rho(h,w) .$$

We shall next introduce the "theta multiplier system" or Kubota symbols [K_2]. Let $(-)$ be the Legendre symbol in k [B.M.S.]. Now, let v be a place of k. There exists a function ε_v on $\{x \in k_v^\times ; |x|_v = 1\}$ so that if $(\ ,\ )_v$ is the quadratic Hilbert symbol,

$$(x,y)_v = \varepsilon_v(xy)/\varepsilon_v(x)\varepsilon_v(y) , \varepsilon_v(x^2) = 1 .$$

For $d \in \mathcal{O}_k$, $(d,2) = 1$, set

$$\varepsilon \to (d) = \Pi\, \varepsilon_v(d) .$$

Although ε is not uniquely characterized by the above formula, ε can be determined by the fact that the equation [K_1]

$$\sum_{x (\bmod\ d)} \left(\frac{x}{d}\right) e\left(\frac{x}{d}\right) = \left(\frac{\delta_k}{d}\right) \epsilon\ (d) N(d)^{\frac{1}{2}}$$

holds, where δ_k is a generator of the different of k, d is coprime to $2\delta_k$, $N(d)$ is the norm of d, and $e(x) = \exp(2\pi i(x+\bar{x}))$, $\bar{x}$ = complex conjugate of x.

Now, define for $g \in \Gamma_o(8)$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\kappa(g) = \begin{cases} \left(\frac{c}{d}\right) \varepsilon(d)^{-1} & (c \neq 0) \\ \varepsilon(d)^{-1} & (c = 0) . \end{cases}$$

Note, however, that $\varepsilon(u) = 1$ if u is a unit of k. We now have the following extension of Kubota's Theorem.

Proposition 2.1. $\kappa(g_1)\kappa(g_2) = \kappa(g_1g_2)$ *for* $g_1, g_2 \in \Gamma_o(8)$.

Proof. We need the following facts which are an immediate consequence of the reciprocity law and our previous discussion:

(i) $\left(\frac{c}{d}\right) = \left(\frac{c}{d'}\right)$ if $d \equiv d' \pmod 8$ and $\pmod c$.

(ii) $\left(\frac{d}{d'}\right)\left(\frac{d'}{d}\right)^{-1} = \frac{\varepsilon(dd')}{\varepsilon(d)\varepsilon(d')}$ if d, d' coprime to 2.

(iii) $\varepsilon(d) = \varepsilon(d')$ if $d \equiv d' \pmod 8$.

Let

$$g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

so that

$$g_1g_2 = \begin{pmatrix} * & * \\ c_1a_2 + d_1c_2 & d_1d_2 + c_1b_2 \end{pmatrix} .$$

Then if $c_1a_2 + d_1c_2 \neq 0$, $c_1 \neq 0$, $c_2 \neq 0$ it follows that

$$\kappa(g_1g_2) = \left(\frac{c_1a_2 + d_1c_2}{d_1d_2 + c_1b_2}\right) \cdot \varepsilon(d_1d_2 + c_1b_2)^{-1} .$$

Since $d_1d_2 + c_1b_2 \equiv d_1d_2 \pmod 8$, $\varepsilon(d_1d_2 + c_1b_2)^{-1} = \varepsilon(d_1d_2)^{-1}$. Then, multiplying through by $((d_2)/d_1d_2 + c_1b_2)$, (assuming $(d_2, c_1) = 1$), we have

$$\kappa(g_1g_2) = \left(\frac{c_1(1+b_2c_2)+d_1c_2d_2}{d_1d_2+c_1b_2}\right)\left(\frac{d_2}{d_1d_2+c_1b_2}\right)^{-1}$$

$$= \left(\frac{c_1}{d_1d_2+c_1b_2}\right)\left(\frac{d_1d_2+c_1b_2}{d_2}\right)\varepsilon(d_2)^{-1}\varepsilon(d_1)^{-1}$$

$$= \left(\frac{c_1}{d_1d_2}\right)\left(\frac{c_1b_2}{d_2}\right)\varepsilon(d_2)^{-1}\varepsilon(d_1)^{-1}$$

$$= \left(\frac{c_1}{d_1}\right)\left(\frac{c_1}{d_2}\right)\left(\frac{c_1}{d_2}\right)\left(\frac{c_2}{d_2}\right)\varepsilon(d_2)^{-1}\varepsilon(d_1)^{-1}$$

$$= \kappa(g_1)\kappa(g_2) \quad .$$

If $(d_2,c_1)\neq 1$, then we can replace g_2 by $g_2\begin{pmatrix}1 & \xi\\ 0 & 1\end{pmatrix}$ for suitable ξ. Then $\kappa(g_1\cdot g_2\begin{pmatrix}1 & \xi\\ 0 & 1\end{pmatrix}) = \kappa(g_1)\kappa(g_2\cdot\begin{pmatrix}1 & \xi\\ 0 & 1\end{pmatrix})$. One easily verifies, however, that $\kappa(g_2\cdot\begin{pmatrix}1 & \xi\\ 0 & 1\end{pmatrix}) = \kappa(g_2)$. The remaining cases are simple to check, and we leave them to the reader.

Now, we can define the Eisenstein series in which we are interested. Let $\mathfrak{a}$ be an ideal of k where $8|\mathfrak{a}$. Then we set

$$\Gamma_\infty = \left\{\begin{pmatrix}a & b\\ o & d\end{pmatrix}\in\Gamma_o(\mathfrak{a})\right\}\subset\Gamma_o(\mathfrak{a}) \quad .$$

Let $w=(w_v)\in H$, $w_v = x_v+iy_v+kt_v$, and let $e\in\otimes_{v\in S_\infty} w_v$. For a unit u in k, let $\tau_v\,(v\in S_\infty)$ be a set of real numbers such that

$$\prod_{v\in S_\infty}|u|_v^{2i\tau_v} = 1 \quad ,$$

and let us define for $\tau=(\tau_v)$, $s\in\mathbb{C}$

$$t^{\tau,s}(w) = \prod_{v\in S_\infty} t_v^{s+i\tau_v} \quad .$$

Also, let χ be a Dirichlet character (not necessarily primitive) on $\mathcal{O}_k/\mathfrak{a}\mathcal{O}_k$, and define for $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_o(\mathfrak{a})$

$$\chi(g) = \chi(d) \quad .$$

Finally, we define the Eisenstein series

$$E(w,s;\tau,\chi,\rho,\mathbf{e}) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_o(\mathfrak{a})} \bar{\kappa}(\gamma)\bar{\chi}(\gamma)t^{\tau,s}(\gamma w)j_\rho(\gamma,w)^{-1}\cdot\mathbf{e} \quad .$$

Recall that ρ is a representation of the unit quanternions into SL_2. This series converges absolutely for $\mathrm{Re}(s|>2$.

Our main results involve the Fourier expansions of the Eisenstein series about the "cusps". The remainder of this section will be devoted to the computation of these Fourier coefficients.

We shall first of all give a group-theoretic description of the cusps. If we assume (for simplicity) that k has class number one, the set of cusps can be identified with the set

$$\{g(\infty);\ g\in\Gamma\}$$

and we can make this correspond to $\Gamma_\infty\backslash\Gamma$ by $g\to g(\infty)$. Two cusps $g_1(\infty)$, $g_2(\infty)$ are equivalent under $\Gamma_o(\mathfrak{a})$ if there exists $\gamma\in\Gamma_o(\mathfrak{a})$ so that $\gamma g_1(\infty)=g_2(\infty)$, i.e., g_1, g_2 represent the same coset in $P(\mathfrak{a}) = \Gamma_o(\mathfrak{a})\backslash\Gamma/\Gamma_\infty$.

Let $\Gamma'_\infty=\{\begin{pmatrix}1&\alpha\\0&1\end{pmatrix};\ \alpha\in\mathcal{O}_k\}$, and define $p:\Gamma'_\infty\to\mathcal{O}_k$ by $\begin{pmatrix}1&\alpha\\0&1\end{pmatrix}\to\alpha$. Let $\Lambda(g)$ be the image under p of the subgroup $g^{-1}\Gamma_o(\mathfrak{a})g\cap\Gamma'_\infty$ of Γ'_∞.

Thus $\Lambda(g)$ is a full submodule of $\mathcal{O}_k$. If $\gamma\in\Gamma_o(\mathfrak{a})$, $g\in\Gamma_\infty$, $\delta=\begin{pmatrix}u^{-1}&*\\o&u\end{pmatrix}$ then

$$\Lambda(\gamma g\delta) = u^2\Lambda(g)\ .$$

For $g\in\Gamma$, $c\in k$, define

$$T(\mathfrak{a},g,c) = \{\gamma\in\Gamma_o(\mathfrak{a})g;c(\gamma)=c\}\ ,$$

where $c(\gamma)$ is the 2,1 entry of γ. Let

$$G(\mathfrak{a},\chi,g,c,u) = \sum_{\gamma\in\Gamma'_\infty\backslash T(\mathfrak{a},g,c)/\begin{pmatrix}1&\Lambda(g)\\0&1\end{pmatrix}} \bar{\kappa}(\gamma g^{-1})\bar{\chi}(\gamma g^{-1})e\left(\frac{\mu d(\gamma)}{c(\gamma)}\right)\ ,$$

for $\mu\in\hat{\Lambda}(g)=\{x\in k;e(xy)=1$ for all $y\in\Lambda(g)\}$, where $d(\gamma)$ is the 2,2 entry of γ and $e(x)=\exp(2\Pi i(x+\bar{x}))$. The sum is a finite one and is essentially a quadratic Gauss sum. It will be evaluated in Section 4.

Next let

$$\psi(s,\tau,\chi,\rho,g,\mu,\mathbf{e}) = \sum_{\substack{c\in\mathcal{O}_k \\ c(\text{mod units})}} G(\mathfrak{a},\chi,g,c,\mu)\omega(c)\mathbf{e}N(c)^{-s} ,$$

$$\omega(c) = \rho\left(\frac{c}{\|c\|}\right)^{-1} \prod_{v\in S_\infty} |c|_v^{-2i\tau_v} ,$$

be a Dirichlet series (matrix valued) formed with the Gauss sum defined above.

Our results also depend on the following generalized Bessel function (see Section 3). Put for $\mu\in\mathbb{C}$, $s\in\mathbb{C}$

$$K_1(\mu,s,\rho) = \int_{\mathbb{C}} e(-\mu z) \frac{1}{(1+|z|^2)^s} \rho\left(\frac{z+k}{\|z+k\|}\right)^{-1} dm(z) ,$$

where $e(\zeta) = \exp(2\pi i(\zeta+\bar{\zeta}))$ and m is the Lebesque measure. Furthermore, if $\rho = \otimes\rho_v$, $\mu = (\mu_v)$, $\tau = (\tau_v)$, we set

$$K(\mu,s,\tau,\rho) = \bigotimes_{v\in S_\infty} K_1(\mu_v, s+i\tau_v, \rho_v) .$$

We now state our main result here as

<u>Proposition 2.2</u>. *With the notations above*

$$j_\rho(g,w)^{-1}E(gw,s,\tau,\chi,\rho,e) = \delta(g,\chi,\rho)t^{\tau,s}(w) +$$

$$+ c(\Lambda(g))^{-1} \sum_{\mu\in\tilde{\Lambda}(g)} t^{\tau,2-s}(w)\Psi(s,\tau,\chi,\rho,g,\mu,\mathbf{e})K(\mu t,s,\tau,\rho)e(\mu z) ,$$

the series being absolutely and locally uniformly convergent when $\mathrm{Re}(s) > 2$. *Here*

$$\delta(g,\chi,\rho) = \begin{cases} \chi(g)\kappa(g)I & g\in\Gamma_0(\mathfrak{a}) \\ 0 & g\in\Gamma_0(\mathfrak{a}) , \end{cases}$$

$$C(\Lambda) = m(2(\Lambda)C^N) \qquad (\Lambda \text{ a full module in } k \ (\deg(k) = N)).$$

Proof. That the Eisenstein series converges in $\mathrm{Re}(s) > 2$ is well known. Also, the series represents a real analytic function in w. Moreover, the function on the left of the equation above is invariant under $\{\begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix}; \lambda \in \Lambda(g)\}$. Thus it can be expanded in a Fourier series which has the described qualities of convergence. The Fourier series is obtained by considering the function as a function of z, $z = (z_v)$ when $w = ((2_v, t_v))$ and the t_v are held fixed. It, therefore, only remains to make this series explicit. Let $\mu \in \tilde{\Lambda}(g)$ and we see that

$$j_\rho(g,w)^{-1} E(gw,s,\tau,\chi,\rho)\mathbf{e}) = \Sigma\, x(\mu,t)e(\mu z)$$

where $t = (t_v)$ and

$$x(\mu,t) = \frac{1}{c(\Lambda(g))} \int_{i(\Lambda(g))\backslash C^N} j_\rho(g,(z,t))^{-1} E(g((z,t)),s,\tau,\chi,\rho,\mathbf{e})\, e(\mu z)\, dm(z) .$$

Here we have suppressed most of the variables on which x depends. Into this, we substitute the series expansion for $E(g((z,t)),s,\tau,\chi,\rho,\mathbf{e})$, namely

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_o} \bar{\kappa}(\bar{\gamma})\chi(\gamma) t^{\tau,s}(\gamma g((z,t))) j_\rho(\gamma, g(z,t)) \cdot \mathbf{e} \quad ,$$

and then we interchange the order of summation and integration.

One term plays a peculiar role. If $g \in \Gamma_o(\mathfrak{a})$ then in the series of E above, the term $\gamma = g^{-1}$ yields 0 if $\mu \neq 0$ and

$$\kappa(g)\chi(g)\, I t^{\tau,s}(w)\mathbf{e}$$

otherwise. This yields the first term in Proposition 2.2.

For the other terms, we observe if $g \notin \Gamma_o(\mathfrak{a})$, $\gamma \in \Gamma_o(\mathfrak{a})g$ then the cosets

$$\Gamma_\infty \gamma \begin{pmatrix}1 & \lambda\\ 0 & 1\end{pmatrix} \qquad \lambda \in \Lambda(g)$$

are distinct from one another. Thus the integral representing $x(\mu,t)$ can be written as

$$c(\Lambda(g))^{-1} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_o(\mathfrak{a})g/\begin{pmatrix}1 & \Lambda(g)\\ 0 & 1\end{pmatrix}} \bar{\chi}(\gamma g^{-1})\bar{\kappa}(g^{-1})$$

$$\cdot \int_{C^N} j_\rho(\gamma(z,t))^{-1} \mathbf{e} \cdot t^{\tau,s}(\gamma(z,t)) e(-\mu z)\, dm(z) .$$

For all the γ appearing here, the 2,1 entry is nonzero. Let $\gamma = \begin{pmatrix} a(\gamma) & b(\gamma) \\ c(\gamma) & d(\gamma) \end{pmatrix}$ and replace z by $z(d(\gamma)/c(\gamma))$. The integral then becomes

$$C(\Lambda(g))^{-1} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_o(\mathfrak{a})/\begin{pmatrix} 1 & \Lambda(g) \\ 0 & 1\end{pmatrix}} \bar{\chi}(\gamma g^{-1})\kappa(\gamma g^{-1})e(\mu d(\gamma)/c(\gamma))$$

$$\cdot \int_{C^N} j_\rho\left(\begin{pmatrix} * & -c^{-1} \\ c & - \end{pmatrix}, (z,t)\right)^{-1} \mathbf{e} t^{\tau,s}\left(\begin{pmatrix} * & -c^{-1} \\ c & 0 \end{pmatrix}(z,t)\right)$$

$$\cdot e(-\mu z)\ dm(z)\ .$$

It is clear that in neither of the two terms where the 1,1 entry appears is it relevant. We now separate γ according to the different values of c by regarding $\Gamma_o(\mathfrak{a})g$ as $UT(\mathfrak{a},g,c)$ ($u\cdot\{I\}$ if $g\in\Gamma_o(\mathfrak{a})$). Thus, in view of the notations we have already established this is

$$\Lambda(g)^{-1} \sum_{c(\text{mod units})} G(\mathfrak{a},\chi,g,c,\mu) \int_{C^N} j_\rho\left(\begin{pmatrix} 0 & -c^{-1} \\ c & 0 \end{pmatrix}, (z,t)\right)^{-1} \mathbf{e}\ \cdot$$

$$\cdot\ t^{\tau,s}\left(\begin{pmatrix} 0 & -c^{-1} \\ c & 0 \end{pmatrix}(z,t)\right) e(-\mu z)\ dm(z)\ .$$

Then, on replacing z by $tz(=t_v z_v)$ and using the homogeneity properties of the functions we get

$$c(\Lambda(g))^{-1} \sum_{c(\text{mod units})} G(\mathfrak{a},\chi,g,c,\mu) \int_{C^N} j_\rho\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, (z,1)\right)^{-1} \cdot$$

$$\cdot \prod_v (1+|z_v|^2)^{-i\tau_v - s}\ e(-\mu t z)\ dm(z)\ \cdot$$

$$\cdot\ j_\rho\left(\frac{c}{\|c\|}\right)^{-1} \mathbf{e} \prod_v t_v^{2-i\tau_v - s} \cdot N(c)^s \prod |\rho_v(c)|^{-i\tau_v}$$

which is

$$c(\Lambda(g))^{-1} \sum_{c(\text{mod units})} t^{-\tau,2-s}(w)K(\mu t,s,\tau,\rho)G(\mathfrak{a},\chi,g,c,\mu)N(c)^{-s}\omega(c)\mathbf{e}\ .$$

This is the result which we quoted once we introduce the definitions of $\Psi(s,\tau,\chi,\rho,g,\mu,\mathbf{e})$.

3. Bessel Functions

We shall now describe the representations of $SU_2(\mathbb{C})$ and the associated functions $K_1(\mu,s,\tau,\rho)$. The irreducible ρ form a family parametrized by an integral $\ell \geq 0$. This representation acts on $\mathbb{C}^{\ell+1}$, and with respect to the standard basis of $\mathbb{C}^{\ell+1}$, numbered $e_0 = (1,0,\ldots,0)$, $e_1 = (0,1,0,\ldots,0),\ldots e_\ell = (0,0,\ldots,0,\ell)$; ρ_ℓ has coefficients $c_{i,j}(g)$ determined by

$$\sum_{0\leq j\leq \ell} c_{i,j}(g) X^j Y^{\ell-j} = (aX+bY)^i(-\bar{b}X+\bar{a}Y)^{\ell-i}$$

if $g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a}\end{pmatrix}$. This is the ℓ-th symmetric power of the standard representation; it extends to $SL_2(\mathbb{C})$. Recall that

$$K_1(\mu,s,\rho) = \int_{\mathbb{C}} e(-\mu z)\,\frac{1}{(1+|z|^2)^s}\,\rho\left(\frac{z+k}{\|z+k\|}\right)^{-1} dm(z) .$$

The following proposition evaluates those coefficients of $K_1(\mu,s,\rho_\ell)$ with respect to the basis $e_0,\ldots,e_\ell$, that we shall need.

We shall also write

$$M_1(s,\rho) = K_1(0,s,\rho_\ell) .$$

<u>Proposition 3.1.</u> *With the notations above*

$$\Gamma\left(s+\frac{\ell}{2}\right) K_1(\mu,s,\rho_\ell)$$

can be analytically continued as analytic functions in s, *not identically zero in* μ *for any* s, *such that*

$$\Gamma\left(s+\frac{\ell}{2}\right)\cdot K(\mu,s,\rho) \sim \pi^{3/2} e^{-4\pi|\mu|}\,(2\pi|\mu|)^{s+\frac{\ell}{2}-1}\,|\mu|^{-\tau}\rho^*\left(\frac{\mu}{|\mu|}, \frac{\bar{\mu}}{|\mu|}\right),$$

where ρ^* *is an* $(\ell+1)\times(\ell+1)$ *matrix which depends polynomially on its arguments.*

$$\text{(a)}\quad M_1(s,\rho_\ell)_{ij} = \begin{cases} 0 & i+j\neq \ell \\[2ex] \pi(-1)^j\,\dfrac{\Gamma\left(s+\frac{\ell}{2}-1-i\right)\Gamma\left(s+\frac{\ell}{2}-1-j\right)}{\Gamma\left(s+\frac{\ell}{2}\right)\Gamma\left(s-\frac{\ell}{2}-1\right)} & i+j=\ell \end{cases}$$

(b) *The following evaluations hold:*

$$\Gamma\left(s+\frac{\rho}{2}\right) K_1(\mu,s,\rho_\ell)_{0j} = 2\pi(2\pi|\mu|)^{s+\frac{\ell}{2}-1}$$

$$(-2\bar{\mu}/|\mu|)^{\ell-j} \frac{\ell!}{j!(\ell-j)!} K_{s-\frac{\ell}{2}+j-1}(|4\pi|\mu) ,$$

$$\Gamma\left(s+\frac{\ell}{2}\right) K_1(\mu,s,\rho_\ell)_{\ell j} = 2\pi(2\pi|\mu|)^{s+\frac{\ell}{2}-1}$$

$$(i\mu/|\mu|)^j \frac{\ell!(-1)^\ell}{j!(\ell-j)!} K_{s+\frac{\ell}{2}-j-1}(4\pi|\mu|) ,$$

$$\Gamma\left(s+\frac{\ell}{2}\right) K_1(\mu,s,\rho_\ell)_{j0} = 2\pi(2\pi|\mu|)^{s+\frac{\ell}{2}-1} (i\bar{\mu}/|\mu|)^{\ell-j}(-1)^\ell$$

$$K_{s-\frac{\ell}{2}+j-1}(4\pi|\mu|) ,$$

$$\Gamma\left(s+\frac{\ell}{2}\right) K_1(\mu,s,\rho_\ell)_{j\ell} = 2\pi(2\pi|\mu|)^{s+\frac{\ell}{2}-1} (-i\mu/|\mu|)^j$$

$$K_{s+\frac{\ell}{2}-j-1}(4\pi|\mu|) .$$

Here $K_\nu(x)$ *is the usual Bessel function, which we shall take as being defined by*

$$K_\nu(x) = \frac{1}{2}\int_0^\infty t^\nu e^{-\frac{x}{2}(t+t^{-1})} t^{-1}\, dt .$$

Proof. Recall that

$$K(\mu,s,\rho) = \int \rho\left(\frac{z+\mathbf{k}}{\|z+\mathbf{k}\|}\right)^{-1} e(-\mu z)(1+|z|^2)^{-s}\, dm(z) ,$$

and we are identifying $SU_2(C)$ with the unit quaternions through

$$1 \to \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \mathbf{i} \to \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix} \; \mathbf{k} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

and so

$$\frac{z+k}{\|z+k\|} \quad \text{corresponds to} \quad \begin{pmatrix} z & -1 \\ 1 & \bar{z} \end{pmatrix}(|z|^2+1)^{-1/2} .$$

The inverse of this matrix is $\begin{pmatrix} \bar{z} & 1 \\ -1 & z \end{pmatrix}(|z|^2+1)^{-1/2}$. Thus the ij-th coefficient of $\rho_\ell((z+\mathbf{k})/\|z+\mathbf{k}\|)^{-1}$ is $\rho \cdot j(z)$ where

$$\sum \gamma_{ij}(z)X^jY^{\ell-j} = (\bar{z}X+Y)^i(-X+zY)^{\ell-i}/(|z|^2+1)^{\ell/2}$$

and we can write

$$\gamma_{ij}(z) = \sum_{\substack{0\leqslant a\leqslant \ell-i \\ 0\leqslant b\leqslant i}} d_{ij}(a,b)z^a\bar{z}^b\cdot(|z|^2+1)^{-\ell/2}$$

and $d_{ij}(a,b)$ are certain integers. As examples

$$\gamma_{0j}(z) = (-1)^j \frac{\ell!}{j!(\ell-j)!} z^{\ell-j}/(|z|^2+1)^{\ell/2} ,$$

$$\gamma_{\ell j}(z) = \frac{\ell!}{j!(\ell-j)!} \bar{z}^j/(|z|^2+1)^{\ell/2} ,$$

$$\gamma_{j0}(z) = z^{\ell-i}/(|z|^2+1)^{\ell/2} ,$$

$$\gamma_{j\ell}(z) = (-1)^{\ell-i}\bar{z}^i/(|z|^2+1)^{\ell/2} .$$

Hence the ij-th entry of $K_1(\mu,s,\rho_\ell)$ is

$$\sum d_{ij}(a,b) \int_{\mathbb{C}} \frac{z^a\bar{z}^b \, e(-\mu z)}{(1+|z|^2)^{s+\ell/2}} \, dm(z) .$$

Now, as

$$\int_0^\infty e^{-u(1+|z|^2)} u^{s+\frac{\ell}{2}-1} \, du = \Gamma\left(s+\frac{\ell}{2}\right)/(1+|z|^2)^{s+\frac{\ell}{2}} ,$$

$\Gamma\left(s+\frac{\ell}{2}\right) K_1(\mu,s,\rho_\ell)ij$ is given by

$$\sum_{a,b} d_{ij}(a,b) \iint z^a\bar{z}^b \, e^{-u(1+|z|^2)^2} \, e(\mu z) u^{s+\frac{\ell}{2}-1} \, du \, dm(z) .$$

If we now let $\partial = \partial/\partial\mu$, $\bar{\partial} = \partial/\partial\bar{\mu}$ this is

$$\sum_{a,b} d_{ij}(a,b)(2\pi i)^{-a-b}\partial^a\bar{\partial}^b \iint e^{-u(1+|z|^2)-2\pi i(\mu z+\bar{\mu}\bar{z})} u^{s+\frac{\ell}{2}-1} \, du \, dm(z).$$

The integral in z can now be evaluated by the usual method. This yields

$$\sum_{a,b} d_{ij}(a,b)(2\pi i)^{-a-b}\partial^a\bar{\partial}^b \int_0^\infty e^{-u-4\pi^2|\mu|^2/u} u^{s+\frac{\ell}{2}-2} \, du \cdot \pi .$$

It is now straightforward to derive Proposition 3.3(a) from this. We can obtain the analytic continuation at once. To obtain the asymptotic estimate, carry out the differentiations and then replace u by $2\pi|\mu|u$. One obtains expressions of the form

$$\int_0^\infty e^{-2\pi|\mu|(u+u^{-1})} u^{s+\frac{\ell}{2}-N-1} \, du \, |\mu|^{s+\frac{\ell}{2}-N} \mu^m \bar{\mu}^{m'}$$

$(m+m' \leqslant N \leqslant a+b)$ with equality achieved for one term with $m=a$, $m=b$. The asymptotic estimates now follow by Watson's lemma [W.W.].

If $a=0$ (or $b=0$) then it is easy to write down the derivative explicitly. Suppose $b=0$, then we find

$$\partial^a \int_0^\infty e^{-u-4\pi^2|\mu|^2/u} u^{s+\frac{\ell}{2}-2} \, du$$

$$= (-4\pi\mu)^a \int_0^\infty e^{-u-4\pi^2|\mu|^2/u} u^{s+\frac{\ell}{2}-a-2} \, du$$

$$= (-\bar{\mu}/|\mu|)^a (2\pi|\mu|)^{s+\frac{\ell}{2}-1} \int_0^\infty e^{-2\pi|\mu|(u+u^{-1})} u^{s+\frac{\ell}{2}-a-2} \, du$$

$$= (-\bar{\mu}/|\mu|)^a (2\pi|\mu|)^{s+\frac{\ell}{2}-1} \cdot \frac{1}{2} \cdot K_{s+\frac{\ell}{2}-a-1}(4\pi|\mu|) \quad .$$

With this result, the corresponding result for $a=0$.

Finally, we prove the formula of Proposition 2.3(b). The j-th coefficient is

$$\sum d_{ij}(a,b) \int_{\mathbb{C}} z^a \bar{z}^b (1+|z|^2)^{-s-\frac{\ell}{2}} \, dm(z) \quad .$$

All the terms here vanish except for those with $a = b$. One has

$$d_{ij}(a,a) = \begin{cases} 0 & (i+j \neq \ell) \\ \dfrac{i!}{(i-a)!a!} \dfrac{j!}{(j-a)!a!} (-1)^{j-a} & (i+j = \ell) \ . \end{cases}$$

If we convert to polar coordinates (finally $x = |z|^2$),

$$\int_{\mathbb{C}} \frac{|z|^{2a}}{(1+|z|^2)^{s+\ell/2}} \, dm(z) = \pi \int_0^\infty \frac{x^a \, dx}{(1+x)^{s+\ell/2}}$$

$$= \pi \frac{\Gamma(a+1)\Gamma\left(s+\frac{\ell}{2}-a-1\right)}{\Gamma\left(s+\frac{\ell}{2}\right)} \quad .$$

Thus, if $i+j = \ell$,

$$M_1(s,\rho)_{ij} = \pi \sum \frac{i!}{(i-a)!a!} \frac{j!}{(j-a)!a!} \frac{a!\Gamma\left(s+\frac{\ell}{2}-a-1\right)}{\Gamma\left(s+\frac{\ell}{2}\right)} (-1)^{j-a}$$

$$= \pi \cdot \Gamma\left(s+\frac{\ell}{2}\right)^{-1} i!j! \sum_{\substack{0 \leqslant a \leqslant i \\ 0 \leqslant a \leqslant j}} \frac{\Gamma\left(s+\frac{\ell}{2}-a-1\right)}{(i-a)!(j-a)!a!} (-1)^{j-a} \quad .$$

It is a consequence of Gauss' theorem on $F(a,b',c,1)$ that this latter sum is (cf. [W.W.])

$$\pi \, \Gamma\left(s+\frac{\ell}{2}\right)^{-1} (-i)^j \frac{\Gamma\left(s+\frac{\ell}{2}-1-2\right)\Gamma\left(s+\frac{\ell}{2}-1-j\right)}{\Gamma\left(s-\frac{\ell}{2}-1\right)} \quad .$$

Alternatively, this may be proved by the method of partial functions; but we shall leave this as an exercise for the reader. The proof of Proposition 3.1 is now complete.

4. Gauss Sums

In this section, we shall discuss the Gauss sums $G(\mathfrak{a},g,c,\mu)$. The dependence on c is nearly multiplicative and we shall make this precise. Moreover, the dependence on primes not dividing $\mathfrak{a}$ is comparatively simple; the dependence on primes dividing $\mathfrak{a}$ is complicated, and in this case we shall content ourselves with partial results, which suffice for our purposes.

Before we begin, we recall the definition of the function $\varepsilon(d)$, which satisfies

$$\text{(i)} \qquad \varepsilon(dd') = \varepsilon(d)\varepsilon(d') \prod_{v|2} (d,d')_v \qquad (d \text{ coprime to } 2)$$

$$\text{(ii)} \qquad \varepsilon(d)N(d)^{1/2} = \sum_{x(d)} \left(\frac{x}{d}\right) e\left(\frac{x}{\delta_k d}\right) \qquad (d \text{ sq. free})$$

where δ_k is a generator of the difference of k, d coprime to $2\delta_k$. The particular ε depends on the choice of δ_k, but once it is chosen ε is entirely determined (it depends on d modulo 4).

As a consequence of this, if d,d' are coprimes ,

$$\left(\frac{d}{d'}\right)\left(\frac{d'}{d}\right) = \frac{\varepsilon(dd')}{\varepsilon(d)\varepsilon(d')},$$

and, by the law of quadratic reciprocity, the left-hand side is

$$\prod_{v|2} (d,d')_v$$

and, by (i), this is also the right-hand side.

Now let

$$g(\mu,c) = \sum_{x(c)} \left(\frac{x}{c}\right) e\left(\frac{\mu x}{\delta_k c}\right) \qquad (c \text{ coprime to } 2).$$

Then, a standard argument using the Chinese Remainder Theorem shows that if $c = c_1,c_2,c_1,c_2$ coprime, then

$$g(\mu,c) = \left(\frac{c_1}{c_2}\right)\left(\frac{c_2}{c_1}\right) g(\mu,c_1)g(\mu,c_2) \quad .$$

Thus for c coprime to 2, $g(\mu,c)/\varepsilon(c)$ is multiplicative. The following lemma will be of use to us.

Lemma 4.1. *Let* $\mu = \pi^m \mu_1$ (μ_1 coprime to π), $c = \pi^t$. *Then*

$$g(\mu,c)/\varepsilon(c) = N(\pi)^{t-\frac{1}{2}}\left(\frac{\mu_1}{\pi}\right) \qquad t \text{ odd}, \quad m = t-1$$

$$= -N(\pi)^{t-1} \qquad t \text{ even}, \quad m = t-1$$

$$= \phi(\pi^t) \qquad t \text{ even}, \quad m \geq t$$

$$= 0 \qquad 0 \text{ otherwise},$$

where ϕ *is the Euler totient function.*

Proof. This is entirely elementary and we sketch the proof. If t is odd then the character is nontrivial and so the sum vanishes if $m \leq t$. If $m < t-1$ the sum also vanishes as $(\overset{*}{\pi})$ has conductor π. This leaves only $m = t-1$. This only depends on $x (\text{mod } \pi)$ and is then (apart from the factor μ_1, which is extracted by replacing by x/μ_1 (mod c)) the Gaussian in (ii). The other cases are analogous but simpler.

Now recall that

$$G(\mathfrak{a},\chi,g,c,\mu) = \sum_{\gamma \in \Gamma'_\infty \backslash T(\mathfrak{a},g,c)/\begin{pmatrix}1 & \Lambda(g)\\ 0 & 1\end{pmatrix}} \bar{\kappa}(\gamma g^{-1})\bar{\chi}(\gamma g^{-1}) e\left(\frac{\mu d(\gamma)}{c(\gamma)}\right),$$

where g is fixed. Let

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad \gamma = \begin{pmatrix} a(\gamma) & b(\gamma) \\ c & d(\gamma) \end{pmatrix}.$$

Then

$$\gamma g^{-1} = \begin{pmatrix} * & * \\ cD - d(\gamma)C & -cB + d(\gamma)A \end{pmatrix}$$

and $d(\gamma)$ runs through all possibilities which satisfy

$$d(\gamma)C \equiv cD \pmod{\mathfrak{a}}$$
$$d(\gamma) \pmod{c\,\Lambda(g)}$$
$$d(\gamma),\ c \ \text{coprime}.$$

It is moreover easy to see that $\lambda \in \Lambda(g)$ if and only if

$$g\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} g^{-1} \in \Gamma_0(\mathfrak{a}).$$

Thus

$$\lambda c^2 \equiv \mathfrak{a}$$

and hence if $C \neq 0$

$$\Lambda(g) = C^{-2}\mathfrak{a} \cap \mathcal{O}$$

and, if $C = 0$ $\Lambda(q) = \mathcal{O}$.

Now

$$\bar{\kappa}(\gamma g^{-1}) = \left(\frac{cD - dC}{-cB + dA}\right) \varepsilon(-cB + dA) ,$$

$$\bar{\chi}(\gamma g^{-1}) = \bar{\chi}(-cB + dA) .$$

To effect the summation we let, if $A \neq 0$,

$$d_1 = -cB + dA ,$$

and, if $A = 0$,

$$d_1 = cD - dC .$$

In the first case the range of summation of d_1 is

$$cD - dC = \frac{1}{A}(c - d_1 C)$$

$$d_1 C \equiv c \ (\mathfrak{a}A)$$

$$d_1 \quad \text{mod } cA\,\Lambda(g)$$

$$d_1 \quad \text{coprime to } A^{-1}(c - d_1 C) ,$$

and in the second case, in which $-cB$ is coprime to $\mathfrak{a}$,

$$d_1 \equiv 0 \ (\mathfrak{a})$$

$$d_1 \quad \text{mod } C\,\Lambda(g)$$

$$d_1 \quad \text{coprime to } cB.$$

In the first case the sum is

$$\left\{\sum_{d_1} \left(\frac{A^{-1}(c - d_1 C)}{d_1}\right) \varepsilon(d_1)\bar{\chi}(d_1) e\left(\frac{\mu d_1}{A_c}\right)\right\} e\left(\frac{\mu B}{A}\right)$$

and in the second

$$\left\{\sum_{d_1} \left(\frac{d_1}{Bc}\right) e\left(-\frac{d_1}{Cc}\right)\right\} e\left(\frac{\mu D}{C}\right) \ e(-Bc)^{-}(-Bc) .$$

Now, these can be evaluated in general, but we shall not need this. We shall, in fact, only make use of these when $g = I$ and $g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

<u>Case 1</u>: <u>$g = I$</u>. Now the upper formula simplifies to

$$\sum_{d_1} \left(\frac{c}{d_1}\right) \cdot \varepsilon(d_1) \cdot \bar{\chi}(d_1)\ e\left(\frac{\mu d_1}{c}\right) \quad ,$$

where we require $c \equiv 0(\mathfrak{a})$ and d_1 satisfies

$$d_1 \text{ mod.} c, \qquad d_1 \text{ coprime to } c.$$

To evaluate this we let

$$c = c_1 c_2\ , \qquad (c_2, \mathfrak{a}) = (1), \qquad c_1 |_{\mathfrak{a}}^{\infty} .$$

Then there exist U_1, U_2 so that

$$U_1 c_2 + U_2 c_1 \ = \ 1, \qquad U_2 \equiv 0(\mathfrak{a}) \ .$$

Also let

$$d_1 \ = \ U_1 c_2 \delta_1 + U_2 c_1 \delta_2$$

δ_1 runs $\text{mod.} c_1$, δ_2 $\text{mod.} c_2$ and $(\delta_1, c_1) = (1)$, $(\delta_2, c_2) = (1)$.

Now

$$\left(\frac{c_1 c_2}{d_1}\right) = \left(\frac{c_1}{d_1}\right) \cdot \left(\frac{c_2}{d_4}\right)$$

and, if we use the law of quadratic reciprocity and the symbol $(\ ,\ )_{\mathfrak{a}}$ defined by

$$(\alpha, \beta)_{\mathfrak{a}} \ = \ \prod_{v|\mathfrak{a}} (\alpha, \beta)_v \ ,$$

here

$$\left(\frac{c_1}{d_1}\right) \ = \ (c_1, \delta_1)_{\mathfrak{a}} \ ,$$

$$\left(\frac{c_2}{d_1}\right) \ = \ \left(\frac{d_1}{c_2}\right) \cdot (c_2, \delta_1)_{\mathfrak{a}} \ ,$$

and thus the sum becomes

$$\left\{\sum_{\delta_1} (c_1\cdot c_2,\delta_1)_{\mathfrak{a}}\varepsilon(\delta_1)\bar{\chi}(\delta_1)e\left(\frac{\mu\delta_1 U_1}{c_1}\right)\right\}\cdot\left\{\sum_{\delta_2}\left(\frac{\delta_2}{c_2}\right)e\left(\frac{\mu\delta_2 U_2}{c_2}\right)\right\}$$

and, making use of the usual substitution, this becomes

$$(c_1,c_2)_{\mathfrak{a}}\left\{\sum_{\delta_1}(c_1,c_2,\delta_1)_{\mathfrak{a}}\varepsilon(\delta_1)\bar{\chi}(\delta_1)e\left(\frac{\mu\delta_1 U_1}{c_1}\right)\right\}g(\delta_k\mu,c_2)\ .$$

Moreover, replacing δ_1 by $\delta_1 c_2$ in the term in brackets we obtain

$$\bar{\chi}(c_2)\varepsilon(c_2)^{-1}g(\delta_k\mu,c_2)\left\{\sum_{\delta_1}(c_1,\delta_1)\varepsilon(\delta_1)\bar{\chi}(\delta_1)e\left(\frac{\mu\delta_1}{c_1}\right)\right\}\ .$$

The term in braces is complicated and there is no point in investigating it more closely here; but simply denote it by $\Gamma(\chi,\mu,c_1)$. Thus

$$G(\mathfrak{a},g,c_1c_2,\mu) = \bar{\chi}(c_2)\cdot\varepsilon(c_2)^{-1}g(\delta_k\mu,c_2)\Gamma(\chi,\mu,c_1)\ .$$

Case 2: $g=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$. In this case the formula simplifies to

$$\left\{\sum_{d_1}\left(\frac{d_1}{c}\right)\cdot e\left(-\left|\frac{\mu d_1}{c}\right|\right)\right\}\varepsilon(-c)\bar{\chi}(-c)\ ,$$

where $d_1\equiv 0(\mathfrak{a})$, d_1 mod c, $(d_1,c)=(1)$ in particular c is coprime to $\mathfrak{a}$ and we obtain

$$\varepsilon(-c)\bar{\chi}(-c)g(\delta_k\mu,-c)\ .$$

We write this finally as

$$\varepsilon(-c)^{-1}g(\delta_k\mu,-c)\cdot(-1,c)_{\mathfrak{a}}\bar{\chi}(-c)\ .$$

We summarize the results of these computations in the following.

Proposition 4.2. *With the notations above*

$$G(\mathfrak{a},\chi,1,c_1c_2,\mu) = \bar{\chi}(c_2)\varepsilon(c_2)^{-1}g(\delta_k\mu,c_2)\Gamma(\chi,\mu,c_1)\ ,$$

where

(a) $|c_1|^\infty$, $(c_2,\mathfrak{a}) = (1)$, *and*

$$\Gamma(\chi,\mu,c_1) =.\sum_{\delta(c_1)}(c_1,\delta)\varepsilon(\delta)\bar{\chi}(\delta)e\left(\frac{\mu\delta}{c_1}\right)$$

$$\text{(b)} \qquad G\left(\mathfrak{a},\chi\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix},c,\mu\right) = \begin{cases} 0 & \text{unless} \quad (c,\mathfrak{a}) = (1) \\ \varepsilon(-c)^{-1} g(\delta_k\mu,-c),(-1,-c)_{\mathfrak{a}}\bar{\chi}(-c) \\ \text{if} \quad (c,\mathfrak{a}) = (1) . \end{cases}$$

Now, let ω_1 be a Grössencharakter of k. We define

$$L(e,\omega_1,\mathfrak{a}) = \prod_{\mathfrak{p}|\mathfrak{a}} \left(1 - \frac{\omega_1(\mathfrak{p})}{(N\mathfrak{p})^s}\right)^{-1} .$$

<u>Proposition 4.3</u>. *With the notation of Section 2, recall that*

$$\omega(c) = \rho\left(\frac{c}{\|c\|}\right)^{-1} \prod_{v\in S_\infty} |c|_v^{-2i\tau_v} .$$

Now, suppose e *satisfies* $\chi(c)\omega(c)e = \omega_1(c)\mathbf{e}$ *where* ω_1 *is a Grössencharakter of* k. *Then if* $u \neq 0$, *and* f_μ *is the conductor of the Grössencharakter associated with* $c \to (-\delta_k\mu/c)$, *we have*

$$\Psi(s,\tau,\chi,\rho,1,\mu,\mathbf{e}) = \left\{\sum_{c_1} \Gamma(\chi,\mu,c_1)\omega_1(c_1)\right\}$$

$$\cdot \prod_{\substack{\pi|\mu\delta \\ \pi\nmid\mathfrak{a}}} P_\pi(s,\omega_1,\mu)L(s-1/2,\omega_1\eta_\mu,\mathfrak{a}f_\mu)L(2s-1,\omega_1^2,\mathfrak{a})^{-1}\mathbf{e} ,$$

$$\Psi\left(s,\tau,\chi,\rho,\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\mathbf{e}\right) = \chi(-1) \prod_{\substack{\pi|\mu\delta \\ \pi\nmid\mathfrak{a}}} P_\pi(s,\omega_1,\mu)L(s-1/2,\omega_1\eta_\mu,\mathfrak{a}f_\mu) \; L(2s-1,\omega_1^2,\mathfrak{a})^{-1}\mathbf{e} ,$$

and

$$\Psi(s,\tau,\chi,\rho,1,0,\mathbf{e}) = \left\{\sum_{c_1} \Gamma(\chi,0,c_1)\omega_1(c_1)\right\} L(2s-2,\omega_1^2,\mathfrak{a})L(s-1,\omega_1^2,\mathfrak{a})^{-1}\mathbf{e} ,$$

$$\Psi\left(s,\tau,\chi,\rho,\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, 0,\mathbf{e}\right) = \chi(-1)L(2s-2,\omega_1^2,\mathfrak{a})L(s-1,\omega_1^2,\mathfrak{a})^{-1}\mathbf{e} .$$

Let $m = \text{ord}_\pi(\mu\cdot\delta_k)$. *Then*

$$P_\pi(s,\omega_1,\mu) = (1-N(\pi)^{2-2s}\cdot\omega_1(\pi)^2)^{-1}(1-N(\pi)^{1-2s}\omega_1(\pi)^2)^{-1}$$
$$\cdot(1+\eta_\mu(\pi)N(\pi)^{1/2-s}\omega_1(\pi)+N(\pi)^{m+1/2-(m+1)s}$$
$$\cdot\omega_1(\pi)^{m+1}(\eta_\mu(\pi)+N(\pi)^{3/2-s}\omega_1(\pi)))e \qquad (m\equiv 0(2), m>0)$$
$$= (1-N(\pi)^{2-2s}\omega_1(\pi)^2)^{-1}(1-N(\pi)^{(m+1)(1-s)}\omega_1(\pi)^{m+1})e$$
$$(m\equiv 1(2)) \quad .$$

Proof. These result from combining Lemma 4.1 and Proposition 4.2. We begin with

$$\Psi(s,\tau,\chi,\rho,I,\mu,e) = \left(\sum_{c_1}\Gamma(\chi,\mu,c_1)\omega_1(c)\right)\left(\sum_{c_2}\varepsilon(c_2)^{-1}g(\delta_k\mu,c_2)\omega_1(c_2)\right)e \ .$$

Likewise

$$\Psi\left(s,\tau,\chi,\rho,\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\mu,e\right) = \chi(-1)\left(\sum_c\varepsilon(c)^{-1}g(\delta_k\mu,c)(-1,c)\omega_1(c)\right)e \quad .$$

In the first sum the second factor is up to $(-1,\ )_{\mathfrak{a}}$ the same as the main term in the second sum. Thus we can treat these together. Moreover by Lemma 4.1, the terms in these sums are multiplicative and so reduce to Euler products over the primes not dividing $\mathfrak{a}$. There are three cases to consider (if $\mu\neq 0$).

(1) π a prime, $\pi\nmid\mu$. Then the Euler factor is

$$1+N(\pi)^{1/2-s}\left(\frac{-\delta_k\mu}{\pi}\right)\omega_1(\pi)e = (1-N(\pi)^{1-2s}\omega_1(\pi)^2)1-N(\pi)^{1/2}\left(\frac{-\delta_k\mu}{\pi}\right)\omega_1(\pi)^{-1}e \, .$$

(2) π a prime, $m=\mathrm{ord}_\pi(\delta_k\mu)\equiv 0(2)$, $\pi|\mu$. Then the Euler factor is

$$\sum_{\substack{t\leqslant m\\ t\equiv 0(2)}}\phi(\pi^t)\omega_1(\pi^t)N(\pi^t)^{-s}+N(\pi)^{-(m+1)s}\left(\frac{-\delta_k\mu/\pi^m}{\pi}\right)\omega_1(\pi^{m+1})e$$
$$= (1-N(\pi)^{2-2s}\omega_1(\pi)^2)^{-1}\left(1-\left(\frac{-\delta_k\mu/\pi^m}{\pi}\right)N(\pi)^{1/2-s}\omega_1(\pi)\right)\cdot$$
$$\cdot\left(1+\left(\frac{-\delta_k\mu/\pi^m}{\pi}\right)N(\pi)^{1/2-s}\omega_1(\pi)+N(\pi)^{m+1/2-(m+1)s}\omega_1(\pi)^{m+1}\right)\cdot$$
$$\cdot\left(\left(\frac{-\delta_k\mu/\pi^m}{\pi}\right)+N(\pi)^{3/2-s}\omega_1(\pi)\right)\Big)\, e \quad .$$

(3) π a prime, $m = \mathrm{ord}_\pi(\delta_k\mu) \equiv 1(2)$. Then the Euler factor is

$$\sum_{\substack{t\leqslant m\\ t\equiv 0(2)}} \phi(\pi^t)\omega_1(\pi^t)N(\pi^t)^{-s} - N(\pi)^{m-(m+1)s}\omega_1(\pi^{m+1}))$$

$$= (1-N(\pi)^{2-2s}\omega_1(\pi^2))^{-1}(1-N(\pi)^{1-2s}\omega_1(\pi)^2-N(\pi)^{m+1(1-s)}$$
$$\omega_1(\pi)^{m+1}(1+N(\pi)^{1-2s}\omega_1(\pi^2)))e$$

$$= (1-N(\pi)^{2-2s}\omega_1(\pi)^2)^{-1}(1-N(\pi)^{1-2s}\omega_1(\pi)^2)(1-N(\pi)^{(m+1)(1-s)}$$
$$\omega_1(\pi)^{m+1})\ e \quad .$$

Now consider $\mu = 0$. One has

$$\varepsilon(c)^{-1}g(0,c) = \begin{cases} 0 & c \neq \text{square} \\ \phi(c) & c = \text{square} \end{cases} .$$

Then

$$G(\mathfrak{a},\chi,1,c_1c_2,0) = \begin{cases} \bar{\chi}(c_2)\Gamma(\chi,0,c_1)\phi(c_2) & c_2 = \text{square} \\ 0 & \text{otherwise}, \end{cases}$$

$$G\left(\mathfrak{a},\chi,\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},c,0\right) = \begin{cases} \bar{\chi}(-c)\phi(c) & (-c) = \text{square} \\ 0 & \text{otherwise}. \end{cases}$$

Thus

$$\Psi(s,\tau,\chi,\rho,1,0,e) = \left\{\sum_{c_1}\Gamma(\chi,0,c_1)\omega_1(c_1)N(c_1)^{-s}\right\}$$
$$\cdot L(2-2s,\omega_1^2,\mathfrak{a})L(2-1,\omega_1^2,\mathfrak{a})^{-1}\ e\ ,$$

$$\Psi(s,\tau,\chi,\rho,1,0,e) = \chi(-1)L(2s-2,\omega_1^2,\mathfrak{a})L(2s-1,\mathfrak{a})^{-1}\ e$$

which completes the proof of the Proposition.

Before we leave this section, it is probably worthwhile to add a few further remarks about the ε-function. The theory of theta-functions yields the formula [K_2].

$$\varepsilon(d) = N(2)^{-1/2} \sum_{x(\mathrm{mod}\ 2)} e\left(-\frac{dx^2}{4\delta_k}\right) .$$

Now ε depends on $d(\mathrm{mod}\ 4)$. Thus we can expand it in terms of multiplicative characters on $(\mathcal{O}/4))^{\times}$. Let

$$\gamma(\theta) = \sum \theta(d)\varepsilon(d)$$

for any such character θ. Let θ_0 be such that $\gamma(\theta_0) \neq 0$. Then

$$\begin{aligned}\gamma(\theta(\delta,\cdot)_{\mathfrak{a}}) &= \sum \theta_0(d)(\delta,d)_{\mathfrak{a}}\varepsilon(d)\\ &= \varepsilon(\delta)^{-1}\sum\theta(d)\varepsilon(d\delta)\\ &= \theta(\delta)^{-1}\varepsilon(\delta)^{-1}\cdot\gamma(\theta) .\end{aligned}$$

Thus consider

$$\phi(\psi)^{-1}\gamma(\theta_0)\sum_{\delta}\theta_0(\delta)^{-1}\varepsilon(\delta)^{-1}(\delta,d)\ \ \bar{\theta}_0(d)$$

which is, *a priori*, a partial sum of the Fourier expansion. Writing $(\delta,d) = \varepsilon(\delta)\varepsilon(d)/\varepsilon(\delta d)$ we obtain

$$\phi(\psi)^{-1}\gamma(\theta_0)\sum_{\delta}\theta_0(d\delta)^{-1}\varepsilon(d\delta)^{-1}\cdot\varepsilon(d) .$$

Hence this must be the complete sum, and

$$\gamma(\theta_0(-1,\cdot)_{\mathfrak{a}})\cdot\gamma(\theta_0) = \phi(\psi) .$$

Rewrite this

$$(\theta_0\varepsilon)(d) = c\sum(\theta_0\varepsilon)(\delta)^{-1}(\delta,d) ,$$

where

$$c = \left\{\sum_{\delta}\theta_0\varepsilon(\delta)^{-1}\right\}^{-1} .$$

The character θ_0 is, in general, unknown. As

$$\varepsilon(dx^2) = \varepsilon(d)$$

it is of order 2, and hence is of the form

$$\theta_0(c) = (y,c)$$

for some y, not necessarily coprime to 2. However, observe that the prime decomposition over 2, by the considerations above, is entirely determined by ε. Then one can also note that

$$\overline{\varepsilon(d)} = N(2)^{-1/2} \sum_x e\left(-\frac{dx^2}{4\delta_k}\right)$$

and also

$$\sum \overline{(\theta_0\varepsilon)}(d) = N(2)^{-1/2} \sum_k \sum_d \overline{\theta_0(d)}\, e\left(+\frac{dx^2}{4\delta_k}\right)$$

$$= N(2)^{-1/2} \sum_d \theta_0(d) e\left(\frac{d}{4\delta_k}\right) \qquad .$$

Thus there exists one θ_0 having the property that this Gauss sum is nonzero; also all others are of the form $\theta_0(\delta,\cdot)_{\mathfrak{a}}$. To obtain more precise results would involve an examination of the local reciprocity law and we shall not go into this more deeply at present. One should observe that these essentially evaluate $\Gamma(\chi,0,c_1)$; this is nonzero when $(c_1,\cdot)_{\mathfrak{a}}\bar{\chi} = \theta_0(\delta,\cdot)$. Thus $\Gamma(\chi,0,c_1) = 0$ unless χ is of order 2; in this case we see that the sum is over a set $c_1 = c_1^* x^2$ where c_1^* is fixed and x turns through $\{\pi_1^{n_1}\ldots\pi_k^{n_r};\ \pi_j|2, n_j \geqslant 0\}$.

5. Dirichlet Series

In this section we shall construct and study two families of Dirichlet series which are derived from the Eisenstein series of Section 2. The first is essentially the L-series associated by Hecke to such a form, and Hecke's methods are applicable here. The second are associated with a subfield $k_0 \subset k$. $[k:k_0] = 2$, and we shall suppose that k_0 is real, as this simplifies the discussion and is also the case which arises in the cases in which we apply our results to L-series of Diophantine interest.

First of all, we shall let ω_s be a Grössencharakter of k which has conductor dividing $\mathfrak{a}$ and which satisfies

$$\omega_s((\alpha)) = N(\alpha)^{-s} \prod_{v\in S_\infty} (i_v(\alpha)|i_v(\alpha)|)^{-m_v} |\alpha|_v^{-i\tau_v} \bar{\chi}(\alpha) \qquad (\alpha,\mathfrak{a}) = 1.$$

We shall write this as $\omega_s(\alpha)$, the m_v and $\bar{\chi}(u) = 1$ $(u \in U_k)$. Now let $\ell_v = |m_v|$ and *we shall assume* for simplicity that

$$\ell_v > 0 \quad (v \in S_\infty) \ ;$$

this is a restriction on ω, which is also justified by the applications. It is not essential. Now let

$$i(v) = 0 \qquad (m_v < 0)$$
$$= \ell_v \qquad (m_v > 0)$$

and

$$j(v) = \ell_v - i(v) \quad .$$

Then if

$$\mathbf{e} = \bigotimes_{v \in S} \varepsilon_i(v) \ , \qquad \mathbf{e}' = \bigotimes_{v \in S_\infty} \mathbf{e}_{j(v)}$$

one has, in the notations of Section 2

$$\omega_s(c)\mathbf{e} = N(c)^{-s}\bar{\chi}(c)\omega(c)\mathbf{e} \ ,$$

if $p = \bigotimes \rho_{\ell(\nu)}$, and so also $\mathbf{e} \in W(\chi,\tau)$.

Let

$$P(\omega_s,\mu) = \prod_{\substack{y | \mu\nu \\ y \nmid \mathfrak{a}}} P_y(\omega_s,\mu) \quad ,$$

where $P_y(\omega_s,\mu)$ is defined as in Proposition 4.3 and $\mathfrak{a}$ is fixed as before. Let

$$F(u,\omega_s) = \sum_{\substack{\mu (\mathrm{mod}\ U_k^2) \\ \mu \in \mathfrak{a}^{-1} r^{-1}}} P(\omega_s,\mu)L(\omega_{s-1/2}\eta_\mu, \mathfrak{a}f_\mu)N(\mu)^{-u} \ ,$$

where $U_k^2 = \{u^2 : u \in U_k\}$, it is of finite index in U_k.

<u>Theorem 5.1</u>. *The series defining* $F(u,\omega_s)$ *converges if* $\mathrm{Re}(s) > 1/2\ (\mathrm{Re}(s)+1)$ *and can be continued to a meromorphic function of finite order in* $\mathbf{C}$*. It has no poles unless there exists* τ *such that* $\tau_v = \tau$ *(all* $v \in S_\infty$*) (in which case we may assume that* $\tau = 0$*, without any loss of generality). In this case there are simple poles at* $u = 0,1$*; the one at* $u = 1$ *has residue*

$$\frac{1}{2}\,[U_k : U_k^2]R_k e_k^{-1}\chi(-1)N(\mathfrak{a})|D_k|^{1/2}L(N\omega_s^2,\mathfrak{a}) \prod_{v \in S_\infty} \{2\Gamma(1+\ell(v)/2)\} \quad .$$

Here R_k, D_k *are the regulator and discuminant of* k, $U_k^2 = \{u^2 : u \in U_k\}$ *and* e_k *is the number of roots of unity in* k*. For every* $e > 0$ *one has that*

$$|F(u,\omega_s)| \ll |I_m(u)|^{\frac{1}{2}[k\cdot Q](1+\varepsilon-\mathrm{Re}(u))} \quad .$$

Proof. As we indicated above this will be carried out by a suitable modification of the Hecke method. Let us first observe that V_ρ carries a Hermitian inner product, which we denote by (,), and $\|x\| = (x,x)^{1/2}$. Let us fix s and consider the set $X_t = \{(z,t)$ $z \in \Pi_{v\in S_\infty} C\}$. Then Γ^1_∞ acts on X_t. Write $E(w)$ for $E(w,s,\tau,\chi,\rho,e)$. Then let for a cusp $\gamma(\infty)$. $C_\gamma(A)$ be the homosphere $\gamma\{(z,t)\Pi t_v \geqslant A\}$, and A is some *fixed real* number. From reduction theory one knows that $\Gamma\backslash H - \cup_{\gamma\in\Gamma} C_\gamma(A)$ is compact. Thus on $H - \cup_{\gamma\in\Gamma} C_\gamma(A), \|E(\omega)\|$ is bounded. On the other hand, on $C_1(A)$ one has from Propositions 2.2 and 3.1 that

$$\|E(w)\| << |t|^{Re(s)} + |t|^{Re(2-s)}$$

and

$$|t| = \prod_{v\in S_\infty} t_v \quad .$$

Thus on $C_\gamma(A)$ one has

$$\|E(w)\| << |t(\gamma^{-1}w)|^{Re(s)} + |t(\gamma^{-1}w)|^{2-Re(s)} \quad .$$

For simplicity let us assume now that $Re(s) \geqslant 1$, so that only the first term matters. Then let $\gamma^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; if $w \in X_t \cap C_\gamma(A)$ then one verifies easily that $N(c) \leqslant A^{-1}|t|^{-1}$. Let us restrict our considerations to such Γ. Then one has

$$\|E(v)\|^2 << |t|^{2\ Re(s)} \Pi \ \|i_v(c)w_v + i_v(d)\|^{-2\ Re(s)} \quad .$$

We consider $\gamma \in \Gamma^1_\infty\backslash\Gamma/\Gamma_\infty$. Thus one sees that by regarding $\Gamma^1_\infty\backslash X_t$ as split into the images of $C_\gamma(A) \cap X_t$ and the test, $\int_{\Gamma^1_\infty\backslash X_t} \|E(w)\|^2\, dm(z)$ is bounded by

$$O(1) + O(|t|^{2\ Re(s)}) \sum_{\substack{\gamma^{-1} \\ N(c(\gamma) \leqslant |t|^{-1}A^{-1})}} \int (|i_v(c)z_v + z_v(d)|^2 + |i_v(c)|^2 t_v^2)^{-Re(s)}\, dm(z)$$

$$= O(1) + O\left(|t|^{2-2\ Re(s)} \sum_{N(c)\leqslant |t|^{-1}A^{-1}} \phi(c) N(c)^{-2\ Re(s)}\right) \quad .$$

From this one obtains that, for any $\varepsilon > 0$, generally

$$\int_{\Gamma^1_\infty\backslash X_t} \|E(w)\|^2\, dm(z) << 1 + |t|^{-|2-2\ Re(s)|+2}$$

if $|Re(s)-1|>\varepsilon$ the "ε" in the estimate may be dropped. Note that if s has a certain value, we may have in $C_1(A)$ that $\|E(w)\| \ll |t| \log|t|$, but this makes no difference to our final estimate. Now if we use Proposition 2.2 we see that if $Re(s)>1$

$$\sum \|\psi(s,\tau,\chi,\rho,g,\mu,e)\|^2 \ll N(u)^{1+\varepsilon}$$

if we sum over μ such that $2^{-1} < |2(\mu)_v|t_v < 2 \quad (v \in S_\infty)$ for fixed t.

Now consider

$$f_\infty(\omega) = \gamma(\mathbf{e}, E(\omega))$$

$$f_0(\omega) = \gamma(\mathbf{e}', j_\rho((\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}),w)^{-1} E((\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}),w)) ,$$

where

$$\gamma = \prod_{v \in S_\infty} \Gamma(s+i\tau(v)+\ell(v)/2) .$$

Then, as $\rho(k)^{-1}\mathbf{e} = (-1)^L \mathbf{e}'$, where $L = \Sigma_{m(0)<0}\, m(\nu)$, one has

$$f_\infty(0,t^{-1}) = (-1)^L f_0(0,t) ,$$

where $t^{-1} = (t_v^{-1})$.

On the other hand, by Propositions 2.2 and 3.1 one sees that $f_\infty(0) = f_\infty^{(1)}(\omega) + f_\infty^{(2)}(\omega)$ where (as $\ell(v)>0 \quad (v \in S_\infty)$)

$$f_\infty^{(1)}(t) = \gamma \prod_{v \in S_\infty} t_v^{s+2\tau(v)}$$

and

$$f_\infty^{(2)}(t) = c(\nu^{-1})^{-1} \sum_{\substack{\mu \in \nu^{-1} \\ \mu \neq 0}}$$

$$\prod_{v \in S_\infty} \Big\{ t_v^{2-s-i\tau(v)} \cdot 4\pi \cdot (2\pi t_v |i_v(\mu)|)^{s+i\tau(v)+\Pi(v)/2-1}$$
$$\cdot i^{\ell(\nu)} (i_v(\mu)/|i_v(\mu)|)^{-m(v)} \cdot K_{s+i\tau(v)-\ell(v)/2-1}(4\pi|i_v(\mu)|t_v) \Big\}$$
$$\cdot(e, \psi(s,\tau,\chi,\mu, \ ,\rho,e,1)) ,$$

and

$$f_0(0,t) = f_0^{(1)}(t) + f_0^{(2)}(t)$$

where

$$f_0^{(1)}(t) = c(\mathfrak{a}^{-1}\nu^{-1})^{-1}(-1)^L \prod_{v\in S_\infty} 2\pi(s+i\tau(v)+\frac{1}{2}\ell(v)-1)^{-1}$$
$$\cdot\left(e,\psi\left(s,\tau,\chi,0,\mathfrak{a},\rho,\begin{pmatrix}0 & -1\\ 1 & 1\end{pmatrix}\prod_{v\in S_\infty} t_v^{2-s-i\tau(v)}\right)\right)$$

and

$$f_0^{(2)}(t) = c(\mathfrak{a}^{-1}\nu^{-1})^{-1}(-1)^L \sum_{\substack{\mu\in\mathfrak{a}^{-1}\nu^{-1}\\ \mu\neq 0}}$$
$$\cdot \prod_{v\in S_\infty}\left\{t_v^{2-s-i\tau(v)}\pi\cdot(2\pi|i_v(\mu)|t_v)^{s+i\tau(v)+\ell(v)/2-1}\right.$$
$$\left.K_{s+i\tau_v-\ell(v)/2-1}(4\pi|i_v(\mu)|t_v)\right\}\left(e,\psi\left(s,\tau,\chi,\mu,\mathfrak{a},e\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right)\right) .$$

One should observe here that ([ ], p. 91)

$$c(\mathfrak{a}^{-1})^{-1} = |D_k|^{1/2}N(\mathfrak{a})^{-1} .$$

Now, in view of the formula

$$\int_{R_t^\times} K_\alpha(4\pi t)t^{-1+s}\,dt = \frac{1}{4}(2\pi)^{-s}\Gamma\left(\frac{s+\alpha}{2}\right)\Gamma\left(\frac{s-\alpha}{2}\right)$$

one has

$$\int_{i(U_k^2)\backslash\prod_{v\in S_\infty}\mathbb{R}_t^\times} f_0^{(2)}(t)\pi t_v^{s+i\tau(v)+2u-2}\,d^\times t = C(u)F(u,\omega_s),$$

and this converges if $\mathrm{Re}(u) > 1$. Here

$$C(u) = \chi(-1)c(\mathfrak{a}^{-1}\nu^{-1})^{-1}(-1)^L L(N\omega_s^2,\mathfrak{a})^{-1}$$
$$\cdot\prod_{v\in S_\infty}\left\{\pi(2\pi)^{1-2u}\Gamma(u+\rho_v/2)\Gamma(u+s+i\tau_v-1)\right\} .$$

Now let $V = \{t\in\prod_{v\in S_\infty}\mathbb{R}_t^\times : \prod_{v\in S_\infty} t_v = 1\}$ and let ν be an invariant form on $\prod_{v\in S_\infty}\mathbb{R}_t^\times$ such that, with a fixed order on S_∞, and $\nu_0\in S_\infty$

$$\bigwedge_{v\in S_\infty} t_v^{-1}\,dt_v = t_{v_0}^{-1}\,dt_{v_0}\wedge\nu \quad .$$

Let $|\nu|$ be the corresponding measure. Then, on the quotient space,

$$|\nu|\left(i(U_k^2)\backslash \prod_{v\in S_\infty}\mathbb{R}_t^\times\right) = [U_k:U_k^2]\cdot R_k\cdot 2^{-1/2[k:Q]}\, e_k^{-1} \quad .$$

The rest of the proof is now clear. In the traditional fashion one divides

$$\prod_{v\in S_\infty}\mathbb{R}_t^\times \ \text{ into } \ V_+ = \left\{ t\in \prod_{S_\infty}\mathbb{R}_t^\times \ \prod_{v\in S_\infty} t_v \geqslant 1\right\},$$

$$V_- = \prod \mathbb{R}_t^\times - V_+ \quad .$$

Then one writes the integral above, on V_- as

$$(-1)^L f_\infty^{(1)}(t^{-1}) = f_0^{(1)}(t) + (-1)^L f_\infty^{(2)}(t^{-1}) \ .$$

These show that

$$C(u)F(u,\omega_s) = (-1)^L \int_{i(u_k^2)\backslash V_+} f_\infty^{(1)}(t) \prod_{v\in S_\infty} t_v^{-(s+1\tau(v)+2u-2)}\, d^\times t$$

$$- \int_{i(u_k^2)\backslash V_-} f_0^{(1)}(t) \prod_{v\in S_\infty} t_v^{s+i\tau(v)+2u-2}\, d^\times t$$

$$+ \int_{i(u_k^2)\backslash V_+} f_0^{(2)}(t) \prod_{v\in S_\infty} t_v^{s+i\tau(v)+2u-2}\, d^\times t$$

$$- \int_{i(u_k^2)\backslash V_+} f_\infty^{(2)}(t) \prod_{v\in S_\infty} t_v^{-(s+i\tau(v)+2u-2)}\, d^\times t \ .$$

The first term is 0 unless for all $v\in S_\infty, \tau(v)=\tau$, when it is

$$(-1)^L|\nu|\left(2(U_k^2)\backslash \prod_{v\in S_\infty}\mathbb{R}_+^\times\right)\gamma\cdot\frac{1}{2}\cdot(u-1)^{-1} \ ,$$

and the second is

$$-c(\mathfrak{a}^{-1}\nu^{-1})(-1)^L \prod_{v\in S_\infty} 2\pi(s+i\tau(v)+\frac{1}{2}\ell(v)-1)^{-1}$$
$$\frac{1}{2}\cdot u^{-1}\left(e,\psi\left(s,\tau,\chi,0,\mathfrak{a},\rho,e,\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right)\right) .$$

The quoted results follow from the functional equation implicit here, the Phragmén-Lindelöf principle and the fact that the residue of $F(u,\omega_s)$ at $u=1$ is

$$\frac{1}{2}\gamma(-1)^L|\nu|\left(i(U_k^2)\backslash \prod_{v\in S_\infty} \mathbb{R}_t^\times\right)\cdot C(1)^{-1} .$$

Our second result requires an additional assumption, namely, that there is a totally real subfield $k_0\subset k$ with $[k:k_0]=2$. This always occurs in the applications we have in mind. Let χ_0 be the character of k_0 associated by class-field theory to the extension k/k_0. Let now $\beta\in k$ and set

$$F_\beta(u,\omega_s) = \sum_{\substack{\mu(\mathrm{mod}\ U_{k_0}^2)\\ \mu\beta^{-1}\in k_0\\ \mu\in\mathfrak{a}^{-1}\nu^{-1}}} P(\omega_s,\mu)L(\omega_{s-1/2}\eta_\mu,\mathfrak{a}f_\mu)N(\mu)^{-u/2} .$$

Observe that if $\beta=1$ then $\mu\in k_0$ and $N(\mu)^{1/2}$ is the norm of μ in k_0.

We shall also write $\chi_\beta(x)=(N_{k/k_0}(\beta)/x_0)_{k_0}$, where $(\ /\ )_{k_0}$ is the Legendre symbol in k_0. Suppose k can be represented as $k_0(\sqrt{\sigma})$ where $\delta\in\mathcal{O}_{k_0}$; then we shall write $\alpha=\beta\sqrt{\delta}$ and we shall assume that

$$\alpha^{-1}\in\mathcal{O}_k,\qquad \alpha\in\mathfrak{a}^{-1} .$$

Observe that this is not a very strong restriction.

Theorem 5.2. *The series defining* $F_\beta(u,\omega_s)$ *converges in* $\mathrm{Re}(u)>1$ *and can be continued to a meronomorphic function in* $\mathbb{C}$. *Suppose that* $\mathrm{Re}(s)\geq 1$ *then the only poles in* $\mathrm{Re}(u)>3/2-\mathrm{Re}(s)$ *are at* $u=1,2,$ $-(s+i\tau)$ *and (possibly)* 0 *or* $3-2(s+i\tau)$. *Here, as before, the notation means that there is no pole at* "$2-(s+i\tau)$" *or* "$3-2(s+i\tau)$" *unless all the* $\tau(v)$ *are equal, when* τ *is the common value. These poles are simple unless* $s+i\tau=1$ *and* $\omega_{s-1}\chi\cdot\chi_\beta$ *considered as a Grössencharakter of* k_0*) is trivial, when the pole may be of order* 2. *Moreover, for*

$\varepsilon > 0$. $3/2 - \mathrm{Re}(s) + \varepsilon < \mathrm{Re}(u) < 1 + \varepsilon$, *and* $|\mathrm{Im}(u)| > 1 + |\mathrm{Im}(s+i\tau)|$ *one has*

$$|F_\beta(u,\omega_s)| \ll |\mathrm{Im}(u)|^{[k_0:\ ](1+\varepsilon-\mathrm{Re}(u))/(\mathrm{Re}(s)-1/2)} .$$

Proof. The proof requires a number of preparations. Note that S_∞ can also be regarded by restriction as the set of infinite places of k_0. Let now

$$H_0 = \prod_{v\in S_\infty} \mathbb{H}^2 ,$$

where $\mathbb{H}^2$ is the usual upper half-plane. Let $G_0 = \Pi_{v\in S_\infty} SL_2(\mathbb{R})$. Define, for $\alpha \in k^\times$,

$$\theta_\alpha : H_0 \to H; \quad (x_v + iy_v) \to (i_v(\alpha)x_v + |i_v(\alpha)|y_v\mathbf{k}) .$$

Let $i_v : SL_2(\mathcal{O}_{k_0}) \to SL_2(\mathcal{O}_k)$ be the natural injection. Let $D_\alpha = \begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}$. Then, under the assumption that $\alpha^{-1} \in \mathcal{O}_k$, $\alpha \in \mathfrak{a}^{-1}$ one has

Lemma 5.3. *The stabilizer* $\theta_\alpha(H_0)$ *in* $\Gamma_0(\mathfrak{a})$ *is* $D_\alpha \Gamma(\mathfrak{a},\alpha) D_\alpha^{-1}$ *where*

$$\Gamma(\mathfrak{a},\alpha) = i_0\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in SL_2(\mathcal{O}_{k_0}) : \right.$$

$$\left. (a,d,\mathfrak{a}) = (1),\quad b \in k_0 \cap \alpha^{-1}\mathcal{O}_k,\ c \in k_0 \cap \alpha\mathfrak{a} \right\} .$$

If $\gamma \in \Gamma(\mathfrak{a},\alpha)$ *then*

$$\kappa(D_\alpha \gamma D_\alpha^{-1}) = \chi_{\alpha^{-1}}(d) \qquad \textit{if} \quad \gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix} .$$

Proof. Clearly $\theta_\alpha(z) = D_\alpha \theta_1(z)$. Thus, observing that the stabilizer of $\theta_1(H_0)$ in G is G_0, we see that the required group is $i^{-1}(i(\Gamma_0(\ )) \cap D_\alpha G_0 D_\alpha^{-1})$; a simple computation shows that this is $D_\alpha \Gamma(\ ,\alpha) D_\alpha^{-1}$

$$\kappa(D_\alpha \gamma D_\alpha^{-1}) = (c\alpha^{-1}/d)_k \varepsilon(d)^{-1} \qquad (c \neq 0)$$

$$= \varepsilon(d)^{-1} \qquad (c = 0) .$$

By well-known properties of the Hilbert norm-residue symbol (cf. [BMS]

$$(c/d)_k = 1, \qquad \varepsilon(d_1 d_2) = \varepsilon(d_1)\varepsilon(d_2) \qquad (c,d,d_1,d_2 \in \mathcal{O}_{k_0})$$

and

$$(\alpha^{-1}/d)_k = \left(N_{k/k_0}(\alpha)/d_{k_0}\right) = \chi_{\alpha^{-1}}(d) \qquad \left(d \in \mathcal{O}_{k_0}\right) .$$

Thus it only remains to compute $\varepsilon(d)$. This is a character of order 2 as $\varepsilon(d)^2 = (-1/d)_k = 1$, also $\varepsilon(d) = 1$ if $d \equiv 1 \pmod 4$. Recall that

$$\varepsilon(d)N(d)^{1/2} = \sum_{x(d)} (x/d)_k e(x/d\delta_k) .$$

Note that $\varepsilon(du) = \varepsilon(d)$ as we have already verified that $\varepsilon(u) = 1$ if $u \in U_k$. Now suppose that δ_0 is the relative different of k/k_0; then $\Delta = \delta_k/\delta_0$ is such that $(\Delta) = \mathrm{conorm}(\nu_{k_0})$ where ν_{k_0} is the different of k_0 (cf. [W1], p. 156). Suppose that $k - k_0$ $(\sqrt{\delta_1})$, then $(\delta_0/\sqrt{\delta_1})$ is a square and hence, by taking a suitable choice of δ_0 one has $\delta_0 = \sqrt{\delta_1}\, m^2$.

Now

$$\varepsilon(d)N(d)^{1/2} = (\delta_k/d)_k \sum_{x(d)} (x/d)_k e(x/d) .$$

Let us assume that $\bar{\chi}_0(d) = -1$, then there exists a prime π of k_0 so that $d \equiv \pi \pmod 4$, we may assume that π avoids a finite set of primes. We can write $x = \alpha + \sqrt{\delta_1}\,\beta$ where α, β are summed $\pmod \pi$. Since $(\alpha/d)_k = 1$ and $e(x/d) = e(\alpha/d)$ one can easily evaluate the Gauss sum, which yields $\varepsilon(\pi) = (\sqrt{\delta_1}\delta_k/\pi)_k$ $\varepsilon(d)$ is independent of the choice of δ_k. For if δ_k is replaced by a δ_k, $\varepsilon(d)$ is replaced by $\varepsilon(du^{-1}) = \varepsilon(d)\varepsilon(u^{-1}) = \varepsilon(d)$.

Now we return to the theorem proper. Let $\theta_\alpha : \Gamma(\mathfrak{a},\alpha) \to \Gamma_0(\mathfrak{a})$ be the map $\gamma \to \begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\gamma\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}^{-1}$. Then consider $j_\rho(\theta_\alpha(\gamma),\theta_\alpha(z))$. This is itself a factor of automorphy which extends to G_0 and so one sees that it can be written as

$$\times_{v \in S_\infty} A_v \,\mathrm{diag}\left(((z_v(c)z_v + i_v(d))/|i_v(c)z_v + i_v(d)|)^{m(v,i)}A_v^{-1}\right.$$

where $m(v,i) \in \mathbf{Z}$ $(1 \leqslant i \leqslant \dim(\rho_v))$ and $\mathrm{diag}(c_i)$ is the diagonal matrix with entries $\delta_{ij}c_i$, and A_v is some matrix which depends on ρ_v. It is easy to determine that A_v, $m(v,i)$ explicitly given ρ_v.

Now let $E(w)$ be as before with $\mathrm{Re}(s) \geqslant 1$ and let

$$f_\alpha(z) = j\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \theta_\alpha(z)\right)^{-1} E\left(\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\theta_\alpha(z)\right) .$$

Then

$$f_\alpha(\gamma z) = \chi_\alpha(\gamma) j_\rho(\theta_\alpha(\gamma),\theta_\alpha(z)) f_\alpha(z) \qquad (\gamma \in \Gamma'(\mathfrak{a},\alpha)) ,$$

where

$$\chi_\alpha(\gamma) = \left(N_{k/k_0}(\alpha^{-1})/a\right)_{k_0} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and

$$\Gamma'(\mathfrak{a},\alpha) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Gamma(\mathfrak{a},\alpha) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For convenience, let

$$j_0(\gamma,z) = j_\rho(\theta_\alpha(\gamma),\theta_\alpha(z)) \quad .$$

As ε is already known to be a character we see that

$$\varepsilon(d) = (\sqrt{\delta_1}\delta_k/d)_k = (m^2\Delta/d)_k \ .$$

But since $(\Delta) = \mathrm{conorm}(\nu_{k_0})$, and as k has class-number 1, $N_{k/k_0}(\Delta)$ is a square. Thus $\varepsilon(d) = 1$, as required. This proves Lemma 5.3.

Now let us return to the theorem proper. Let $\theta_\alpha \Gamma(\mathfrak{a},\alpha) \to \Gamma_0(\mathfrak{a})$ be $\gamma \to D_\alpha \gamma D_\alpha^{-1}$. Then consider $j_\rho(\theta_\alpha(\gamma),\theta_\alpha(z))$ which is itself a factor of automorphy extending to G_0. It can be written as

$$\bigotimes_{v \in S_\infty} A_v(\mathrm{diag}(((z_v(c)z_v + iz_v(d))/|i_v(c)z_v + i_v(d)|))^{m(i,v)})A_v^{-1} ,$$

where $m(v,i) \in \mathbb{Z}$ $(i \leqslant i \leqslant \dim(\rho_v))$ and $\mathrm{diag}(\theta_i)$ represents the diagonal matrix with entries $a_i\delta_{ij}$, and A_v is some matrix which would be found explicitly. Note that if $\rho_v = \rho_{\ell(v)}$ then

$$\bigotimes_{v \in S_\infty} A_v(\mathrm{diag}((i_v(c)/|i_v(c)|)^{m(i,v)})A_v^{-1} = \prod_{v \in S_\infty} \mathrm{sgn}(i_v(c))^{\ell(v)} I ,$$

where sgn is the usual signum function $(c \in k_0^\times)$. This observation will later be reflected in a simplification in our study of Eisenstein series.

Now let $E(\omega)$ be as before, with $\mathrm{Re}(s) \geqslant 1$ and let

$$f_\alpha(\omega) = j\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\theta_\alpha(z)\right)^{-1} E\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\theta_\alpha(z)\right) \quad .$$

Then

$$f_\alpha(\gamma z) = \chi\chi_\alpha(\gamma) j_\rho(\theta_\alpha(\gamma),\theta_\alpha(z)) f_\alpha(z) \qquad (\gamma \in \Gamma(\mathfrak{a},\alpha)) ,$$

where

$$\chi_\alpha(\gamma) = \chi_\alpha(d) \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

and

$$\Gamma'(\mathfrak{a},\alpha) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Gamma(\mathfrak{a},\alpha) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} .$$

Let

$$j_0(\gamma,z) = j_\rho(\theta_\alpha(\gamma),\theta_\alpha(z)),$$

and

$$\phi_A^u(z) = \prod_{v\in S_\infty} y_v^{s+i\tau(\nu)+u-1} \quad \text{if} \quad \prod_{v\in S_\infty} y_\nu \leqslant A,\ z=(x_v+iy_v)$$

$$= \quad 0 \quad \text{otherwise} .$$

Furthermore, let

$$C_0(A) = \left\{(x_v+iy_v) \in H_0 : \prod y_0 \geqslant A\right\}$$

and we shall choose A large enough that $\gamma C_0(A) \cap C_0(A) = \phi$ if $\gamma \in \Gamma'(\mathfrak{a},\alpha) - \Gamma'_\infty(\mathfrak{a},\alpha)$, where $\Gamma'_\infty(\mathfrak{a},\alpha) = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma'(\mathfrak{a},\alpha)\}$.

Now we shall consider the integral, whose convergence will soon be clear,

$$I_A(u) = \gamma \int_{\Gamma'_\infty(\mathfrak{a},\alpha)\backslash H_0} (\mathbf{e}',\phi_A^u(z) f_\alpha(z)) \quad d\sigma(z) \quad ,$$

where $d\sigma = \otimes\, d\sigma_\nu$, $d\sigma_\nu = |dx_v \wedge dy_\nu|/u_v^2$. Then we can treat this in two different ways. First of all let

$$E^A(z,u) = \sum_{\gamma\in\Gamma'_\infty(\mathfrak{a},\alpha)\Gamma'(\mathfrak{a},\alpha)} \overline{\chi\chi_\alpha(\gamma)\phi_A^{\bar{u}}(\gamma z)}\ j_0(\gamma,z)^{-1}\ \mathbf{e}' \quad ,$$

so that, by the usual Rankin transformation,

$$I_A(u) = \gamma \int_{H_0} (E^A(z,\bar{u}),\ f_\alpha(z))\ d\sigma(z) \quad .$$

In view of the usual estimates one can easily verify that this converges of $\operatorname{Re}(u) > 1$; this will become clear from our subsequent discussion. On the other hand, on making use of Proposition 2.3, we see that $I_n(u)$ is the sum of

$$\gamma c(\mathfrak{a}^{-1}\nu^{-1})^{-1}(-1)^L \int_{\Gamma_\infty(\mathfrak{a},\alpha)\backslash(H_0-C_0(A))} \prod_{v\in S_\infty} \Big\{(|i_v(\alpha)|y_v)^{2-s-i\tau(\nu)}$$

$$y_v^{s+i\tau(\nu)+u-1}\Big\} \otimes y_v^{-2}|dx_v \cap dy_v \cdot \prod_{v\in S_\infty} \{2\pi(s+i\tau(v)+\tfrac{1}{2}\ell(v)-1)^{-1}\}$$

$$\left(e,\psi\left(s,\tau,\chi,0,\mathfrak{a},\rho,e,\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right)\right)$$

and

$$\sum_{\substack{\mu\in\mathfrak{a}^{-1}\nu^{-1}\\ \mu\neq 0}} c(\mathfrak{a}^{-1}\nu^{-1})^{-1}(-1)^L \int_{\Gamma_\infty(\mathfrak{a},\alpha)\backslash(H_0-C_0(A))}$$

$$\prod_{v\in S_\infty} \Big\{(|i_v(\alpha)|y_v)^{2-s-i\tau v} y_v^{s+i\tau(\nu)+u-1} \cdot 4\pi$$

$$\cdot(2\pi|i_v(\alpha\mu)|y_v)^{s+i\tau(\nu)+\ell(\nu)/2-1} K_{s+i\tau(\nu)-\ell(\nu)/2-1}(4\pi|i_v(\alpha\mu)y_v)\Big\}$$

$$e(\alpha\mu x)\,d\sigma\left(e,\psi\left(s,\tau,\chi,\mu,\ ,e,\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right)\right) \quad .$$

The point of our method is now that $\int e(\mu\alpha x)\otimes dx_v = 0$ unless $T_{vk/k_0}(\alpha\mu) = 0$, in which case it is $m = m(i(k_0 \cap \alpha^{-1}\mathcal{O}_1)\backslash \mathbb{R})$, where m is the usual Lebesgue measure. If now $k = k_0(\sqrt{\delta})$ then $T_r(\alpha\mu) = 0$ means that $\mu \in \alpha^{-1}\sqrt{\delta}\, k_0$. Hence if we take $\alpha = \sqrt{\delta}\,\beta$ this means that $\beta\mu \in k_0 \cap \mathfrak{a}^{-1}\nu^{-1}$. The first integral can easily be evaluated and it is

$$m_0\gamma c(\mathfrak{a}^{-1}\nu^{-1})^{-1}(-1)^L \prod_{v\in S_\infty} \Big\{2\pi(s+i\tau(v) + \tfrac{1}{2}\ell(v)-1)^{-1}|i_v(\alpha)|^{2-s-i\tau(v)}\Big\}$$

$$R_0 A^u u^{-1} \cdot \chi(-1) L(N^2\omega_s^2,\mathfrak{a}) L(N\omega_s^2,\mathfrak{a})^{-1} ,$$

where R_0 is the regulator of k_0.

To deal with the second term we write

$$J_A(u) = \sum_{\mu} c(\mathfrak{a}^{-1}\nu^{-1})^{-1}(-1)^L \int_{\Gamma_\infty(\mathfrak{a},a)\backslash C_0(A)}$$

$$\prod_{v\in S_\infty} \Big\{|i_v(\alpha)|y_v)^{2-s-i\tau(v)} y_v^{s+i\tau(v)+u-1}\ 4\pi$$

$$\cdot(2\pi|i_v(\alpha_\mu)|y_v^{s+i\tau(v)+\ell(v)/2-1}$$

$$\cdot K_{s+i\tau(v)-\ell(v)/2-1}(4\pi|i_v(\alpha_\mu)|y_v)y_v^{-2}\Big\} \otimes dy_v\ \ (e,\psi) \quad .$$

From the exponential decay of $K_\alpha(x)$ it follows easily that this series is absolutely convergent and represents an entire function. Then, if

$$G_0(u) = m_0 c(\mathfrak{a}^{-1}\nu^{-1})^{-1}(-1)^L \chi(-1) \prod_{v\in S_\infty} \left\{ |i_v(\alpha)|^{2-u-s-i\tau(v)} \cdot \pi \cdot (2\pi)^{-u} \Gamma\left(\frac{u}{2} + \frac{\ell(v)}{2}\right) \Gamma\left(\frac{u}{2} + s + i\tau(v) - 1\right)\right\} L(N\omega_s^2, \mathfrak{a})^{-1} ,$$

one sees that the second term is $C_0(u)F_\beta(u,\omega_s) - J_A(u)$. Thus, we have

$$I_A(u) = \gamma \int_{\Gamma(\mathfrak{a},\alpha)\backslash H_0} (E^A(z,\bar{u}), f_\alpha(z))\, d\sigma(z)$$

$$= G_1 A^u u^{-1} + G_0(u) F_\beta(u,\omega_s) - J_A(u) \qquad ,$$

where

$$G_1 = m_0 \gamma R_0 c(\mathfrak{a}^{-1},\nu^{-1})^{-1}(-1)^L \chi(-1) L(N^2\omega_s^2,\mathfrak{a}) L(N\omega_s^2,\mathfrak{a})^{-1} \prod_{v\in S_\infty} \left\{ 2\pi |i_v(\alpha)|^{2-s-i\tau(v)} (s+i\tau(v) + \tfrac{1}{2}\ell(v) - 1)^{-1} \right\} .$$

The series defining $E^A(z,u)$ converges in $\mathrm{Re}(u) > 2 - \mathrm{Re}(s)$. To analyze the integral further we have to examine the behavior of $E^A(z,u)$ at all the cusps. Thus let P be the set of cusps of $\Gamma'(\mathfrak{a},\alpha)$ and for each $p \in P$ we choose $\sigma_p \in SL_2(k_0)$ so that $\sigma_p(\infty) = p$. Each of these is also a cusp of $\begin{pmatrix}0 & P_1\\ 1 & 0\end{pmatrix}^{-1} D(\alpha)^{-1}\Gamma_0(\mathfrak{a}) D(\alpha) \begin{pmatrix}0 & P^{-1}\\ 1 & 0\end{pmatrix}$. Let $P_0 \subset P$ be the set of cusps equivalent under $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}^{-1} D(\alpha)^{-1}\ \Gamma_0(\mathfrak{a})\ D(\alpha)\ \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ to 0. Both $\Gamma'(\mathfrak{a},\alpha)\backslash P$ and $\Gamma'(\mathfrak{a},\alpha)\backslash P_0$ are finite. Also assume that if $p = \gamma p'\ (\gamma \in \Gamma'(\mathfrak{a},\alpha))$ then $\sigma_p = \gamma\sigma_{p'}$. As in Proposition 2.3 one has that

$$j_0(\sigma_p, z)^{-1}\ E^A(z,u)$$

has a Fourier expansion of the same type (apart from the difference incurred from the "truncation"). We shall only need the "constant term" which is, if $z \in \sigma_p^{-1}\ (C_0(A))$, $z = (x_v + iy_v)$,

$$\prod_{v\in S_\infty} \left\{ y_v^{2-u-(\overline{s+i\tau(v)})} \right\} \cdot \phi(u,p) ,$$

where the $\phi(u,p)$ can be computed as before. The only facts which we need are that if $\Lambda(t,\omega^{o}_{s-1}\chi_\alpha)$ is the Hecke L-series with Grössencharakter, with the corresponding gamma-functions and ω^o_{s-1} is the restriction of ω_{s-1} to k_0 then $\Lambda(2u,\overline{\omega}^o_{s-1}\chi_\alpha)\ \phi(u,p)$ has no poles unless $\overline{\omega}^o_{s-1}\chi_\alpha = N_0^{-t}$ (where N_0 is the norm in k_0) and then the only poles in $\mathrm{Re}(u) > 1/2\ \mathrm{Re}(s)-1)$ are those at $u = 1 - 1/2\ t$ and $1/2(1-t)$. Moreover, $\Lambda(2u,\overline{\omega}^o_{s-1}\chi_\alpha)E(z,u)$ is regular at $u = 1/2(1-t)$. These remarks, as the conscientious reader will verify, follow from the observation concerning j_0 made above.

Now we define $\tilde{E}^A(z,u)$ (cf. [Se]).

$$\tilde{E}^A(z,u) = E^A(z,u) - j_0(\sigma_p^{-1},z) \prod_{v\in S_\infty} \left\{ y_v(\sigma_p^{-1}z)^{2-u-(\overline{s+i\tau(v)})} \right\} \phi(u,p)$$

$$(\text{if } z\in\sigma_p((C_0(A))) \quad .$$

Then $\tilde{E}^A(\cdot,u)$ is square-integrable and meromorphic in $\mathcal{C}$ (in the L^2 sense). Moreover, it decays exponentially at the cusps. We shall now examine

$$\tilde{I}_A(u) = \gamma\int(\tilde{E}^A(z,\bar{u}),f_\alpha(z))\ d\sigma(z)\ ,$$

which exists wherever $\tilde{E}(z,\bar{u})$ is anti-analytic. Let $\Gamma'_p(\mathfrak{a},\alpha)$ be the stabilizer of ∞ in $\sigma_p^{-1}\Gamma'(\mathfrak{a},\alpha)\sigma_p$. Then

$$I_A(u) - \tilde{I}_A(u) = \gamma \sum_{p\in\Gamma'(\mathfrak{a},\alpha)\backslash P} \int_{\Gamma'_p(\mathfrak{a},\alpha)\backslash C_0(A)}$$

$$\prod_{v\in S_\infty} \left\{ y_v^{2-u-(s+i\tau(v))} \right\} (\phi(\bar{u},p),j_0(\sigma_p,z)^{-1}f_\alpha(\sigma_p(z))\ d\sigma\ .$$

In the region $C_0(A)$ we apply Proposition 2.3 to $j_0(\sigma_p,z)^{-1}f_\alpha(\sigma_p(z))$ which can be written as a sum of

$$\prod_{v\in S_\infty} \left\{ y_v^{2-s-i\tau(v)} \right\} c_p + \prod_{v\in S_\infty} \left\{ y_v^{2-s-i\tau(v)} \right\} c_p^* \qquad (\text{if } p\in P_0)$$

or

$$\prod_{v\in S_\infty} \left\{ y_0^{2-s-i\tau(v)} \right\} c_p^* \qquad (\text{if } p\in P-P_0)$$

and a second term which we denote by $f_p^{(1)}(z)$, and which decays exponentially at ∞. Write now

$$J_A^0(u,p) = \gamma \int_{\Gamma_p'(\mathfrak{a},a)\backslash C_0(A)} \prod_{v \in S_\infty} \left\{ y_v^{2-u-s-i\tau(v)} \right\} f_p^{(1)}(z)\ d\sigma(2)$$

and one sees that this is an entire function. Thus

$$I_A(u) - \tilde{I}_A(u) = \gamma \sum_{p \in \Gamma'(\mathfrak{a},\alpha)\backslash P} (\phi(\bar{u},p), J_A^0(u,p))$$

$$+ \gamma \sum_{p \in \Gamma'(\mathfrak{a},\alpha)\backslash P_0} (\phi(\bar{u},p), c_p) m_p' A^{1-u}(u-1)^{-1}$$

$$+ \gamma \sum_{p \in \Gamma'(\mathfrak{a},\alpha)\backslash P} (\phi(\bar{u},p), c_p^*) m_p' A^{3-u-2(s+i\tau)} / (u-3+2(s+i\tau)) ,$$

where the convention of the enumeration of the theorem applies to the last term. This we rewrite as

$$G_0(u) F_p(u,\omega_s) = \gamma \int_{\Gamma'(\mathfrak{a},\alpha)\backslash H_0} (\tilde{E}^A(z,\bar{u}), f_\alpha(z))\ d\sigma(z) - G_1 A^u u^{-1}$$

$$- \gamma \sum_{p \in \Gamma'(\mathfrak{a},\alpha)\backslash P_0} (\phi(\bar{u},p), c_p) m_p' A^{1-u}/(u-1)$$

$$- \gamma \sum_{p \in \Gamma'(\mathfrak{a},\alpha)\backslash P} (\phi(\bar{u},p), c_p^*) m_p' A^{30u-2(s+i\tau)}/(u+3+3(s+i\tau))$$

$$- \gamma \sum_{p \in \Gamma'(\mathfrak{a},\alpha)\backslash P} (\phi(\bar{u},p), J_A^0(u,p)) + J_A(u) .$$

Here the m_p' are certain measures, analogous to $R_0 m_0$ at the cusp ∞, which are complicated to define and which will not be further used. They can be computed by any reader desirous to know them. This technique, a generalization of the well-known Rankin-Selberg method, was introduced in [P].

With this formula we have essentially reached our goal. The assertions about the poles can now be read off directly. This leaves only the assertion about the growth in the vertical strips. One observes that the functional equation for the Eisenstein series gives a relation between $\tilde{E}^A(z,u)$ and similar ones evaluated at $3-2s-u$, in which $(\tau(\nu))$ has been replaced by $(-\tau(\nu))$. This induces a functional equation for

$$\Lambda(2u,\bar{\omega}_{s-1},\chi_\alpha)G_0(u)F_\beta(u,\omega_s)$$

relating it to a similar series evaluated at $3-2s-u$. Then as our integral expression (modulo some "trivial" poles) shows that this of finite order, the assertion concerning the rate of growth folllows as usual from the Phragmén-Lindelöf principle.

Remark. It would be interesting to develop these methods to look at

$$F_\beta(\chi,\omega) = \sum_{\substack{\mu\in\mathfrak{a}^{-1}\nu^{-1} \\ \mu \bmod U_{k_0}^2 \\ \mu\in\beta k_0}} P(\omega,\mu)L(\omega\eta_\mu,f_\mu\mathfrak{a})\chi(\mu) ,$$

where χ is also a Grössencharakter. One suspects that if "$\mathrm{Re}(\omega)\geqslant 1/2$" then this has a double pole at χ_1 if

$$\chi_1\circ N_{k/k_0} = \omega\chi_\beta$$

and a similar statement *vis-à-vis* Theorem 5.1. This would subsume our present picture in a satisfactory fashion.

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