FOURIER EXPANSIONS OF $GL(2)$ NEWFORMS AT VARIOUS CUSPS

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Abstract. This paper studies the Fourier expansion of Hecke-Maass eigenforms for $GL(2,\mathbb{Q})$ of arbitrary weight, level, and character at various cusps. It is shown that the Fourier coefficients at a cusp satisfy certain very explicit multiplicativity relations. As an application, it is proved that a local representation of $GL(2,\mathbb{Q}_p)$ which is isomorphic to a local factor of a global cuspidal automorphic representation generated by the adelic lift of a newform of arbitrary weight, level N, and character $\chi \pmod N$ cannot be supercuspidal if χ is primitive. Furthermore, it is supercuspidal if and only if at every cusp (of width m and cusp parameter $= 0$) the mp^ℓ Fourier coefficient, at that cusp, vanishes for all sufficiently large positive integers ℓ. In the last part of this paper a three term identity involving the Fourier expansion at three different cusps is derived.

§1. Introduction

Consider the group

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{Z},\ ad-bc=1,\ c \equiv 0 \pmod N \right\}.$$

Then $\Gamma_0(N)$ acts on the upper half-plane $\mathfrak{h} := \{x+iy \mid x \in \mathbb{R},\ y>0\}$ by linear fractional transformations.

We fix an integer k (called the weight), an integer $N \geq 1$ (called the level), and a Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z}) \to \mathbb{C}^\times$. For any function $f : \mathfrak{h} \to \mathbb{C}$, and any matrix $\gamma \in GL(2,\mathbb{R})$ of positive determinant, define the slash operator

$$(1.1) \qquad \left(f\big|_k \gamma\right)(z) := \left(\frac{cz+d}{|cz+d|}\right)^{-k} f\left(\frac{az+b}{cz+d}\right),$$

and the character $\widetilde{\chi} : \Gamma_0(N) \to \mathbb{C}^\times$ defined by

$$(1.2) \qquad \widetilde{\chi}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \chi(d).$$

An automorphic function of weight k, and character χ for $\Gamma_0(N)$ is a smooth function $f : \mathfrak{h} \to \mathbb{C}$ which satisfies the automorphy relation

$$(1.3) \qquad \left(f\big|_k \gamma\right)(z) = \widetilde{\chi}(\gamma) f(z), \qquad (z \in \mathfrak{h})$$

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for all $\gamma \in \Gamma_0(N)$. Note that this forces f to be identically zero unless k and χ satisfy the compatibility condition $\chi(-1) = (-1)^k$. Clearly, an automorphic function of weight k and character χ for $\Gamma_0(N)$ is also an automorphic function of weight k and character χ' for $\Gamma_0(M)$ whenever N divides M and χ' is the Dirichlet character $\pmod M$ obtained by pulling χ back via the natural projection $(\mathbb{Z}/M\mathbb{Z})^\times \to (\mathbb{Z}/N\mathbb{Z})^\times$. Such automorphic functions are said to be "old". An automorphic function is said to be of level N if it is an automorphic function of weight k and level χ for $\Gamma_0(N)$, for some k, χ, and it is not an automorphic function of level M for any M dividing N.

Now fix $\nu \in \mathbb{C}$. A Maass form of weight k, type ν, level N and character $\chi \pmod N$ is an automorphic function of weight k, level N and character χ which has moderate growth and which is also an eigenfunction of the weight k Laplace operator (see §2) with eigenvalue $\nu(1-\nu)$. A Maass form is said to be a "new form" if it lies in the orthogonal complement (with respect to the Petersson inner product) of the space of old forms (see [2]). A Hecke newform is a newform which is an eigenfunction of all the Hecke operators. The main result of Atkin-Lehner theory [2] is that the space of newforms has a basis where each basis element is an eigenfunction of all the Hecke operators.

Cusps are defined to be elements of $\mathbb{Q} \cup \{\infty\}$. We define an action of $GL(2, \mathbb{Q})$ on cusps by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathfrak{a} = \begin{cases} \frac{a}{c}, & \mathfrak{a} = \infty, \ c \neq 0, \\ \infty, & \mathfrak{a} = \infty, \ c = 0, \\ \frac{a\cdot\mathfrak{a}+b}{c\cdot\mathfrak{a}+d}, & \mathfrak{a} \in \mathbb{Q}, \ c\cdot\mathfrak{a}+d \neq 0, \\ \infty, & \mathfrak{a} \in \mathbb{Q}, \ c\cdot\mathfrak{a}+d = 0. \end{cases}$$

In addition, an element of $SL(2, \mathbb{R})$ may be allowed to act on the set of cusps if it is of the form $d \cdot \gamma$ with $\gamma \in GL(2, \mathbb{Q})$ and d diagonal. Two cusps $\mathfrak{a}, \mathfrak{b} \in \mathbb{Q} \cup \{\infty\}$ are $\Gamma_0(N)$- equivalent if $\gamma\mathfrak{a} = \mathfrak{b}$ for some $\gamma \in \Gamma_0(N)$.

The group $SL(2, \mathbb{Z})$ permutes the cusps transitively. Thus, given a cusp $\mathfrak{a}$ it is possible to choose a matrix $\gamma_\mathfrak{a} \in SL(2, \mathbb{Z})$ such that $\gamma_\mathfrak{a}\infty = \mathfrak{a}$. The matrix $\gamma_\mathfrak{a}$ is unique up to an element of the stabilizer, Γ_∞, of ∞ on the right. This group is given explicitly by

$$\Gamma_\infty = \left\{ \begin{pmatrix} \epsilon & n \\ 0 & \epsilon \end{pmatrix} \middle| \, \epsilon \in \{\pm 1\}, \ n \in \mathbb{Z} \right\},$$

and is contained in the group $\Gamma_0(N)$ for every N. In particular, $\gamma_\mathfrak{a}\delta\gamma_\mathfrak{a}^{-1}$ is independent of the choice of $\gamma_\mathfrak{a}$ for each $\delta \in \Gamma_\infty$.

Let $\Gamma_\mathfrak{a} = \{\gamma \in \Gamma_0(N) \mid \gamma\mathfrak{a} = \mathfrak{a}\}$. Then $\gamma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\gamma_\mathfrak{a}$ is a subgroup of finite index in Γ_∞, and contains the scalar matrix -1. As such it is the product of the group of order 2 generated by -1 and an infinite cyclic group generated by $\left(\begin{smallmatrix} 1 & m_\mathfrak{a} \\ 0 & 1 \end{smallmatrix}\right)$ for some positive integer $m_\mathfrak{a}$. This integer $m_\mathfrak{a}$ may be characterized as the least positive power of $\gamma_\mathfrak{a}\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\gamma_\mathfrak{a}^{-1}$ which is in $\Gamma_0(N)$, and as such is independent of the choice of $\gamma_\mathfrak{a}$. Let

$$\sigma_\mathfrak{a} = \gamma_\mathfrak{a} \begin{pmatrix} \sqrt{m_\mathfrak{a}} & 0 \\ 0 & \sqrt{m_\mathfrak{a}}^{-1} \end{pmatrix}, \tag{1.4}$$

then $\sigma_\mathfrak{a}$ has the following key properties:

$$\sigma_\mathfrak{a}\infty = \mathfrak{a} \qquad \sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a} = \Gamma_\infty,$$

The matrix $\sigma_{\mathfrak{a}} \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \sigma_{\mathfrak{a}}^{-1} \in \Gamma_{\mathfrak{a}}$ is independent of the choice of $\sigma_{\mathfrak{a}}$. We denote this element by $g_{\mathfrak{a}}$. It generates $\Gamma_{\mathfrak{a}}$ together with the scalar matrix -1. We define the cusp parameter $0 \le \mu_{\mathfrak{a}} < 1$ determined by the condition

$$\widetilde{\chi}(g_{\mathfrak{a}}) \;=\; e^{2\pi i \mu_{\mathfrak{a}}}. \tag{1.5}$$

It was shown in [15], [16] that a newform of weight k, level N, and character $\chi \pmod N$ for $\Gamma_0(N)$ has a Fourier-Whittaker expansion at every cusp. The expansion takes the following explicit form.

Proposition 1.6 (Fourier-Whittaker expansion at a cusp) *Let f be a newform of weight k, type ν, level N, and character $\chi \pmod N$ for $\Gamma_0(N)$. Let $\mathfrak{a} \in \mathbb{Q} \cup \{\infty\}$ be a cusp and let $\sigma_{\mathfrak{a}}, \mu_{\mathfrak{a}}$ be as in (1.4), (1.5), respectively. Then*

$$\left(f\big|_k \sigma_{\mathfrak{a}}\right)(z) = \sum_{n+\mu_{\mathfrak{a}} \neq 0} a(\mathfrak{a}, n)\, W_{\frac{\mathrm{sgn}(n)k}{2},\, \nu-\frac{1}{2}}\Big(4\pi |n+\mu_{\mathfrak{a}}| \cdot y\Big) e^{2\pi i (n+\mu_{\mathfrak{a}})x}, \qquad (z = x+iy \in \mathfrak{h})$$

where $a(\mathfrak{a}, n) \in \mathbb{C}$ is called the n^{th} Fourier coefficient at the cusp $\mathfrak{a}$ and

$$W_{\alpha,\nu}(y) = \frac{y^{\nu+\frac{1}{2}} e^{-\frac{y}{2}}}{\Gamma\left(\nu - \alpha + \frac{1}{2}\right)} \int_0^\infty e^{-yt}\, t^{\nu-\alpha-\frac{1}{2}} \left(1+t\right)^{\nu+\alpha-\frac{1}{2}}\, dt, \qquad (\alpha \in \mathbb{R},\ \nu \in \mathbb{C})$$

is the Whittaker function.

Remark: In general, the coefficients $a(\mathfrak{a}, n)$ depend on the choice of $\gamma_{\mathfrak{a}}$ and not only on the choice of $\mathfrak{a}$.

It is known by the theory of Hecke operators that the Fourier coefficients (at the cusp infinity) of a Hecke newform of arbitrary weight, character and level satisfy multiplicativity relations. Similar results regarding the Fourier coefficients at an arbitrary cusp have been proven by Asai when the level is squarefree [1] and by Kojima when the level is 4q where q is a prime [14]. The essential reason for this is that if N is squarefree, then the cusps are represented by the quotients $1/t$ with t running over the positive divisors of N, and the corresponding Fricke involution W_t maps the cusp ∞ to $1/t$ and, at the same time, acts as an involution on the space of newforms commuting with all Hecke operators. When N is not squarefree, this reasoning breaks down and very little was known about the Fourier coefficients at cusps other than ∞. The difficulty in dealing with level N when N is not squarefree can also be seen from the adelic theory of automorphic representations, because in this case there may exist supercuspidal representations which appear to be extremely elusive objects.

An aim of this paper is to explicitly relate the Fourier coefficients of a Hecke newform (arbitrary weight, prime power level, and character) at an arbitrary cusp to the Fourier coefficients at the cusp ∞. We shall prove the main theorem below whose proof makes use of the modern theory of automorphic representations.

Theorem 1.7 (Main theorem) *Let q^e be a fixed prime power. Let f be a Hecke newform of level q^e, character $\chi \pmod{q^e}$, weight k, type ν for $\Gamma_0(q^e)$. Assume $\chi = \chi_0 \cdot \chi_{\text{trivial}}$ where χ_{trivial} is the trivial character modulo q and χ_0 is a primitive Dirichlet character of prime power level q^{e_0} (with $0 \le e_0 \le e$). Take S, a set of inequivalent cusps for $\Gamma_0(q^e)$ as*

$$S = \{0, \infty\} \cup \left\{ \frac{1}{cq^l} \;\middle|\; 1 \le l < e,\ \ (c,q) = 1,\ \ 1 \le c < \min\left(q^l, q^{e-l}\right) \right\}.$$

For $\mathfrak{a} \in S$ and $n \in \mathbb{Z}$, let $a(\mathfrak{a}, n)$ denote the n^{th} Fourier coefficient of f at the cusp $\mathfrak{a}$ as in Proposition 1.6. Assume that $a(\infty, 1) = 1$. For any $\mathfrak{a} \in S$ and an arbitrary non-negative integer M let

$$\boxed{\epsilon M + \mu_{\mathfrak{a}} \;=\; \epsilon m_{\mathfrak{a}} p_1^{m_1} \cdots p_n^{m_n} \cdot q^m}$$

where $\epsilon = \pm 1$, $m_1, \ldots, m_n$ are positive integers, $m \in \mathbb{Z}$, $p_1, \ldots, p_n$ are distinct primes different from q, and $\mu_{\mathfrak{a}}$ is the cusp parameter given in (1.5). Set $M_0 = p_1^{m_1} \cdots p_n^{m_n}$. For each cusp $\mathfrak{a} = \frac{1}{cq^l} \in S$, there exists a unique cusp $\frac{1}{c'q^l} \in S$, determined by the conditions

$$1 \le c' \le \min\left(q^{e-l}, q^l\right), \qquad c'\epsilon M_0 \equiv c \left(\bmod \min\left(q^{e-l}, q^l\right)\right).$$

If $l \le e/2$ then there is, in addition, a unique integer j determined by the conditions

$$0 \le j < q^{e-2l}, \qquad cc'j \;\equiv\; (c'\epsilon M_0 - c) \cdot q^{-l} \pmod{q^{e-2l}}.$$

Let $\alpha \ge 0$ denote the greatest integer such that $q^\alpha \mid M$. Then $m, \mu_{\mathfrak{a}}$ and $a(\mathfrak{a}, \epsilon M)$ are given as follows.

• $\underline{\mathfrak{a} = \infty :}$ *In this case $\mu_\infty = 0$, $m = \alpha$, and*

$$a(\infty, \epsilon M) = a(\infty, \epsilon) a(\infty, p_1^{m_1}) \cdots a(\infty, p_n^{m_n}) a(\infty, q^m).$$

• $\underline{\mathfrak{a} = 0 :}$ *In this case $\mu_0 = 0$, $m = \alpha - e$, and*

$$a(0, \epsilon M) = a(\infty, \epsilon M_0) a(0, q^{e+m}) \chi(\epsilon M_0)^{-1}.$$

• $\underline{\mathfrak{a} = \frac{1}{cq^l},\ \text{and } \mu_{\mathfrak{a}} \neq 0 :}$ *In this case $e_0 > \max(l, e-l)$, $m = -e_0 + l$, and*

$$a\left(\frac{1}{cq^l},\, \epsilon M\right) = \begin{cases} a(\infty, \epsilon M_0) a\left(\frac{1}{c'q^l}, 0\right) e_\infty\left(q^m \cdot j\right) \chi\left(jc'q^l + 1\right)^{-1}, \\ \qquad \text{if } l \le \frac{e}{2}, \\ a(\infty, \epsilon M_0) a\left(\frac{1}{c'q^l},\, 0\right), \\ \qquad \text{if } l > \frac{e}{2}, \end{cases}$$

Furthermore, the cusp parameter of $\frac{1}{c'q^l} \in S$ is $\min(q^{e_0 - l}, q^{e_0 - e + l})^{-1}$. (If $e_0 = e$, i.e., if χ is primitive, then there is a unique cusp $\mathfrak{a}_0 = \frac{1}{c_0 q^l} \in S$ having this property, so that $c' = c_0$, independently of c, ϵ, and M!)

• $\underline{\mathfrak{a} = \frac{1}{cq^l},\ \text{and } \mu_{\mathfrak{a}} = 0 :}$ *In this case $m = \alpha - \max(e - 2l, 0)$ and*

$$a\left(\frac{1}{cq^l},\, \epsilon M\right) = \begin{cases} a(\infty, \epsilon M_0) a\left(\frac{1}{c'q^l},\, q^{e-2l+m}\right) e_\infty\left(q^m \cdot j\right) \chi\left(jc'q^l + 1\right)^{-1}, \\ \qquad \text{if } l \le \frac{e}{2}, \\ a(\infty, \epsilon M_0) a\left(\frac{1}{c'q^l},\, q^m\right), \\ \qquad \text{if } l > \frac{e}{2}. \end{cases}$$

In theorem 3.8, we obtain an extension of theorem 1.7 to the case of $\Gamma_0(N)$ where N is a product of prime powers. We then apply our methods further to obtain an interesting application. In theorem 3.9 it is shown that a representation of $GL(2,\mathbb{Q}_p)$ which is isomorphic to the local representation factor of an irreducible automorphic representation of $GL(2,\mathbb{A})$ (generated by the adelic lift of a classical newform) is supercuspidal if and only if the $m_{\mathfrak{a}}p^{\ell}$ Fourier coefficient vanishes for all sufficiently large integers ℓ for each cusp $\mathfrak{a} \in \mathbb{Q} \cup \{\infty\}$. Here $m_{\mathfrak{a}}$, also called the width of the cusp, is an integer given by (1.4). It is well-known (see [3, 7] and also proposition 5.1 of [17]) that every supercuspidal representation of $GL(2,\mathbb{Q}_p)$ can be realized as a component of a global irreducible cuspidal automorphic representation of $GL(2,\mathbb{A})$, and that every such representation may be associated to a unique normalized Maass-Hecke newform (for a detailed proof, see [9]). Theorem 3.9 gives a criterion for recognizing those Maass-Hecke newforms which are connected with supercuspidals in this way. In corollary 3.10 it is further shown that a local representation of $GL(2,\mathbb{Q}_p)$ cannot be supercuspidal if it isomorphic to a local factor of a global cuspidal automorphic representation generated by the adelic lift of a newform of weight k, level N and character $\chi \pmod N$ if χ is a primitive character. This shows that supercuspidal representations can only arise from newforms of level N with imprimitive characters (mod N). In this connection, we would also like to mention a result of Casselman (see [5]), which implies that supercuspidal representations of $GL(2,\mathbb{Q}_p)$ can only arise from newforms of level N such that $p^2 \mid N$. For more detail see the remark which follows the proof of corollary 3.10.

Finally, in §7, we attempt to derive relations between the Fourier coefficients at different cusps for Maass forms of arbitrary weight level and character by using a purely classical approach. The main result is given in Theorem 7.4 where a three term relation (involving three different cusps) is obtained.

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§2. Newforms and Hecke operators

Let f be an automorphic function of weight k, level N and character χ. An automorphic function f is a Maass form of type $\nu \in \mathbb{C}$, if

$$(\Delta_k f)(z) = \nu(1-\nu) f(z)$$

where

$$\Delta_k = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \frac{\partial}{\partial x}$$

is the weight k Laplace operator.

Let f be an automorphic function of weight k, level N and character χ. For any positive integer n, the Hecke operator (twisted by a character χ) is denoted T_n^{χ}, and is defined by

$$(T_n^{\chi} f)(z) := \frac{1}{\sqrt{n}} \sum_{ad=n, a,d>0} \chi(a) \sum_{b=0}^{d-1} \left(f \Big|_k \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right)(z).$$

Let f be an automorphic function of weight k, level N and character χ. Then f is cuspidal if

$$\int_0^1 (f|_k \sigma_{\mathfrak{a}})(x+iy)\,dx = 0$$

for any cusp $\mathfrak{a}$. A Maass cusp form f of weight k, type ν, level N, and character χ is a newform if f is an eigenfunction of the Hecke operator T_n^χ for every positive integer n.

Let f be an automorphic function of weight k, level N and character χ. Define the T_{-1} operator by

$$(T_{-1}f)(z) := f(-\overline{z}), \qquad (\forall z \in \mathfrak{h}).$$

Then $T_{-1}f$ is an automorphic function of weight $-k$.

Proposition 2.1 *Let f be a newform of weight k, type ν, level N and character χ. Then f has a Fourier-Whittaker expansion at infinity of the type*

$$f(z) = \sum_{n\neq 0} a(\infty, n) W_{\frac{sgn(n)k}{2}, \nu-\frac{1}{2}}(4\pi|n|y)e^{2\pi i n x}.$$

Assume f is normalized so that $a(\infty,1) = 1$. If $n = \mathrm{sgn}(n)p_1^{m_1}p_2^{m_2}\cdots p_r^{m_r}$ for distinct primes $p_1, \cdots, p_r$ and positive integers $m_1, \cdots, m_r$, then we have

$$a(\infty, n) = a(\infty, sgn(n)) \prod_{i=1}^{r} a(\infty, p_i^{m_i}).$$

Proof: The Fourier-Whittaker expansion is given in proposition 1.6. Since f is a newform, it is an eigenfunction of all the Hecke operators T_n^χ for every positive integer n. It follows from [16] (see also [8]) that for $n = p_1^{m_1}\cdots p_n^{m_n}$,

$$a(\infty, n) = \prod_{i=1}^{r} a(\infty, p_i^{m_i}).$$

Since $(T_n^\chi T_{-1} f)(z) = (T_{-1}T_n^\chi f)(z)$ for any positive integer n, we have

$$a(\infty, -n) = a(\infty, -1)a(\infty, n). \quad \square$$

§3. Whittaker functions for the adelic lift of a newform

Let $\mathbb{A}$ be the ring of adeles over $\mathbb{Q}$. A place of $\mathbb{A}$ is defined to be either a rational prime or ∞. For a finite prime p and for $x \in \mathbb{Q}_p$ let

$$\{x\} := \begin{cases} \sum\limits_{i=-k}^{-1} a_i p^i, & \text{if } x = \sum\limits_{i=-k}^{\infty} a_i p^i \in \mathbb{Q}_p, \text{ with } k > 0,\ 0 \le a_i \le p-1; \\ 0, & \text{otherwise.} \end{cases}$$

We now define an additive character e_v at each place v. If $x \in \mathbb{Q}_v$, we let

$$e_v(x) := \begin{cases} e^{2\pi i x}, & \text{if } v = \infty; \\ e^{-2\pi i \{x\}}, & \text{if } v < \infty. \end{cases}$$

Furthermore, for $x = \{x_v\}_v \in \mathbb{A}$, define a global additive character for $\mathbb{A}$ as

$$e(x) := \prod_{v \le \infty} e_v(x_v).$$

Let χ be a Dirichlet character modulo N and let $\chi_{\text{idelic}} : \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ be the idelic lift of χ as in [9]. For the convenience of the reader we repeat the definition

Definition (Idelic lift of a Dirichlet character) *Let χ be a Dirichlet character of conductor p^f where p^f is a fixed prime power. We define the idelic lift of χ to be the unitary Hecke character $\chi_{\text{idelic}} : \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times \to \mathbb{C}^\times$ defined as*

$$\chi_{\text{idelic}}(g) := \chi_\infty(g_\infty) \cdot \chi_2(g_2) \cdot \chi_3(g_3) \cdots , \qquad \left(g = \{g_\infty, g_2, g_3, \dots\} \in \mathbb{A}_{\mathbb{Q}}^\times\right),$$

where

$$\chi_\infty(g_\infty) = \begin{cases} 1, & \chi(-1) = 1, \\ 1, & \chi(-1) = -1, \ g_\infty > 0, \\ -1, & \chi(-1) = -1, \ g_\infty < 0, \end{cases}$$

and where

$$\chi_v(g_v) = \begin{cases} \chi(v)^m, & \textit{if } g_v \in v^m \mathbb{Z}_v^\times \textit{ and } v \neq p, \\ \chi(j)^{-1}, & \textit{if } g_v \in p^k \left(j + p^f \mathbb{Z}_p\right) \textit{ with } j, k \in \mathbb{Z}, \ (j,p) = 1 \textit{ and } v = p. \end{cases}$$

More generally, every Dirichlet character χ of conductor $q = \prod_{i=1}^{r} p_i^{f_i}$, where $p_1, p_2, \dots p_r$ are distinct primes and $f_1, f_2, \dots f_r \geq 1$, can be factored as $\chi = \prod_{i=1}^{r} \chi^{(i)}$, where $\chi^{(i)}$ is a Dirichlet character of conductor $p_i^{f_i}$. It follows that χ may be lifted to a Hecke character χ_{idelic} on $\mathbb{A}_{\mathbb{Q}}^\times$ where $\chi_{\text{idelic}} = \prod_{i=1}^{r} \chi_{\text{idelic}}^{(i)}$.

For each $p|N$, let

$$I_{p,N} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z}_p) \;\middle|\; c \equiv 0 \pmod{N} \right\}.$$

Here $c \equiv 0 \pmod{N}$ means $c \in N \cdot \mathbb{Z}_p$. Then we can define

$$K_0(N) := \left(\prod_{p|N} I_{p,N} \right) \left(\prod_{p \nmid N} GL(2, \mathbb{Z}_p) \right).$$

For each $p|N$, define a character $\widetilde{\chi_p} : I_{p,N} \to \mathbb{C}^\times$, where

$$\widetilde{\chi_p}\left(\begin{pmatrix} a_p & b_p \\ c_p & d_p \end{pmatrix} \right) := \chi_p(d_p).$$

Define a character $\widetilde{\chi}_{\text{idelic}} : K_0(N) \to \mathbb{C}^\times$ such that

$$\widetilde{\chi}_{idelic}(k) := \prod_{p|N} \widetilde{\chi_p}(k_p)$$

for all $k = \{k_p\}_p \in K_0(N)$. If χ is primitive, then the kernel of $\widetilde{\chi}_{idelic}$ is given by

$$\mathrm{Ker}\,(\widetilde{\chi}_{\text{idelic}}) = \prod_{p\nmid N} GL(2,\mathbb{Z}_p)\cdot \prod_{p|N}\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in I_{p,N}\;\middle|\; d\equiv 1 \left(\mathrm{mod}\ |N|_p^{-1}\right)\right\}.$$

Let $\mathbb{A}_f$ denote the finite adeles. For $g = \{g_v\}_v \in GL(2,\mathbb{A})$ or $GL(2,\mathbb{A}_f)$, define $(g)_v := g_v$ where v is a place of $\mathbb{Q}$. Then for $v \le \infty$, we have the inclusion map $i_v : GL(2,\mathbb{Q}_v) \to GL(2,\mathbb{A})$ defined by

$$(i_v(g_v))_w := \begin{cases} \left(\begin{smallmatrix} 1 & 0\\ 0 & 1\end{smallmatrix}\right), & \text{if } w \neq v;\\ g_v, & \text{if } w = v.\end{cases}$$

We shall define the diagonal embedding map $i_{diag} : GL(2,\mathbb{Q}) \to GL(2,\mathbb{A}_{\mathbb{Q}})$ by

$$i_{diag}(\gamma) \;:=\; \{\gamma,\gamma,\gamma,\;\ldots\}, \qquad (\forall\gamma\in GL(2,\mathbb{Q})).$$

By strong approximation, it follows that for any $g \in GL(2,\mathbb{A})$, there exist $\gamma \in GL(2,\mathbb{Q})$, $g_\infty \in GL(2,\mathbb{R})^+$, $k \in K_0(N)$ such that

$$g = i_{diag}(\gamma)i_\infty(g_\infty)i_{\text{finite}}(k),$$

where i_{finite} denotes the diagonal embedding of $GL(2,\mathbb{Q})$ into $GL(2,\mathbb{A}_{\text{finite}})$, the group of finite adeles.

Let f be a cusp form of weight k, type ν, level N and character χ modulo N. By the above decomposition, we may define a function $f_{\text{adelic}} : GL(2,\mathbb{A}) \to \mathbb{C}$ as

$$f_{\text{adelic}}(g) := f_{\text{adelic}}(i_{diag}(\gamma)i_\infty(g_\infty)i_{\text{finite}}(k)) := \left(f\big|_k g_\infty\right)(i)\cdot \widetilde{\chi}_{idelic}(k). \tag{3.1}$$

It follows from (1.3) that f_{adelic} is well-defined. Further, f_{adelic} is an adelic automorphic form with a central character χ_{idelic}.

The function f_{adelic} has a Fourier expansion,

$$f_{\text{adelic}}(g) = \sum_{\alpha\in\mathbb{Q}^\times} W_f\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} g\right)$$

where

$$W_f(g) := \int_{\mathbb{Q}\backslash\mathbb{A}} f_{\text{adelic}}\left(\begin{pmatrix} 1 & u\\ 0 & 1\end{pmatrix} g\right) e(-u)du$$

for all $g \in GL(2,\mathbb{A})$. The function $W_f(g)$ is called a global Whittaker function for f_{adelic}.

Theorem 3.2 *Let $f : \Gamma_0(N)\backslash\mathfrak{h} \to \mathbb{C}$ be a cusp form of weight k, type ν, level N, and character χ (mod N). Let S be a set of representatives for the $\Gamma_0(N)$-equivalence classes of cusps. Then f has the Fourier-Whittaker expansion at every cusp as in Proposition 1.6. By the Iwasawa decomposition, every $g \in GL(2,\mathbb{A})$ has a decomposition*

$$g = \begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} r & 0\\ 0 & r\end{pmatrix} i_{diag}\left(\begin{pmatrix}\epsilon & 0\\ 0 & 1\end{pmatrix}\right) k,$$

where $x \in \mathbb{A}$, r, $y = \{y_v\}_v \in \mathbb{A}^\times$, *with* $y_\infty > 0$, $k = \{k_v\}_v \in K = SO(2,\mathbb{R}) \prod_{p<\infty} GL(2,\mathbb{Z}_p)$, $k_\infty = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$, $(0 \le \theta < 2\pi)$, *and* $\epsilon = \pm 1$. *Let* $t := \prod_p |y_p|_p^{-1}$. *Then*

$$W_f(g) = \begin{cases} a\big(\mathfrak{a}, \epsilon m_{\mathfrak{a}} t - \mu_{\mathfrak{a}}\big)\, W_{\frac{\epsilon k}{2}, \nu - \frac{1}{2}}(4\pi y_\infty)\, \chi_{\text{idelic}}(r)\, e(x)\, e_\infty\left(\frac{k\theta}{2\pi} + tj\epsilon\right) \widetilde{\chi}_{\text{idelic}}(k_0), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \epsilon m_{\mathfrak{a}} t - \mu_{\mathfrak{a}} \in \mathbb{Z}, \\ 0, \quad \text{otherwise.} \end{cases}$$

Here $k_0 \in K_0(N)$, *the cusp* $\mathfrak{a} \in S$, *and an integer* $0 \le j < m_{\mathfrak{a}}$ *are uniquely determined by*

$$i_{\text{finite}}\left(\gamma_{\mathfrak{a}} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}\right) \prod_{p|N} i_p\left(\begin{pmatrix} t^{-1} y_p & 0 \\ 0 & 1 \end{pmatrix} k_p\right) = k_0 \ \in \ K_0(N).$$

Proof: See [9]. $\square$

Definition 3.3 *Let* $f : \Gamma_0(N)\backslash\mathfrak{h} \to \mathbb{C}$ *be a cusp form of weight* k, *type* ν, *level* N *and character* $\chi \pmod N$. *For each place* v, *we define a function* $W_{f,v}(g_v) : GL(2,\mathbb{Q}_v) \to \mathbb{C}$ *as follows.*

- <u>$v = \infty$</u>:
$$W_{f,\infty}(g_\infty) := W_f\big(i_\infty(g_\infty)\big)$$
- <u>$v = p < \infty$</u>:
$$W_{f,p}(g_p) := \frac{W_f\big(i_p(g_p)\big)}{W_{\frac{k}{2}, \nu - \frac{1}{2}}(4\pi)}.$$

Corollary 3.4 *Let* f *be a Maass form of weight* k, *type* ν, *level* N *and character* $\chi \pmod N$. *Let* S *be a set of representatives for the* $\Gamma_0(N)$*-equivalence classes of cusps. Let* $\mathfrak{a} \in \mathbb{Q}$ *be a cusp for* $\Gamma_0(N)$. *For each integer* n, *let* $a(\mathfrak{a}, n)$ *be the* n*-th Fourier coefficient of* f *at the cusp* $\mathfrak{a}$ *as in Proposition 1.6. For every place* v *of* $\mathbb{Q}$ *and any* $g_v \in GL(2,\mathbb{Q}_v)$, *we have a decomposition*

$$g_v = \begin{pmatrix} 1 & x_v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_v & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_v & 0 \\ 0 & r_v \end{pmatrix} k_v,$$

where $x_v \in \mathbb{Q}_v$ *and* y_v, $r_v \in \mathbb{Q}_v^\times$. *If* $v = \infty$, *then* $y_\infty > 0$, $\epsilon = \pm 1$ *and* $k_\infty \in \left(\begin{smallmatrix} \epsilon & 0 \\ 0 & 1 \end{smallmatrix}\right) SO(2,\mathbb{R})$. *If* v *is finite, then* $k_v \in GL(2,\mathbb{Z}_v)$.

For each finite prime p, *fix* $y_p \in \mathbb{Q}_p^\times$ *and* $k_p \in GL(2,\mathbb{Z}_p)$. *Then there exists a cusp* $\mathfrak{a}_p \in S$, *an integer* $0 \le j_p < m_{\mathfrak{a}_p}$, *and* $k_{p,0} \in K_0(N)$, *which are uniquely determined by* y_p *and* k_p *such that*

$$i_{\text{finite}}\left(\gamma_{\mathfrak{a}_p} \begin{pmatrix} 1 & j_p \\ 0 & 1 \end{pmatrix}\right) i_p\left(\begin{pmatrix} |y_p|_p y_p & 0 \\ 0 & 1 \end{pmatrix} k_p\right) = k_{p,0} \in K_0(N).$$

Then by Definition 3.3, for each place v *for* $\mathbb{Q}$, *we have the following:*

- <u>$v = \infty$</u>:

$$W_{f,\infty}(g_\infty) = a(\infty, \epsilon)\, W_{\frac{\epsilon k}{2},\, \nu - \frac{1}{2}}(4\pi y_\infty)\, \chi_\infty(r_\infty)\, e_\infty\left(x_\infty + \frac{k\theta}{2\pi}\right).$$

- $\underline{v = p \nmid N}$:

$$W_{f,p}(g_p) = \begin{cases} a\big(\infty, |y_p|_p^{-1}\big)\, \chi_p(r_p)\, e_p(x_p), & \textit{if } y_p \in \mathbb{Z}_p, \\ 0, & \textit{otherwise.} \end{cases}$$

- $\underline{v = p \mid N}$:

$$W_{f,p}(g_p) = \begin{cases} a\Big(\mathfrak{a}_p, m_{\mathfrak{a}_p}|y_p|_p^{-1} - \mu_{\mathfrak{a}_p}\Big)\, \chi_p(r_p)\, e_p(x_p)\, e_\infty\left(|y_p|_p^{-1} j_p\right)\, \widetilde{\chi}_{idelic}(k_{p,0}), \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{if } \; m_{\mathfrak{a}_p}|y_p|_p^{-1} - \mu_{\mathfrak{a}_p} \in \mathbb{Z}, \\ 0, \;\; \textit{otherwise.} \end{cases}$$

Proof Use Definition 3.3 and Theorem 3.2. □

Remark 3.5 *In Corollary 3.4, assume that f is a normalized Maass form, i.e., $a(\infty, 1) = 1$. Then for $p \nmid N$,*

$$W_{f,p}(k_p) = 1, \; \textit{for all } k_p \in GL(2, \mathbb{Z}_p),$$

and for $p \mid N$,

$$W_{f,p}(k_p) = 1, \; \textit{for all } k_p \in I_{p,N}.$$

Theorem 3.6 *Let $f : \Gamma_0(N)\backslash\mathfrak{h} \to \mathbb{C}$ be a cusp form of weight k, type ν, level N and character $\chi \pmod N$. If f is a normalized Hecke newform, i.e., its first Fourier coefficient at ∞ is 1, then*

$$W_f(g) = \prod_v W_{f,v}(g_v).$$

Proof: If f is a newform then f_{adelic} generates an irreducible subspace

$$V \;\subset\; \mathcal{A}_0\Big(GL(2,\mathbb{Q})\backslash GL(2,\mathbb{A}), \;\; \chi_{\text{idelic}}\Big),$$

and an irreducible automorphic cuspidal representation (π, V), under the actions of $GL(2, \mathbb{A}_f)$ and $(\mathfrak{g}, K_\infty)$ for $GL(2, \mathbb{R})$, where $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{C})$ and $K_\infty = O(2, \mathbb{R})$ (see [4] theorem 3.6.1). By [6] there are local representations (π_v, V_v) for $GL(2, \mathbb{Q}_v)$ for each place v of $\mathbb{Q}$ such that

- π_∞ is an irreducible and admissible $(\mathfrak{g}, K_\infty)$-module;
- π_v is an irreducible and admissible representation for $GL(2, \mathbb{Q}_v)$ for all finite places v. Furthermore, for almost all v, we know that V_v contains a non-zero K_v-fixed vector where $K_v = GL(2, \mathbb{Z}_v)$, and

$$(\pi, V) \;\cong\; \bigotimes_v{}' (\pi_v, V_v), \qquad \text{(restricted tensor product)},$$

where the restricted tensor product is taken with respect to some choice of non-zero K_v-fixed vectors in all but finitely many of the spaces V_v. (Different choices may give rise to different restricted tensor products, but the representations obtained are all isomorphic to one another and to (π, V).)

We shall use the existence and uniqueness of Whittaker models for (π, V) and for each of the local representations (π_v, V_v) (see [4] section 3.5). For each place v, let $\mathcal{W}(\pi_v, e_v)$ denote the Whittaker space, corresponding to (π_v, V_v) and the additive character e_v introduced at the beginning of this section. For each place v such that (π_v, V_v) contains a K_v-fixed vector, the space of K_v-fixed vectors is one dimensional (see [4] theorems 2.4.2 , 4.6.2), and contains a unique element which takes the value 1 on all of K_v (the existence of such an element in the non-archimedean case is proved in [4], proposition 3.5.2; existence in the Archimedean case can be proved along the same lines but we do not need it here).

Let $\mathcal{W}_{\text{tensor}}$ denote the restricted tensor product of the spaces $\mathcal{W}(\pi_v, e_v)$ with respect to the K_v-fixed vectors which take the value 1 on K_v. Then pointwise multiplication gives an isomorphism between this space and the unique Whittaker model of the original representation π.

Indeed, suppose that $g = \{g_v\}_v \in GL(2, \mathbb{A})$ and that $\otimes_v W_v$ is an element of $\mathcal{W}_{\text{tensor}}$, so that for each v, the Whittaker function W_v is an element of $\mathcal{W}(\pi_v, e_v)$, and $W_v(k_v) = 1$ for all but finitely many v (for all $k_v \in K_v$). Then the infinite product $\prod_v W_v(g_v)$ is convergent, because all but finitely many of its terms are 1.

Let $\prod_v \mathcal{W}(\pi_v, e_v)$ be the space of complex valued functions on $GL(2, \mathbb{A})$ spanned by $\prod_v W_v(g_v)$ where $W_v(g_v) \in \mathcal{W}(\pi_v, e_v)$ such that $W_v(k_v) = 1$ for all $k_v \in GL(2, \mathbb{Z}_v)$ for almost all $v < \infty$. (i.e., where $\otimes_v W_v \in \mathcal{W}_{\text{tensor}}$.)

Then
$$\bigotimes_v{}' (\pi_v, V_v) \;\cong\; \prod_v \mathcal{W}(\pi_v, e_v)$$
and $\prod_v \mathcal{W}(\pi_v, e_v)$ is a global Whittaker function space.

Let
$$\mathcal{W}(\pi, e) := \left\{ W_\phi(g) := \int_{\mathbb{Q}\backslash\mathbb{A}} \phi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) e(-u)du, \text{ for all } g \in GL(2, \mathbb{A}) \;\middle|\; \phi \in V \right\}.$$
Then $\mathcal{W}(\pi, e)$ is a Whittaker model isomorphic to (π, V). By the global uniqueness of Whittaker models,
$$(\pi, V) \cong \mathcal{W}(\pi, e) = \prod_v \mathcal{W}(\pi_v, e_v) \;\cong\; \bigotimes_v{}' (\pi_v, V_v).$$
The statement of theorem 3.6, amounts to the assertion that the element of the restricted tensor product $\bigotimes_v V_v$ corresponding to the element f_{adelic} of V is a pure tensor $\otimes_{v \le \infty} \xi_v$. This may be deduced from uniqueness of the "local new vector" at each place [5]. However, we shall take a different approach.

There is a bijection between (π, V) and $\mathcal{W}(\pi, e) = \prod_v \mathcal{W}(\pi_v, e_v)$ such that for any $\phi \in V$,
$$W_\phi(g) := \int_{\mathbb{Q}\backslash\mathbb{A}} \phi\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) e(-u)du \;\in\; \mathcal{W}(\pi, e),$$
and for any $W \in \mathcal{W}(\pi, e)$,
$$\phi_W(g) := \sum_{\alpha \in \mathbb{Q}^\times} W\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right) \in V.$$

Since $f_{\text{adelic}} \in V$, it follows that for some fixed integer $M \geq 0$,

$$W_f(g) = \sum_{i=1}^{M} c_i W_i(g), \qquad \big(g \in GL(2,\mathbb{A})\big),$$

where $c_i \in \mathbb{C}$, $W_i(g) = \prod_v W_{i,v}(g_v)$ with $W_{i,v} \in \mathcal{W}(\pi_v, e_v)$ for every place v of $\mathbb{Q}$. Furthermore, $W_{i,v}(k_v) = 1$ for all $k_v \in GL(2,\mathbb{Z}_v)$ for almost all $v \neq \infty$. For each place w of $\mathbb{Q}$,

$$W_f(i_w(g_w)) = \sum_{i=1}^{M} c_i \left(\prod_{w \neq v} W_{i,v}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\right) \cdot W_{i,w}(g_w), \qquad \big(g_w \in GL(2,\mathbb{Q}_w)\big),$$

which implies $W_f(i_w(g_w)) \in \mathcal{W}(\pi_w, e_w)$. Therefore, $W_{f,v}(g_v) \in \mathcal{W}(\pi_v, e_v)$ since

$$W_{f,v}(g_v) = \begin{cases} W_f(i_v(g_v)), & \text{if } v = \infty \\ \frac{W_f(i_v(g_v))}{W_{\frac{k}{2},\nu-\frac{1}{2}}(4\pi)}, & \text{if } v < \infty \end{cases}$$

for all $g_v \in GL(2,\mathbb{Q}_v)$ and for each place v.

Let

$$W(g) := \prod_v W_{f,v}(g_v). \tag{3.7}$$

Then $W(g) \in \mathcal{W}(\pi, e)$.

Definition (The functions ϕ_W, F_{ϕ_W}) *Let $W \in \mathcal{W}(\pi, e)$ be given by (3.7). Then we define*

$$\phi_W := \sum_{\alpha \in \mathbb{Q}} W\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right) \in V, \qquad F_{\phi_W}(x+iy) := \phi_W\left(i_\infty\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right)\right),$$

for $x, y \in \mathbb{R}$ and $y > 0$.

Then $F_{\phi_W}(x+iy)$ is a Maass cusp form. Furthermore,

$$\begin{aligned} F_{\phi_W}(x+iy) &= \phi_W\left(i_\infty\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right)\right) \\ &= \sum_{\alpha \in \mathbb{Q}^\times} W\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} i_\infty\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right)\right) \\ &= \sum_{\alpha \in \mathbb{Q}^\times} W_{f,\infty}\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) \cdot \prod_{p<\infty} W_{f,p}\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right). \end{aligned}$$

Next, we evaluate the Whittaker functions appearing above at every place.

- $\underline{v = \infty}$:

$$\begin{aligned} W_{f,\infty}\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) &= W_{f,\infty}\left(\begin{pmatrix} 1 & \alpha x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} |\alpha| y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \text{sign}(\alpha) & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= a\big(\infty, \text{sign}(\alpha)\big)\, W_{\frac{\text{sign}(\alpha)k}{2},\, \nu-\frac{1}{2}}(4\pi|\alpha|y)\, e_\infty(\alpha x). \end{aligned}$$

• $\underline{v = p < \infty}$:

$$W_{f,p}\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right) = \begin{cases} a\left(\infty, |\alpha|_p^{-1}\right), & \text{if } \alpha\in\mathbb{Z}_p,\\ 0, & \text{otherwise.}\end{cases}$$

Therefore,

$$W\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} i_\infty\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix}\right)\right)$$

$$= \begin{cases} a\big(\infty, \operatorname{sign}(\alpha)\big)\, W_{\frac{\operatorname{sign}(\alpha)k}{2},\, \nu-\frac12}(4\pi|\alpha|y)\, e_\infty(\alpha x)\, \prod_p a\left(\infty, |\alpha|_p^{-1}\right),\\ \qquad\qquad \text{if } \alpha\in\mathbb{Z} \text{ and } \alpha\neq 0;\\ 0, \ \text{otherwise.}\end{cases}$$

Then we have

$$F_{\phi_W}(x+iy) = \sum_{n\in\mathbb{Z},\ n\neq 0} a\big(\infty, \operatorname{sign}(n)\big)\, W_{\frac{\operatorname{sign}(n)k}{2},\, \nu-\frac12}(4\pi|n|y)\, e_\infty(nx) \prod_p a\left(\infty, |n|_p^{-1}\right).$$

Since f is a newform, it follows from Proposition 2.1 that,

$$a(\infty, n) = a(\infty, \operatorname{sign}(n)) \prod_{p|n} a\left(\infty, |n|_p^{-1}\right).$$

This implies that for $\mathfrak{a} = \infty$, we have a Fourier-Whittaker expansion for f, as follows:

$$\begin{aligned} f(x+iy) &= \sum_{n\in\mathbb{Z}, n\neq 0} a(\infty, n) W_{\frac{\operatorname{sign}(n)k}{2},\, \nu-\frac12}(4\pi|n|y) e^{2\pi i n x}\\ &= \sum_{n\in\mathbb{Z},\ n\neq 0} a\big(\infty, \operatorname{sign}(n)\big)\, W_{\frac{\operatorname{sign}(n)k}{2},\, \nu-\frac12}(4\pi|n|y)\, e_\infty(nx) \prod_p a\left(\infty, |n|_p^{-1}\right).\end{aligned}$$

Therefore, $F_{\phi_W}(x+iy) = f(x+iy)$ for $x, y\in\mathbb{R}$ and $y>0$. This implies that

$$\phi_W\left(i_\infty\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix}\right)\right) = f_{\text{adelic}}\left(i_\infty\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix}\right)\right)$$

for $x, y\in\mathbb{R}$ and $y>0$. For any $g\in GL(2,\mathbb{A})$, there exist a unique $\gamma\in GL(2,\mathbb{Q})$, such that

$$g_\infty = \begin{pmatrix}1 & x_\infty\\ 0 & 1\end{pmatrix}\begin{pmatrix}y_\infty & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}r_\infty & 0\\ 0 & r_\infty\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\in GL(2,\mathbb{R})^+,$$

with $0\le\theta<2\pi$, real numbers $x_\infty, y_\infty, r_\infty$, with $y_\infty>0$ and $r_\infty\neq 0$, and $k_f\in K_0(N)$ such that

$$g = i_{\text{diag}}(\gamma) i_\infty\left(\begin{pmatrix}1 & x_\infty\\ 0 & 1\end{pmatrix}\begin{pmatrix}y_\infty & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}r_\infty & 0\\ 0 & r_\infty\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\right) i_{\text{finite}}(k_0).$$

Then for $\alpha\in\mathbb{Q}^\times$, we have

$$\begin{aligned} W\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} g\right) &= W_{f,\infty}\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x_\infty\\ 0 & 1\end{pmatrix}\begin{pmatrix}y_\infty & 0\\ 0 & 1\end{pmatrix}\right) e_\infty\left(\frac{k\theta}{2\pi}\right)\chi_\infty(r_\infty)\\ &\qquad\cdot\prod_{p<\infty} W_{f,p}\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)\widetilde{\chi}_{\text{idelic}}(k_f)\\ &= W\left(i_{\text{diag}}\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right) i_\infty\left(\begin{pmatrix}1 & x_\infty\\ 0 & 1\end{pmatrix}\begin{pmatrix}y_\infty & 0\\ 0 & 1\end{pmatrix}\right)\right) e_\infty\left(\frac{k\theta}{2\pi}\right)\chi_\infty(r_\infty)\widetilde{\chi}_{\text{idelic}}(k_f).\end{aligned}$$

Therefore

$$\begin{aligned}
\phi_W(g) &= \sum_{\alpha\in\mathbb{Q}} W\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} g\right)\\
&= \sum_{\alpha\in\mathbb{Q}} W\left(i_{diag}\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right) i_\infty\left(\begin{pmatrix}1 & x_\infty\\ 0 & 1\end{pmatrix}\begin{pmatrix}y_\infty & 0\\ 0 & 1\end{pmatrix}\right)\right)\cdot e_\infty\left(\frac{k\theta}{2\pi}\right)\chi_\infty(r_\infty)\widetilde{\chi}_{\text{idelic}}(k_f)\\
&= \phi_W\left(i_\infty\left(\begin{pmatrix}1 & x_\infty\\ 0 & 1\end{pmatrix}\begin{pmatrix}y_\infty & 0\\ 0 & 1\end{pmatrix}\right)\right)\cdot e_\infty\left(\frac{k\theta}{2\pi}\right)\chi_\infty(r_\infty)\widetilde{\chi}_{\text{idelic}}(k_f),
\end{aligned}$$

and for all $g\in GL(2,\mathbb{A})$, $r_\infty\in\mathbb{R}^\times$, $0\le\theta<2\pi$ and $k_f\in K_0(N)$,

$$\begin{aligned}
W_f&\left(g\cdot i_\infty\left(\begin{pmatrix}r_\infty & 0\\ 0 & r_\infty\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\right) i_{\text{finite}}(k_f)\right)\\
&= W_f(g)e_\infty\left(\frac{k\theta}{2\pi}\right)\chi_\infty(r_\infty)\widetilde{\chi}_{\text{idelic}}(k_f).
\end{aligned}$$

It follows that for all $g\in GL(2,\mathbb{A})$ as above, we have

$$\begin{aligned}
\phi_W(g) &= \phi_W\left(i_\infty\left(\begin{pmatrix}1 & x_\infty\\ 0 & 1\end{pmatrix}\begin{pmatrix}y_\infty & 0\\ 0 & 1\end{pmatrix}\right)\right) e_\infty\left(\frac{k\theta}{2\pi}\right)\chi_\infty(r_\infty)\widetilde{\chi}_{\text{idelic}}(k_f)\\
&= f_{\text{adelic}}\left(i_\infty\left(\begin{pmatrix}1 & x_\infty\\ 0 & 1\end{pmatrix}\begin{pmatrix}y_\infty & 0\\ 0 & 1\end{pmatrix}\right)\right) e_\infty\left(\frac{k\theta}{2\pi}\right)\chi_\infty(r_\infty)\widetilde{\chi}_{idelic}(k_f)\\
&= \sum_{\alpha\in\mathbb{Q}^\times} W_f\left(i_{diag}\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right) i_\infty\left(\begin{pmatrix}1 & x_\infty\\ 0 & 1\end{pmatrix}\begin{pmatrix}y_\infty & 0\\ 0 & 1\end{pmatrix}\right)\right) e_\infty\left(\frac{k\theta}{2\pi}\right)\chi_\infty(r_\infty)\widetilde{\chi}_{\text{idelic}}(k_f)\\
&= \sum_{\alpha\in\mathbb{Q}^\times} W_f\left(i_{diag}\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right) g\right)\\
&= f_{\text{adelic}}(g).
\end{aligned}$$

Therefore for all $g=\{g_v\}_v\in GL(2,\mathbb{A})$, we have an equality of Whittaker functions

$$W(g) = \prod_v W_{f,v}(g_v) = W_f(g). \quad \square$$

Theorem 3.8 *Fix a positive integer $N = q_1^{e_1}\cdots q_h^{e_h}$ where $q_1^{e_1},\cdots,q_h^{e_h}$ are powers of distinct primes. Let S be a set of inequivalent cusps for $\Gamma_0(N)$. Fix an integer $k\ge 1$ and let f be a normalized Hecke newform of weight k, type ν, level N and character $\chi=\prod_{1\le i\le h}\chi_i$, where χ_i is a Dirichlet character $\pmod{q_i^{e_i}}$ for $1\le i\le h$. Fix one cusp $\mathfrak{a}\in S$ and let M be a positive integer such that*

$$\begin{aligned}
\epsilon M + \mu_{\mathfrak{a}} = \epsilon m_{\mathfrak{a}} p_1^{m_1}\cdots p_n^{m_n}\cdot q_1^{m_1'}\cdots q_h^{m_h'},&\\
\big(\text{for } m_i, m_j'\in\mathbb{Z},\ m_i>0,\ \text{with } 1\le i\le n,\ 1\le j\le h\big)&,
\end{aligned}$$

where $p_1,\dots,p_n$ are distinct primes which do not divide N.

Then for each $i = 1, \dots, h$, there exist a unique cusp $\mathfrak{b}_i \in S$ and a unique integer $1 \le j_i < m_{\mathfrak{b}_i}$ such that

$$\gamma_{\mathfrak{b}_i} \begin{pmatrix} 1 & j_i \\ 0 & 1 \end{pmatrix} =: \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix},$$

$$\gamma_{\mathfrak{b}_i} \begin{pmatrix} 1 & j_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon M_i & 0 \\ 0 & 1 \end{pmatrix} \gamma_{\mathfrak{a}}^{-1} = \begin{pmatrix} a_{q_i} & b_{q_i} \\ c_{q_i} & d_{q_i} \end{pmatrix}, \qquad \left(\text{for } M_i := \frac{\epsilon M + \mu_{\mathfrak{a}}}{\epsilon m_{\mathfrak{a}} q_i^{m_i'}} \in \mathbb{Q} \right),$$

where $\gamma_{\mathfrak{b}_i} \in SL(2,\mathbb{Z})$ with $\gamma_{\mathfrak{b}_i}\infty = \mathfrak{b}_i$. For all $1 \le u \le h$ with $u \ne i$, we have $\left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right) \in \Gamma_0(q_u^{e_u})$. Let $\delta_i := \prod_{u \ne i} q_i^{\max(0,-m_u')}$ and δ_i' be an integer such that $\delta_i'\delta_i \equiv 1 \pmod{q_i^{e_i}}$. Then we have $\delta_i a_{q_i},\ \delta_i b_{q_i},\ \delta_i c_{q_i},\ \delta_i d_{q_i} \in \mathbb{Z}$ and $\delta_i c_{q_i} \equiv 0 \pmod{q_i^{e_i}}$.

Let $a(\mathfrak{a}, \epsilon M)$ denote the ϵM^{th} Fourier coefficient of f as in proposition 1.6. Then we have

$$a(\mathfrak{a}, \epsilon M) = a(\infty, \epsilon) \prod_{i=1}^{n} a\left(\infty, p_i^{m_i}\right)$$
$$\cdot \prod_{i=1}^{h} \left(a\Big(\mathfrak{b}_i, m_{\mathfrak{b}_i} q_i^{m_i'} - \mu_{\mathfrak{b}_i}\Big) e_\infty(q_i^{m_i'} j_i) \left(\prod_{u \ne i,\, u=1}^{h} \chi_u(d_i)^{-1} \right) \chi_i\Big(\delta_i \delta_i' d_{q_i}\Big)^{-1} \right)$$

if $m_{\mathfrak{b}_i} q_i^{m_i'} - \mu_{\mathfrak{b}_i} \in \mathbb{Z}$ for all $i = 1, \dots, h$. Otherwise $a(\mathfrak{a}, \epsilon M) = 0$.

Proof: Fix one cusp $\mathfrak{a} \in S$. Then there exists $\gamma_{\mathfrak{a}} \in SL(2,\mathbb{Z})$ such that $\gamma_{\mathfrak{a}}\infty = \mathfrak{a}$. For $y = \{y_v\}_v \in \mathbb{A}^\times$, $y_\infty = 1$, let $t = \prod_p |y_p|_p^{-1}$ and for each $q \mid N$, let

$$k_q := \begin{pmatrix} t y_q^{-1} & 0 \\ 0 & 1 \end{pmatrix} \gamma_{\mathfrak{a}}^{-1}.$$

Then $k_q \in GL(2,\mathbb{Z}_q)$ for every $q \mid N$. Let $\epsilon = \pm 1$ and take

$$g = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} i_{diag}\left(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \right) \prod_{q \mid N} i_q(k_q) \in GL(2,\mathbb{A}).$$

Then

$$i_{\text{finite}}(\gamma_{\mathfrak{a}}) \prod_{q \mid N} i_q \left(\begin{pmatrix} t^{-1} y_q & 0 \\ 0 & 1 \end{pmatrix} k_q \right) \in K_0(N)$$

since

$$\gamma_{\mathfrak{a}} \begin{pmatrix} t^{-1} y_q & 0 \\ 0 & 1 \end{pmatrix} k_q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in I_{q,N} \text{ for each } q \mid N$$

and $\gamma_{\mathfrak{a}} \in GL(2,\mathbb{Z}_p)$ for any prime p.

By Theorem 3.6, we know that

$$W_f \left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} i_{diag}\left(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \right) \prod_{q \mid N} i_q(k_q) \right)$$
$$= W_{f,\infty}\left(\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \right) \left(\prod_{p \nmid N} W_{f,p}\left(\begin{pmatrix} y_p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \right) \right)$$
$$\cdot \left(\prod_{q \mid N} W_{f,q}\left(\begin{pmatrix} y_q & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} k_q \right) \right).$$

Then by Corollary 3.4, for each fixed prime $q \mid N$, there exists a cusp $\mathfrak{b}_q \in S$ and an integer $0 \le j_q < m_{\mathfrak{b}_q}$ which are uniquely determined by

$$i_{\text{finite}}\left(\gamma_{\mathfrak{b}_q}\begin{pmatrix}1 & j_q\\ 0 & 1\end{pmatrix}\right) i_q\left(\begin{pmatrix}|y_q|_q y_q & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\epsilon & 0\\ 0 & 1\end{pmatrix} k_q\right) =: k_{q,0} \in K_0(N).$$

This is equivalent to

$$(k_{q,0})_q = \gamma_{\mathfrak{b}_q}\left(\begin{smallmatrix}1 & j_q\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}\epsilon t|y_q|_q & 0\\ 0 & 1\end{smallmatrix}\right)\gamma_{\mathfrak{a}}^{-1} \in I_{q,N};$$

for each prime $q' \mid N$ and $q' \neq q$,

$$(k_{q,0})_{q'} = \gamma_{\mathfrak{b}_q}\begin{pmatrix}1 & j_q\\ 0 & 1\end{pmatrix} \in I_{q',N};$$

and for each prime $p \nmid N$,

$$(k_{q,0})_p = \gamma_{\mathfrak{b}_q}\begin{pmatrix}1 & j_q\\ 0 & 1\end{pmatrix} \in GL(2, \mathbb{Z}_p).$$

It follows form Corollary 3.4, that

$$W_f\left(\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix} i_{diag}\left(\begin{pmatrix}\epsilon & 0\\ 0 & 1\end{pmatrix}\right)\prod_{q\mid N} i_q(k_q)\right) = a(\mathfrak{a}, \epsilon m_{\mathfrak{a}} t - \mu_{\mathfrak{a}}) W_{\frac{\epsilon k}{2}, \nu-\frac{1}{2}}(4\pi)$$

$$= \begin{cases} a(\infty,\epsilon) W_{\frac{\epsilon k}{2},\, \nu-\frac{1}{2}}(4\pi)\left(\prod\limits_{q\mid N} a\Big(\mathfrak{b}_q, m_{\mathfrak{b}_q}|y_q|_q^{-1} - \mu_{\mathfrak{b}_q}\Big)\, e_\infty\left(|y_q|_q^{-1} j_q\right)\, \widetilde{\chi}_{idelic}(k_{q,0})\right) \\ \cdot \prod\limits_{p\nmid N} a\left(\infty, |y_p|_p^{-1}\right), \\ \qquad\qquad \text{if } \epsilon m_{\mathfrak{a}} t - \mu_{\mathfrak{a}} \text{ and } \ m_{\mathfrak{b}_q}|y_q|_q^{-1} - \mu_{\mathfrak{b}_q} \in \mathbb{Z} \text{ for all } q \mid N; \\ 0, \quad \text{otherwise.} \end{cases}$$

Therefore,

$$a(\mathfrak{a}, \epsilon m_{\mathfrak{a}} t - \mu_{\mathfrak{a}})$$

$$= \begin{cases} a(\infty,\epsilon)\left(\prod\limits_{q\mid N} a\Big(\mathfrak{b}_q, m_{\mathfrak{b}_q}|y_q|_q^{-1} - \mu_{\mathfrak{b}_q}\Big)\, e_\infty\left(|y_q|_q^{-1} j_q\right)\, \widetilde{\chi}_{\text{idelic}}(k_{q,0})\right) \\ \cdot \prod\limits_{p\nmid N} a\left(\infty, |y_p|_p^{-1}\right), \\ \qquad\qquad \text{if } \epsilon m_{\mathfrak{a}} t - \mu_{\mathfrak{a}},\ m_{\mathfrak{b}_q}|y_q|_q^{-1} - \mu_{\mathfrak{b}_q} \in \mathbb{Z} \text{ for all } q \mid N, \\ 0, \quad \text{otherwise.} \end{cases}$$

Now let $\epsilon m_{\mathfrak{a}} t - \mu_{\mathfrak{a}} = \epsilon M \in \mathbb{Z}$. Then $\epsilon m_{\mathfrak{a}} t = \epsilon M + \mu_{\mathfrak{a}}$. Since $t = \prod_p |y_p|_p^{-1}$,

$$\epsilon M + \mu_{\mathfrak{a}} = \epsilon m_{\mathfrak{a}} p_1^{m_1} \cdots p_n^{m_n} q_1^{m_1'} \cdots q_h^{m_h'}$$

for distinct primes $p_1, \dots, p_n$ (different from $q_1, \dots, q_h$), and $m_1, \dots, m_n,\ m'_1, \dots, m'_h \in \mathbb{Z}$. This means that for each prime p, we take

$$y_p = \begin{cases} p_i^{m_i}, & \text{if } p = p_i \text{ for some } i = 1, \dots, n, \\ q_i^{m'_i}, & \text{if } p = q_i \text{ for some } i = 1, \dots, h, \\ 1, & \text{otherwise.} \end{cases}$$

It follows that if $m_i < 0$, then $a(\infty, p_i^{m_i}) = 0$. So assume that $m_1, \dots, m_n > 0$. Then for each $i = 1, \dots, h$, we use the uniqueness property above for choosing $\mathfrak{b}_i$ and integers $1 \leq j_i < m_{\mathfrak{b}_i}$. □

Theorem 3.9 *Fix a prime p, let V_p be a complex vector space, and let $\pi_p : GL(2, \mathbb{Q}_p) \to GL(V_p)$. Assume (π_p, V_p) is an irreducible and admissible representation of $GL(2, \mathbb{Q}_p)$.*

Let f be a normalized Hecke newform of weight k, type ν, level N, and character χ (mod N) for $\Gamma_0(N)$. For each cusp $\mathfrak{a} \in \mathbb{Q} \cup \{\infty\}$ and $n \in \mathbb{Z}$, let $a(\mathfrak{a}, n)$ denote the n^{th} Fourier coefficient of f at the cusp $\mathfrak{a}$. Let f_{adelic} be the adelic lift of f as in (3.1). If (π_p, V_p) is isomorphic to the local representation factor of the irreducible global automorphic representation of $GL(2, \mathbb{A})$ (which is generated by f_{adelic}), then (π_p, V_p) is supercuspidal if and only if for each cusp $\mathfrak{a} \in \mathbb{Q} \cup \{\infty\}$ with $\mu_{\mathfrak{a}} = 0$, there exists an integer $M_{\mathfrak{a}} \geq 0$, such that

$$a(\mathfrak{a}, m_{\mathfrak{a}} p^m) = 0, \qquad \big(\text{for all } m \in \mathbb{Z},\ m \geq M_{\mathfrak{a}}\big),$$

where $m_{\mathfrak{a}}$, $\mu_{\mathfrak{a}}$ are given by (1.4), (1.5), respectively.

Proof: If $p \nmid N$, then f_{adelic} is fixed by $K_p = GL(2, \mathbb{Z}_p)$. It follows that (π_p, V_p) has a nonzero K_p-fixed vector, and a nonzero Whittaker model, which forces it to be an irreducible principal series representation (see [4], theorem 4.6.4). It follows that if $p \nmid N$ then (π_p, V_p) cannot be a supercuspidal representation. On the other hand, if $p \nmid N$ and f is an eigenfunction of T_p^{χ}, then it follows easily that infinitely many of the coefficients $a(\infty, p^m)$ are nonzero. This proves the equivalence in this case, and henceforth we shall assume that $p \mid N$.

Let W_f denote the global Whittaker function of f_{adelic}. It follows from Theorem 3.6, that there exist local Whittaker functions $W_{f,v}$ on $GL(2, \mathbb{Q}_v)$ at each place of v such that

$$W_f(g) = \prod_v W_{f,v}(g_v), \qquad \big(\forall g = \{g_v\}_{v \leq \infty} \in GL(2, \mathbb{A})\big).$$

For a prime p, assume that (π_p, V_p) is isomorphic to a component of the irreducible automorphic representation of $GL(2, \mathbb{A})$ generated by f_{adelic}. As we have noted before (see proof of theorem 3.6) there exists a Whittaker space $\mathcal{W}_p := \mathcal{W}(\pi_p, e_p)$ associated to (π_p, V_p) and $W_{f,p} \in \mathcal{W}_p$. We also have a corresponding Kirillov space, denoted $\mathcal{K}_p$, where

$$\mathcal{K}_p = \left\{ W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) \;\middle|\; y \in \mathbb{Q}_p^{\times},\ \ W \in \mathcal{W}_p \right\}.$$

It is a theorem of Kirillov (see [9], [12], [13]) that the representation (π_p, V_p) is isomorphic to one and only one Kirillov representation (π', K_p) where the representation $\pi'_p : GL(2, \mathbb{Q}_p) \to GL(\mathcal{K}_p)$ operates in such a way that

$$\pi'_p\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right).W\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right) \;=\; e_p(by)\, W\left(\begin{pmatrix} ay & 0 \\ 0 & 1 \end{pmatrix}\right)$$

for all $a, y \in \mathbb{Q}_p^\times$ for all $b \in \mathbb{Q}_p$, and for all $W \in \mathcal{K}_p$.

Define the Schwartz-Bruhat space

$$S(\mathbb{Q}_p^\times) := \left\{ \phi : \mathbb{Q}_p^\times \to \mathbb{C} \;\middle|\; \begin{array}{c} \phi \text{ is locally constant, and } \exists N_\phi > \epsilon_\phi > 0 \text{ such that} \\ \phi(y) = 0 \text{ if } |y|_p < \epsilon_\phi \text{ or } |y|_p > N_\phi \end{array} \right\}.$$

Then by [9], [11], (π_p, V_p) is supercuspidal if and only if $\mathcal{K}_p = S(\mathbb{Q}_p^\times)$.

Assume that $p \mid N$. For $y \in \mathbb{Q}_p^\times$, define

$$\varphi_p(y) := W_{f,p}\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) \in \mathcal{K}_p.$$

By Corollary 3.4

$$\varphi_p(y) = \begin{cases} a(\infty, |y|_p^{-1}), & \text{if } |y|_p^{-1} \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

Then φ_p is invariant under the action of

$$I_{p,N} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z}_p) \;\middle|\; c \in N\mathbb{Z}_p \right\},$$

and $\mathcal{K}_p$ is spanned by

$$\left\{ \pi'(g).\varphi_p \;\middle|\; g \in GL(2, \mathbb{Q}_p) \right\}.$$

Now fix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2, \mathbb{Q}_p)$. We will compute $\pi'(g)\,.\varphi(y)$ for $y \in \mathbb{Q}_p^\times$ and determine under what conditions this function lies in $S\left(\mathbb{Q}_p^\times\right)$. There are two different cases that need to be considered.

Case (1) $c = 0$ **:**

$$\pi'\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right) .\varphi_p(y) = W_{f,p}\left(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right)$$

$$= \begin{cases} \chi_p(d)\, e_p\left(bd^{-1}y\right) a\left(\infty, |ad^{-1}y|_p^{-1}\right), & \text{if } |ad^{-1}y|_p^{-1} \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

For fixed $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in GL(2, \mathbb{Q}_p)$, the function $\pi'\left(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\right) .\varphi_p \in S(\mathbb{Q}_p^\times)$ if and only if there exists an integer $M \geq 0$ such that $\pi'\left(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\right) .\varphi_p(y) = 0$ if $y \in p^M\mathbb{Z}_p$. Since $\chi_p(d) \neq 0$ and $e_p(bd^{-1}y) \neq 0$, this function vanishes if and only if $a\left(\infty, |ad^{-1}y|_p^{-1}\right) = 0$. It follows that $\pi'\left(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\right) .\varphi_p \in S(\mathbb{Q}_p^\times)$ if and only if there exists a an integer $M_\infty \geq 0$ such that $a(\infty, p^m) = 0$ whenever $m \geq M_\infty$.

Case (2) $c \neq 0$ **:**

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} c^{-2}(ad - bc) & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & c^{-1}d \\ 0 & 1 \end{pmatrix}.$$

Consequently

$$\pi_p'\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right).\varphi_p(y)$$

$$= \pi_p'\left(\begin{pmatrix} c & 0\\ 0 & c\end{pmatrix}\begin{pmatrix} c^{-2}(ad-bc) & ac^{-1}\\ 0 & 1\end{pmatrix}\right)\cdot\left(\pi_p'\left(\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 1 & c^{-1}d\\ 0 & 1\end{pmatrix}\right).\varphi_p\right)(y)$$

$$= \chi_p(c)e_p\left(ac^{-1}y\right)\left(\pi_p'\left(\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 1 & c^{-1}d\\ 0 & 1\end{pmatrix}\right).\varphi_p\right)\left(c^{-2}(ad-bc)y\right).$$

If $c^{-1}d\in\mathbb{Z}_p$ then

$$\left(\pi_p'\left(\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix} 1 & c^{-1}d\\ 0 & 1\end{pmatrix}\right).\varphi_p\right)(c^{-2}(ad-bc)y) = \left(\pi_p'\left(\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\right).\varphi_p\right)(c^{-2}(ad-bc)y)$$

$$= W_{f,p}\left(\begin{pmatrix} c^{-2}(ad-bc)y & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\right).$$

Then by Corollary 3.4, there exists a cusp $\mathfrak{a}\in\mathbb{Q}\cup\{\infty\}$, an integer $0\le j<m_{\mathfrak{a}}$, and $k_0\in K_0(N)$, which are uniquely determined by $g=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ and y such that

$$i_{\text{finite}}\left(\gamma_{\mathfrak{a}}\begin{pmatrix} 1 & j\\ 0 & 1\end{pmatrix}\right)i_p\left(\begin{pmatrix} |c^{-2}(ad-bc)y|_p c^{-2}(ad-bc)y & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\right) = k_0\in K_0(N).$$

Then

$$W_{f,p}\left(\begin{pmatrix} c^{-2}(ad-bc)y & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}\right)$$

$$= \begin{cases} a\Big(\mathfrak{a}, m_{\mathfrak{a}}|c^{-2}(ad-bc)y|_p^{-1}-\mu_{\mathfrak{a}}\Big)\, e_\infty\Big(|c^{-2}(ad-bc)y|_p^{-1}j\Big)\,\widetilde{\chi}_{\text{idelic}}(k_0), \\ \qquad \text{if } m_{\mathfrak{a}}|c^{-2}(ad-bc)y|_p^{-1}-\mu_{\mathfrak{a}}\in\mathbb{Z}, \\ 0, \qquad\qquad \text{otherwise.}\end{cases}$$

Therefore, if $c^{-1}d\in\mathbb{Z}_p$ then

$$\pi_p'\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right).\varphi_p(y)$$

$$= \begin{cases} \chi_p(c)e_p(ac^{-1}y)\, a\Big(\mathfrak{a}, m_{\mathfrak{a}}|c^{-2}(ad-bc)y|_p^{-1}-\mu_{\mathfrak{a}}\Big)e_\infty\Big(|c^{-2}(ad-bc)y|_p^{-1}j\Big)\widetilde{\chi}_{\text{idelic}}(k_0), \\ \qquad\qquad \text{if } m_{\mathfrak{a}}|c^{-2}(ad-bc)y|_p^{-1}-\mu_{\mathfrak{a}}\in\mathbb{Z}, \\ 0, \qquad\qquad \text{otherwise.}\end{cases}$$

If $c^{-1}d \notin \mathbb{Z}_p$,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & c^{-1}d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c^{-1}d & 0 \\ 0 & c^{-1}d \end{pmatrix}\begin{pmatrix} c^2d^{-2} & -cd^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ cd^{-1} & 1 \end{pmatrix}$$

and $\left(\begin{smallmatrix} 1 & 0 \\ cd^{-1} & 1 \end{smallmatrix}\right) \in GL(2,\mathbb{Z}_p)$ since $cd^{-1} \in p\mathbb{Z}_p$. It follows that

$$\begin{aligned}\left(\pi_p'\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & c^{-1}d \\ 0 & 1 \end{pmatrix}\right).\varphi_p\right)&(c^{-2}(ad-bc)y)\\ &= \chi_p\left(c^{-1}d\right) e_p\Big(-c^{-1}d^{-1}(ad-bc)y\Big)\left(\pi_p'\left(\begin{pmatrix} 1 & 0 \\ cd^{-1} & 1 \end{pmatrix}\right).\varphi_p\right)\left(d^{-2}(ad-bc)y\right)\end{aligned}$$

and

$$\left(\pi_p'\left(\begin{pmatrix} 1 & 0 \\ cd^{-1} & 1 \end{pmatrix}\right).\varphi_p\right)\left(d^{-2}(ad-bc)y\right) = W_{f,p}\left(\begin{pmatrix} d^{-2}(ad-bc)y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ cd^{-1} & 1 \end{pmatrix}\right).$$

There exists a cusp $\mathfrak{a} \in \mathbb{Q}\cup\{\infty\}$, an integer $0 \le j < m_{\mathfrak{a}}$ and $k_0 \in K_0(N)$, which are uniquely determined by $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and y such that

$$i_{\text{finite}}\left(\gamma_{\mathfrak{a}}\begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}\right) i_p\left(\begin{pmatrix} |d^{-2}(ad-bc)y|_p d^{-2}(ad-bc)y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ cd^{-1} & 1 \end{pmatrix}\right) = k_0 \in K_0(N).$$

It follows that

$$\begin{aligned} W_{f,p}&\left(\begin{pmatrix} d^{-2}(ad-bc)y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ cd^{-1} & 1 \end{pmatrix}\right)\\ &= \begin{cases} a\Big(\mathfrak{a}, m_{\mathfrak{a}}|d^{-2}(ad-bc)y|_p^{-1} - \mu_{\mathfrak{a}}\Big)e_\infty\Big(|d^{-2}(ad-bc)y|_p^{-1}j\Big)\widetilde{\chi}_{\text{idelic}}(k_0), \\ \qquad \text{if } m_{\mathfrak{a}}|d^{-2}(ad-bc)y|_p^{-1} - \mu_{\mathfrak{a}} \in \mathbb{Z}, \\ 0, \qquad \text{otherwise.} \end{cases}\end{aligned}$$

Therefore, if $c^{-1}d \notin \mathbb{Z}_p$ then

$$\begin{aligned}\pi_p'&\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right).\varphi_p(y)\\ &= \begin{cases} \chi_p(c)e_p\left(ac^{-1}y\right) a\Big(\mathfrak{a}, m_{\mathfrak{a}}|d^{-2}(ad-bc)y|_p^{-1} - \mu_{\mathfrak{a}}\Big)e_\infty\Big(|d^{-2}(ad-bc)y|_p^{-1}j\Big)\widetilde{\chi}_{\text{idelic}}(k_0), \\ \qquad \text{if } m_{\mathfrak{a}}|d^{-2}(ad-bc)y|_p^{-1} - \mu_{\mathfrak{a}} \in \mathbb{Z}, \\ 0, \qquad \text{otherwise.} \end{cases}\end{aligned}$$

For fixed $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2,\mathbb{Q}_p)$ with $c \neq 0$, the function $\pi_p'\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right).\varphi_p \in S(\mathbb{Q}_p^\times)$ if and only if there exists an integer $M \ge 0$ such that $\pi_p'\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right).\varphi_p(y) = 0$ for $y \in p^M\mathbb{Z}_p$. If $\mu_{\mathfrak{a}} \neq 0$ then

$a\left(\mathfrak{a}, m_{\mathfrak{a}}|y|_p^{-1} - \mu_{\mathfrak{a}}\right) \in S(\mathbb{Q}_p^\times)$ already. Therefore, for each cusp $\mathfrak{a} \in \mathbb{Q} \cup \{\infty\}$ with $\mu_{\mathfrak{a}} = 0$, if there exists a non-negative integer $M_{\mathfrak{a}}$ such that $a(\mathfrak{a}, m_{\mathfrak{a}} p^m) = 0$ for any integer $m \geq M_{\mathfrak{a}}$, then by the above computations, $\pi_p'\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right).\varphi_p \in S(\mathbb{Q}_p^\times)$ for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2, \mathbb{Q}_p)$.

Now assume that there exists a cusp $\mathfrak{a} \in \mathbb{Q} \cup \{\infty\}$ with $\mu_{\mathfrak{a}} = 0$ such that for any non-negative integer M, there exists an integer $m \geq M$ and $a(\mathfrak{a}, m_{\mathfrak{a}} p^m) \neq 0$. By Theorem 3.8,

$$a(\mathfrak{a}, m_{\mathfrak{a}} p^m) = \prod_{q|N} W_{f,q}\left(\begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix} \gamma_{\mathfrak{a}}^{-1}\right) \neq 0.$$

So for any non-negative integer M there exists an integer $m \geq M$ such that $W_{f,p}\left(\left(\begin{smallmatrix} p^m & 0 \\ 0 & 1 \end{smallmatrix}\right)\gamma_{\mathfrak{a}}^{-1}\right) = \left(\pi_p'\left(\gamma_{\mathfrak{a}}^{-1}\right) . \varphi_p\right)(p^m) \neq 0$. Therefore

$$\pi_p'\left(\gamma_{\mathfrak{a}}^{-1}\right) . \varphi_p(y) \notin S(\mathbb{Q}_p^\times).$$

Then (π_p, V_p) is not supercuspidal. □

Corollary 3.10 *Let f be a normalized Hecke newform of weight k, type ν, level N, and character χ (mod N) where χ is a primitive Dirichlet character. Let f_{adelic} be the adelic lift of f as in (3.1).*

For a prime p, let (π_p, V_p) be an irreducible and admissible representation of $GL(2, \mathbb{Q}_p)$. If (π_p, V_p) is isomorphic to the local component of the irreducible global automorphic representation of $GL(2, \mathbb{A})$ (which is generated by f_{adelic}), then (π_p, V_p) cannot be supercuspidal.

Proof: If $p \mid N$ and χ is primitive, we know that $a(\infty, p) \neq 0$ by [10]. Since f is a newform, for any positive integer m, $a(\infty, p^m) = a(\infty, p)^m \neq 0$. By Theorem 3.9, (π_p, V_p) cannot be supercuspidal. □

Remark: In the proof of theorem 3.9 we say that if (π_p, V_p) is supercuspidal then $p|N$. In fact, more it true: it follows directly from [5] that if (π_p, V_p) is supercuspidal then $p^2|N$. For the convenience of the reader we briefly show how to deduce this fact from [5]. We shall assume the reader is familiar with the notation of [5] for the remainder of this paragraph. Suppose that the field k considered in [5] is $\mathbb{Q}_p$ and the representation ϱ considered in theorem 1 of [5] comes from a Maass form of level N, such that $p^\alpha | N$ and $p^{\alpha+1} \nmid N$. Then Casselman's conductor $c(\varrho)$ is the ideal $p^\alpha \cdot \mathbb{Z}_p$. It is also shown in [5] (page 304, line 16) that in the supercuspidal case $c(\varrho) = p^{-n_1}$ where n_1 is a certain integral invariant of the representation ϱ which was shown by Jacquet-Langlands to be at most -2. (See [5], p. 303, paragraphs 1 and 2.) Thus $\alpha = -n_1 \geq 2$.

§4. Some cusp representatives

In order to give the proof of theorem 1.7, we wish to describe a convenient set of representatives for the equivalence classes of cusps in the case when $N = q^e$ is a prime power.

Lemma 4.1 *If q is a prime, and e a positive integer, then the set*

$$\{0, \infty\} \cup \left\{ \frac{1}{c_1 q^l} \;\middle|\; \begin{matrix} 1 \leq l < e, \ \gcd(c_1, q) = 1, \\ 1 \leq c_1 < \min(q^l, q^{e-l}), \end{matrix} \right\}$$

is a set of representatives for the $\Gamma_0(q^e)$-equivalence classes of cusps.

Proof: It is well known and easily verified that the group $SL(2,\mathbb{Z})$ permutes the set of cusps transitively. It follows that $\Gamma_0(q^e)$-equivalence classes of cusps are naturally identified with double cosets $\Gamma_0(q^e)\backslash SL(2,\mathbb{Z})/\widetilde{\Gamma}_{\mathfrak{a}}$, where $\tilde{\Gamma}_{\mathfrak{a}}$ denotes the stabilizer in $SL(2,\mathbb{Z})$ of any fixed cusp $\mathfrak{a}$. It is convenient to employ this identification with $\mathfrak{a}=\infty$. As remarked above, the stabilizer $\widetilde{\Gamma}_\infty = \Gamma_\infty$ is independent of N and given explicitly by

$$\Gamma_\infty = \left\{ \begin{pmatrix} \epsilon & n \\ 0 & \epsilon \end{pmatrix} \middle| \, \epsilon \in \{\pm 1\},\ n \in \mathbb{Z} \right\}.$$

Now, it follows easily from the definition of $\Gamma_0(q^e)$ that $\Gamma_0(q^e)\backslash SL(2,\mathbb{Z})$ is naturally identified with $B^1(\mathbb{Z}/q^e\mathbb{Z})\backslash SL(2,\mathbb{Z}/q^e\mathbb{Z})$, where $B^1(R)$ denotes the group of upper triangular matrices with entries in the ring R and determinant equal to 1. The projective line $\mathbb{P}^1(\mathbb{Z}/q^e\mathbb{Z})$ is given by

$$\left\{(x_0,x_1) \in (\mathbb{Z}/q^e\mathbb{Z})^2 \mid \langle x_0, x_1\rangle = \mathbb{Z}/q^e\mathbb{Z}\right\} / \sim .$$

Here, $\langle x_0, x_1\rangle$ denotes the ideal generated by x_0 and x_1, and $\sim$ denotes the equivalence relation given by

$$(x_0,x_1) \sim (x_0',x_1') \iff (x_0',x_1') = (\lambda x_0, \lambda x_1), \text{ some } \lambda \in (\mathbb{Z}/q^e\mathbb{Z})^\times.$$

We write $[x_0 : x_1]$ for the equivalence class of (x_0,x_1). The group $SL(2,\mathbb{Z}/q^e\mathbb{Z})$, has a natural right action on $\mathbb{P}^1(\mathbb{Z}/q^e\mathbb{Z})$ given by

$$[x_0 : x_1]\cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = [ax_0 + cx_1 : bx_0 + dx_1], \qquad [x_0 : x_1] \in \mathbb{P}^1(\mathbb{Z}/q^e\mathbb{Z}), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}/q^e\mathbb{Z}).$$

Clearly, the stabilizer of $[0:1]$ is $B^1(\mathbb{Z}/q^e\mathbb{Z})$. Thus $\mathbb{P}^1(\mathbb{Z}/q^e\mathbb{Z})$ may be identified with the coset space $\Gamma_0(q^e)\backslash SL(2,\mathbb{Z})$. It follows that $\Gamma_0(q^e)\backslash SL(2,\mathbb{Z})/\Gamma_\infty$ is in one-to-one correspondence with orbits for the action of Γ_∞ on $\mathbb{P}^1(\mathbb{Z}/q^e\mathbb{Z})$ via inclusion into $SL(2,\mathbb{Z})$ and then projection to $SL(2,\mathbb{Z}/q^e\mathbb{Z})$. Note that the coset in $\Gamma_0(q^e)\backslash SL(2,\mathbb{Z})$ which corresponds to the element $[x_0 : x_1] \in \mathbb{P}^1(\mathbb{Z}/q^e\mathbb{Z})$ consists of all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $(c,d) \equiv (\lambda x_0, \lambda x_1) \pmod{q^e}$ for some $\lambda \in (\mathbb{Z}/q^e\mathbb{Z})^\times$.

It is clear that

$$\mathbb{P}^1(\mathbb{Z}/q^e\mathbb{Z}) \ = \ \left\{[1:x_1] \,\middle|\, x_1 \in \mathbb{Z}/q^e\mathbb{Z}\right\} \ \cup \ \left\{[x_0 : 1] \,\middle|\, x_0 \in \mathbb{Z}/q^e\mathbb{Z} - (\mathbb{Z}/q^e\mathbb{Z})^\times\right\},$$

and that the action of Γ_∞ permutes the elements of $\{[1 : x_1] \mid x_1 \in \mathbb{Z}/q^e\mathbb{Z}\}$ transitively. It follows that the $\Gamma_0(q^e)$-cosets corresponding to these elements comprise a single double coset in $\Gamma_0(q^e)\backslash SL(2,\mathbb{Z})/\Gamma_\infty$ which is represented by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. This matrix maps ∞ the the cusp 0.

We study the action of Γ_∞ on $\left\{[x_0 : 1] \mid x_0 \in \mathbb{Z}/q^e\mathbb{Z} - (\mathbb{Z}/q^e\mathbb{Z})^\times\right\}$. Writing $x_0 = q^l c_1$ with $1 \le l \le e, 1 \le c_1 < q^{e-l}$ and $\gcd(c_1, q) = 1$, we compute

$$\left[q^l c_1 : 1\right] \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \left[q^l c_1 : q^l c_1 n + 1\right] = \left[q^l c_1 \overline{(q^l c_1 n + 1)} : 1\right],$$

where $\overline{a}$ denotes a^{-1} modulo q^{e-l}. From this we see at once that each part of the partition

$$\bigcup_{l=1}^{e} \left\{ \ [q^l c_1 : 1] \ \middle| \ (c_1, q) = 1,\ 1 \le c_1 < q^{e-l} \right\}$$

is preserved by the action of Γ_∞, and that Γ_∞ acts trivially on

$$\left\{[q^l c_1 : 1] \;\middle|\; (c_1, q) = 1,\ 1 \le c_1 < q^{e-l}\right\}$$

whenever $e - l \le l$, for in this case $\overline{(q^l c_1 n + 1)} \equiv 1 \pmod{q^{e-l}}$, whence $[q^l c_1 \overline{(q^l c_1 n + 1)} : 1] = [q^l c_1 : 1]$. When $l = e$, $\{[q^l c_1 : 1] \mid (c_1, q) = 1,\ 1 \le c_1 < q^{e-l}\} = [0 : 1]$, which corresponds to the element of $\Gamma_0(q^e)\backslash SL(2,\mathbb{Z})$ represented by the identity matrix. The corresponding cusp is ∞. For other values of $l \ge \frac{e}{2}$, we have shown that for each c_1 such that $1 \le c_1 < q^{e-l}$ and $\gcd(c_1, q) = 1$, the coset in $\Gamma_0(q^e)\backslash SL(2,\mathbb{Z})$ corresponding to $[q^l c_1 : 1]$, is in fact a double coset $\Gamma_0(q^e)\backslash SL(2,\mathbb{Z})/\Gamma_\infty$. This coset is represented by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ q^l c_1 & 1 \end{smallmatrix}\right)$ which maps ∞ to $\frac{1}{q^l c_1}$.

When $e - l > l$, the action of $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ on $\{[q^l c_1 : 1] \mid (c_1, q) = 1,\ 1 \le c_1 < q^{e-l}\}$ factors through the function $n \mapsto \overline{(q^l c_1 n + 1)}$. This maps $\mathbb{Z}$ into the group U_l of units in $\mathbb{Z}/q^e\mathbb{Z}$ which are equivalent to 1 modulo q^l. It is easy to see that this function is surjective. More precisely $n \mapsto c_1 n$ is a bijection $\mathbb{Z}/q^{e-l}\mathbb{Z} \to \mathbb{Z}/q^{e-l}\mathbb{Z}$, while $m \mapsto 1 + q^l m$ is a bijection $\mathbb{Z}/q^{e-l}\mathbb{Z} \to U_l$, and $\overline{}$ is a bijection $U_l \to U_l$.

Thus we are reduced to studying the action of U_l on $(\mathbb{Z}/q^{e-l}\mathbb{Z})^\times$. Clearly $c_1 u \equiv c_1 \pmod{q^l}$ for all $c_1 \in (\mathbb{Z}/q^{e-l}\mathbb{Z})^\times$, and $u \in U_l$. Equally clearly, if $c_1, c_1' \in (\mathbb{Z}/q^{e-l}\mathbb{Z})^\times$, and $c_1 \equiv c_1' \pmod{q^l}$ then $c_1'\overline{c_1} \in U_l$. It follows that the orbits for the action of U_l on $(\mathbb{Z}/q^{e-l}\mathbb{Z})^\times$, are precisely the residue classes modulo q^l. This completes the proof. □

§5. Proof of the main theorem

For a prime q and positive integer e, fix $N = q^e$. From lemma 4.1, we can take the complete set of inequivalent cusps for $\Gamma_0(q^e)$ as

$$S \;=\; \{0, \infty\} \cup \left\{ \frac{1}{c_1 q^l} \;\middle|\; \begin{matrix} 1 \le l < e,\ \gcd(c_1, q) = 1, \\ 1 \le c_1 < \min(q^l, q^{e-l}), \end{matrix} \right\}. \tag{5.1}$$

For each cusp $\mathfrak{a} \in S$, we have the following.

$$\begin{cases} \mathfrak{a} = 0: & \gamma_\mathfrak{a} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sigma_\mathfrak{a} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\begin{pmatrix} q^{\frac{e}{2}} & 0 \\ 0 & q^{-\frac{e}{2}} \end{pmatrix},\ g_\mathfrak{a} = \left(\begin{smallmatrix} 1 & 0 \\ q^e & 1 \end{smallmatrix}\right),\ m_\mathfrak{a} = q^e, \\[2ex] \mathfrak{a} = \frac{1}{cq^l},\ l \le \frac{e}{2}: & \gamma_\mathfrak{a} = \begin{pmatrix} 1 & 0 \\ q^l c & 1 \end{pmatrix},\ \sigma_\mathfrak{a} = \begin{pmatrix} 1 & 0 \\ q^l c & 1 \end{pmatrix}\begin{pmatrix} q^{\frac{e}{2}-l} & 0 \\ 0 & q^{l-\frac{e}{2}} \end{pmatrix}, \\ & g_\mathfrak{a} = \begin{pmatrix} 1-q^{e-l}c & q^{e-2l} \\ -q^e c^2 & 1+q^{e-l}c \end{pmatrix},\ m_\mathfrak{a} = q^{e-2l}, \\[2ex] \mathfrak{a} = \frac{1}{cq^l},\ l > \frac{e}{2}: & \gamma_\mathfrak{a} = \begin{pmatrix} 1 & 0 \\ q^l c & 1 \end{pmatrix},\ \sigma_\mathfrak{a} = \begin{pmatrix} 1 & 0 \\ q^l c & 1 \end{pmatrix}, \\ & g_\mathfrak{a} = \begin{pmatrix} 1-q^l c & 1, \\ -q^{2l}c^2 & 1+q^l c \end{pmatrix},\ m_\mathfrak{a} = 1. \end{cases}$$

From the above table, we can easily see that $\mu_0 = \mu_\infty = 0$ since $\widetilde{\chi}(g_0) = \widetilde{\chi}(g_\infty) = 1$. If $\mathfrak{a} \ne 0, \infty$, then $\mathfrak{a} = \frac{1}{cq^l}$, with $1 \le c < \min(q^l, q^{e-l})$ and $(c, q) = 1$ and

$$\widetilde{\chi}(g_\mathfrak{a}) = \chi\big(1 + c \cdot \max(q^l, q^{e-l})\big) = \chi\big(1 + \max(q^l, q^{e-l})\big)^c = e^{2\pi i \mu_\mathfrak{a}}.$$

So $\widetilde{\chi}(g_{\mathfrak{a}})^{\min(q^l,q^{e-l})} = 1$.

Lemma 5.2 *Fix an integer $e \geq 1$. Let χ be a Dirichlet character of prime power level $N = q^e$. Let χ_{trivial} be the trivial character modulo q^e. Let χ_0 be a primitive Dirichlet character of prime power level $N_0 = q^{e_0}$ (with $0 \leq e_0 \leq e$) such that $\chi = \chi_0 \cdot \chi_{\text{trivial}}$. Then the following hold.*

- *For each integer $1 \leq l < e$ with $\max(q^l, q^{e-l}) \geq q^{e_0}$, and any cusp $\mathfrak{a} = \frac{1}{cq^l}$ with $1 \leq c < \min(q^l, q^{e-l})$ and $(c, q) = 1$, the cusp parameter $\mu_{\mathfrak{a}}$ is zero.*
- *For each integer $1 \leq l < e$ with $\max(q^l, q^{e-l}) < q^{e_0}$, there exists a cusp $\mathfrak{a}_0 = \frac{1}{c_0 q^l}$ with cusp parameter $\mu_{\mathfrak{a}_0} = \min\left(q^{e_0-l},\, q^{e_0-e+l}\right)^{-1}$. Here $1 \leq c_0 < \min(q^{e_0-l},\, q^{e_0-e+l})$ and $(c_0, q) = 1$. Then*
$$\widetilde{\chi}(g_{\mathfrak{a}_0}) = e^{2\pi i \min\left(q^{e_0-l},\, q^{e_0-e+l}\right)^{-1}},$$
and for any cusp $\mathfrak{a} = \frac{1}{cq^l}$ with $1 \leq c < \min(q^l, q^{e-l})$ and $(c, q) = 1$, there exists a unique integer $1 \leq r < \min\left(q^{e_0-l},\, q^{e_0-e+l}\right)$ with $(r, q) = 1$ such that
$$c \;\equiv\; rc_0 \left(\bmod\ \min\left(q^{e_0-l},\, q^{e_0-e+l}\right)\right) \quad \textit{and} \quad \mu_{\mathfrak{a}} = r\mu_{\mathfrak{a}_0}.$$

Proof For any integer m with $1 \leq m < e$, let $U_m = \{a \in (\mathbb{Z}/q^e\mathbb{Z})^\times \mid a \equiv 1 \pmod{q)^m}\}$. This is a subgroup. In fact, it is the kernel of the natural projection from $(\mathbb{Z}/q^e\mathbb{Z})^\times$ to $(\mathbb{Z}/q^m\mathbb{Z})^\times$. The integer e_0 is the smallest integer such that χ factors through this projection. Thus the restriction $\chi|_{U_m}$ of χ to U_m is trivial iff $m \geq e_0$.

Use the set of representatives for cusps S in (5.1). As we see from the table above, the lower right entry $d_{\mathfrak{a}}$ of the generator $g_{\mathfrak{a}}$ is an element of $U_{\max l, e-l}$. Since $\mu_{\mathfrak{a}}$ is defined so that $e^{2\pi i \mu_{\mathfrak{a}}} = \chi(d_{\mathfrak{a}})$, we need to study the restriction $\chi|_{U_{\max(l,e-l)}}$.

If $e_0 \leq \max(l, e-l)$, this restriction is trivial and $\mu_{\mathfrak{a}}$ is zero, regardless of c. The function $c \mapsto 1 + cq^{\max(l,e-l)}$ is an isomorphism $\mathbb{Z}/q^{\min(l,e-l)}\mathbb{Z} \to U_{\max(l,e-l)}$. Composing with χ, we obtain a homomorphism φ from $\mathbb{Z}/q^{\min(l,e-l)}\mathbb{Z}$ to $\mathbb{C}$. For any m with $\max(l, e-l) \leq m \leq e$, the preimage of U_m in $\mathbb{Z}/q^{\min(l,e-l)}\mathbb{Z}$ is the cyclic subgroup generated by $q^{m-\max(l,e-l)}$, and these are the only subgroups of $\mathbb{Z}/q^{\min(l,e-l)}\mathbb{Z}$. Since the kernel of χ contains U_{e_0}, but not U_{e_0-1}, it follows that the kernel of $\chi|_{U_{\max l, e-l}}$ is precisely equal to U_{e_0}, and that its image is the $(q^{e_0-\max(l,e-l)})^{\text{th}}$ roots of unity. Furthermore, φ factors through the natural projection $\mathbb{Z}/q^{\min(l,e-l)}\mathbb{Z} \to \mathbb{Z}/q^{e_0-\max(l,e-l)}$. For c_0, we take the least positive element of the residue class which maps to $e^{2\pi i \min\left(q^{e_0-l},\, q^{e_0-e+l}\right)^{-1}}$. $\square$

Proof of Theorem 1.7: Fix $\mathfrak{a} \in S$. Let M be a positive integer and $\epsilon = \pm 1$. Write
$$\epsilon M + \mu_{\mathfrak{a}} = \epsilon m_{\mathfrak{a}} p_1^{m_1} \cdots p_n^{m_n} q^m,$$
for distinct primes $p_1, \cdots, p_n \neq q$, and integers $m_1, \cdots, m_n, m$ with $m_1, \cdots, m_n > 0$. It follows from lemma 5.2 and the discussion preceding it that $\mu_{\mathfrak{a}} = 0$ except when $\mathfrak{a} = \frac{1}{cq^l}$ with $\max(l, e-l) < e_0$. Also, in all cases, $m_{\mathfrak{a}}$ is a power of q. Thus, when $\mu_{\mathfrak{a}} = 0$, the expression
$$M = p_1^{m_1} \cdots p_n^{m_n} \cdot (m_{\mathfrak{a}} q^m)$$

is the prime factorization of M. The expressions for m in terms of α in the various cases now follow easily from the values of $m_{\mathfrak{a}}$ tabulated above.

When $\mathfrak{a} = \frac{1}{cq^l}$ with $\max(l, e-l) < e_0$, it follows from lemma 5.2 that $\mu_{\mathfrak{a}}$ is a rational number of the form $\frac{r}{\min\left(q^{e_0-l},\, q^{e_0-e+l}\right)} = \frac{r}{q^{e_0-\max(l,e-l)}}$ with $gcd(r,q)=1$. Consequently $\epsilon M + \mu_{\mathfrak{a}}$ is a rational number with the same denominator, and a numerator which is congruent to r $\left(\bmod\ \min\left(q^{e_0-l},\, q^{e_0-e+l}\right)\right)$. Since in this case $m_{\mathfrak{a}} = q^{\max(l,e-l)-l}$, we obtain $m = -e_0 + l$.

Let

$$M_0 := \frac{\epsilon M + \mu_{\mathfrak{a}}}{\epsilon m_{\mathfrak{a}} q^m} = p_1^{m_1} \cdots p_n^{m_n}.$$

Then by theorem 3.8, there exists a unique cusp $\mathfrak{b} \in S$ and a unique integer $1 \le j < m_{\mathfrak{b}}$ such that

$$\gamma_{\mathfrak{b}} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} =: \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}),$$

$$\gamma_{\mathfrak{b}} \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon M_0 & 0 \\ 0 & 1 \end{pmatrix} \gamma_{\mathfrak{a}}^{-1} =: \begin{pmatrix} a_q & b_q \\ c_q & d_q \end{pmatrix},$$

where $c_q \equiv 0 \pmod{q^e}$. Then

$$a(\mathfrak{a}, \epsilon M) = a(\infty, \epsilon) \prod_{i=1}^{n} a(\infty, p_i^{m_i}) \left(a(\mathfrak{b}, m_{\mathfrak{b}} q^m - \mu_{\mathfrak{b}}) e_\infty(q^m \cdot j) \chi(d_q)^{-1} \right).$$

(we shall show that $m_{\mathfrak{b}} q^m - \mu_{\mathfrak{b}}$ is always integral).

(1) If $\mathfrak{a} = \infty$, then $\mu_{\mathfrak{a}} = 0$ and $m_{\mathfrak{a}} = 1$. Since $\gamma_\infty \left(\begin{smallmatrix} \epsilon M_0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \gamma_\infty^{-1} = \left(\begin{smallmatrix} \epsilon M_0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in I_{q,N}$. So $\mathfrak{b} = \infty$. Since $m = \alpha > 0$ and $\mu_{\mathfrak{b}} = 0$, it follows at once that $m_{\mathfrak{b}} q^m - \mu_{\mathfrak{b}} \in \mathbb{Z}$. Furthermore,

$$a(\infty, \epsilon M) = a(\infty, \epsilon) \prod_{i=1}^{n} a(\infty, p_i^{m_i}) \cdot a(\infty, q^m).$$

(2) If $\mathfrak{a} = 0$, then $\mu_{\mathfrak{a}} = 0$ and $m_{\mathfrak{a}} = q^e$. Then $\gamma_0 \left(\begin{smallmatrix} \epsilon M_0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \gamma_0^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \epsilon M_0 \end{smallmatrix}\right) \in I_{q,N}$. Therefore, $\mathfrak{b} = 0$ and $j = 0$. Once again $\mu_{\mathfrak{b}} = 0$. Furthermore, $m_{\mathfrak{b}} q^m = q^\alpha \in \mathbb{Z}$. Finally,

$$a(0, \epsilon M) = a(\infty, \epsilon) \prod_{i=1}^{n} a(\infty, p_i^{m_i}) \cdot a(0, q^{e+m}) \chi(\epsilon M_0)^{-1}$$

If $\mathfrak{a} \ne 0, \infty$ then $\mathfrak{a} = \frac{1}{cq^l}$ for some fixed integers $1 \le l < e$ and $1 \le c < \min(q^l, q^{e-l})$ with $(c,q) = 1$. Also, $m_{\mathfrak{a}} = \max(q^{e-2l}, 1)$. Let us explicitly determine $\mathfrak{b}$ in this case. First, assume $l \ge \frac{e}{2}$. Consider the computation

$$\begin{aligned} \gamma_{\mathfrak{b}} \begin{pmatrix} \epsilon M_0 & 0 \\ 0 & 1 \end{pmatrix} \gamma_{\mathfrak{a}}^{-1} &= \begin{pmatrix} 1 & 0 \\ c' q^l & 1 \end{pmatrix} \begin{pmatrix} \epsilon M_0 & \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -cq^l & 1 \end{pmatrix} \\ &= \begin{pmatrix} \epsilon M_0 & 0 \\ q^l (c' \epsilon M_0 - c) & 1 \end{pmatrix}. \end{aligned} \tag{5.3}$$

It is clear that the matrix on the right-hand side is an element of $I_{q,N}$ if and only if $c \equiv c'\epsilon M_0 \pmod{q^{e-l}}$. Thus $\mathfrak{b} = \frac{1}{c'q^l}$ for this particular value of c'. Referring to the table above, we see that $m_\mathfrak{b} = m_\mathfrak{a} = 1$, and $d_\mathfrak{a} \equiv d_\mathfrak{b}^{\epsilon M_0} \pmod{q^e}$, whence $\epsilon M_0 \mu_\mathfrak{b} - \mu_\mathfrak{a} \in \mathbb{Z}$.

Now assume $l < \frac{e}{2}$. Consider the computation

$$\gamma_\mathfrak{b}\begin{pmatrix}1 & j\\ 0 & 1\end{pmatrix}\begin{pmatrix}\epsilon M_0 & 0\\ 0 & 1\end{pmatrix}\gamma_\mathfrak{a}^{-1} = \begin{pmatrix}1 & 0\\ c'q^l & 1\end{pmatrix}\begin{pmatrix}1 & j\\ 0 & 1\end{pmatrix}\begin{pmatrix}\epsilon M_0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -cq^l & 1\end{pmatrix}$$

$$= \begin{pmatrix}\epsilon M_0 - jcq^l & j\\ q^l(c'\epsilon M_0 - jc'cq^l - c) & jc'q^l+1\end{pmatrix}. \tag{5.4}$$

It is clear that the matrix on the right-hand side is an element of $I_{q,N}$ if and only if c' and j are such that $c \equiv c'\epsilon M_0 \pmod{q^{e-l}}$ and $jc'c \equiv \left(\frac{c'\epsilon M_0 - c}{q^l}\right) \pmod{q}^{e-2l}$. This shows that $\mathfrak{b} = \frac{1}{c'q^l}$, where c' is the unique solution to $c \equiv c'\epsilon M_0 \pmod{q^{e-l}}$ in the range $1 \le c' < q^l$. It follows at once that $m_\mathfrak{b} = m_\mathfrak{a} = q^{e-2l}$, and that $\epsilon M_0 \mu_\mathfrak{b} - \mu_\mathfrak{a} \in \mathbb{Z}$.

(3) If $\mu_\mathfrak{a} \neq 0$, then by Lemma 5.2, $\max(q^l, q^{e-l}) < q^{e_0}$ and

$$\mu_\mathfrak{a} = \frac{r}{\min(q^{e_0-l}, q^{e_0-e+l})},$$

for some integer r with $1 \le r < \min(q^{e_0-l}, q^{e_0-e+l})$. Similiarly

$$\mu_\mathfrak{b} = \frac{r'}{\min(q^{e_0-l}, q^{e_0-e+l})},$$

for some integer r' with $1 \le r' < \min(q^{e_0-l}, q^{e_0-e+l})$. Since

$$\epsilon M + \frac{r}{\min(q^{e_0-l}, q^{e_0-e+l})} = \epsilon m_\mathfrak{a} p_1^{m_1} \cdots p_n^{m_n} q^m = \epsilon M_0 \max(q^{e-2l}, 1)\cdot q^m$$

it follows that $\epsilon M \min(q^{e_0-l}, q^{e_0-e+l}) + r = \epsilon M_0 q^{m+e_0-l}$. This implies that $m = -e_0 + l$ and $\epsilon M_0 \equiv r \pmod{\min(q^{e_0-l}, q^{e_0-e+l})}$. Since $\epsilon M_0 \mu_\mathfrak{b} - \mu_\mathfrak{a} \in \mathbb{Z}$, we deduce that $\epsilon M_0 r' \equiv r \pmod{\min(q^{e_0-l}, q^{e_0-e+l})}$, and hence that $r' = 1$. It follows that $\mu_\mathfrak{b} = \min(q^{e_0-l}, q^{e_0-e+l})^{-1} = m_\mathfrak{b} q^m$.

• $l \le e-l$ and $\mu_\mathfrak{a} = r\mu_{\mathfrak{a}_0} \neq 0$: In this case it follows from (5.4) and the definitions of c' and j that

$$a\left(\frac{1}{cq^l}, \epsilon M\right) = a(\infty, \epsilon)\prod_{i=1}^n a(\infty, p_i^{m_i})\left(a\left(\frac{1}{c'q^l}, 0\right) e_\infty\left(q^{-e_0+l}\cdot j\right)\chi\left(jc'q^l+1\right)^{-1}\right),$$

where c' and j are determined by c and ϵM_0 as above.

• $l > e-l$ and $\mu_\mathfrak{a} = r\mu_{\mathfrak{a}_0} \neq 0$: In this case it follows from (5.3) and the definition of c' that

$$a\left(\frac{1}{cq^l}, \epsilon M\right) = a(\infty, \epsilon)\prod_{i=1}^n a\left(\infty, p_i^{m_i}\right) a\left(\frac{1}{c'q^l}, 0\right),$$

where c' is determined by c and ϵM_0 as above.

(4) For a fixed integer $1 \le l < e$, take an integer $1 \le c < \min(q^l, q^{e-l})$ and $(c, q) = 1$. Let $\mathfrak{a} = \frac{1}{cq^l}$. As shown above, $\mathfrak{b} = \frac{1}{c'q^l}$, where c' is the unique solution to

$$1 \le c' < \min(q^l, q^{e-l}) \qquad c'\epsilon M_0 - c \equiv 0 \pmod{\min(q^l, q^{e-l})}.$$

Assume that $\mu_{\mathfrak{a}} = 0$. Then $\mu_{\mathfrak{b}} = 0$ by Lemma 5.2. Furthermore $m_{\mathfrak{b}} = m_{\mathfrak{a}} = q^{\max(e-2l,0)}$. It follows that $q^m m_{\mathfrak{b}} - \mu_{\mathfrak{b}} = q^\alpha$ (where α is the highest pwer of q that divides M as before), which is integral.

- $l \le e - l$ and $\mu_{\mathfrak{a}} = 0$: In this case it follows from (5.4) and the definitions of c' and j that

$$a(\frac{1}{cq^l}, \epsilon M) = a(\infty, \epsilon) \prod_{i=1}^{n} a(\infty, p_i^{m_i}) \left(a(\frac{1}{c'q^l}, q^{e-2l+m}) e_\infty(q^m \cdot j) \chi(jc'q^l + 1)^{-1} \right).$$

- $l > e - l$ and $\mu_{\mathfrak{a}} = 0$: In this case it follows from (5.3) and the definition of c' that

$$a\left(\frac{1}{cq^l},\ \epsilon M\right) = a(\infty, \epsilon) \prod_{i=1}^{n} a(\infty, p_i^{m_i}) \cdot a\left(\frac{1}{c'q^l},\ q^m\right).$$

□

§6. Remarks on choices of $\mathfrak{a}$ and $\gamma_{\mathfrak{a}}$

As remarked in §1, the Fourier coefficients $a(\mathfrak{a}, n)$ of a Maass form f at a cusp $\mathfrak{a}$ actually depend on the matrix $\sigma_{\mathfrak{a}}$, or, equivalently, the matrix $\gamma_{\mathfrak{a}}$ used in its definition, and not only on the choice of $\mathfrak{a}$. Further, while it is intuitively obvious that when considering Fourier expansions at various cusps, it is sufficient to consider a maximal set of $\Gamma_0(N)$-inequivalent cusps, it is also clear that the choice of representative for each $\Gamma_0(N)$-equivalence class will influence the precise numbers considered. In this section we make these dependencies completely explicit and then offer some remarks on choice of representatives for N not a prime power.

Because we wish to study the dependence of the Fourier coefficients on the choice of matrix $\gamma_{\mathfrak{a}}$ used to define them, it is necessary to make this dependence explicit. Thus, we write $a(\gamma_{\mathfrak{a}}, n)$ rather than $a(\mathfrak{a}, n)$.

Lemma 6.1 *Suppose that $\mathfrak{a}$ and $\mathfrak{a}'$ are two $\Gamma_0(N)$-equivalent cusps, and that $\gamma_{\mathfrak{a}}, \gamma_{\mathfrak{a}'}$ are two elements of $SL(2, \mathbb{Z})$ such that $\gamma_{\mathfrak{a}}\infty = \mathfrak{a}$ and $\gamma_{\mathfrak{a}'}\infty = \mathfrak{a}'$. Let $a(\gamma_{\mathfrak{a}}, n)$ (resp. $a(\gamma_{\mathfrak{a}'}, n)$), $n \in \mathbb{Z}$ denote the Fourier coefficients of a Maass form f at $\mathfrak{a}$ (resp. $\mathfrak{a}'$) defined using an element $\sigma_{\mathfrak{a}}$ (resp. $\sigma_{\mathfrak{a}'}$) obtained from $\gamma_{\mathfrak{a}}$ (resp. $\gamma_{\mathfrak{a}'}$) as in §1. Then*

$$a(\gamma_{\mathfrak{a}'}, n) = \widetilde{\chi}(\gamma_0) \cdot e\left((n + \mu_{\mathfrak{a}}) \cdot \frac{j}{\mathfrak{m}_{\mathfrak{a}}}\right) \cdot a(\gamma_{\mathfrak{a}}, n),$$

where $\gamma_0 \in \Gamma_0(N)$ and $j \in \mathbb{Z}$ with $0 \le j < m_{\mathfrak{a}}$ are uniquely determined by the condition that

$$\gamma_{\mathfrak{a}'} = \gamma_0 \cdot \gamma_{\mathfrak{a}} \cdot \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}.$$

Proof: This follows easily from the definitions. □

We would like to extend the idea for choosing explicit representatives for the equivalence classes of cusps described in §4. It is not convenient or necessary to make a completely explicit, choice of cusp representatives. It turns out to be sufficient to specify our representatives only modulo a suitable power of each prime dividing N.

For the remainder of this section and the next, we shall employ the following notation. We take S to be a finite set of primes, denoting a general element of S by q, and a general prime which is not in S by p. For each element q of S we fix a strictly positive integer e_q, and we let $N = \prod_{q \in S} q^{e_q}$.

Lemma 6.2 *For each $q \in S$, let $\pi_q : \mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}/q^{e_q}\mathbb{Z}$ denote the natural projection. The natural map $\mathbb{Z}/N\mathbb{Z} \to \prod_{q\in S} \mathbb{Z}/q^{e_q}\mathbb{Z}$ given by $n \mapsto (\pi_q(n))_{q\in S}$ induces a bijection $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \to \prod_{q\in S} \mathbb{P}^1(\mathbb{Z}/q^{e_q}\mathbb{Z})$. Furthermore, two elements of $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$ are in the same Γ_∞-orbit if and only if their images in $\mathbb{P}^1(\mathbb{Z}/q^{e_q}\mathbb{Z})$ are in the same Γ_∞-orbit for all $q \in S$.*

Proof: The Chinese remainder theorem states that the natural map $\mathbb{Z}/N\mathbb{Z} \to \prod_{q\in S} \mathbb{Z}/q^{e_q}\mathbb{Z}$ is a ring isomorphism. It follows easily that gives a bijection

$$\left\{(x_0, x_1) \in (\mathbb{Z}/N\mathbb{Z})^2 \;\middle|\; \langle x_0, x_1\rangle = \mathbb{Z}/N\mathbb{Z}\right\} \to \prod_{q\in S} \left\{(x_0, x_1) \in (\mathbb{Z}/q^{e_q}\mathbb{Z})^2 \;\middle|\; \langle x_0, x_1\rangle = \mathbb{Z}/q^{e_q}\mathbb{Z}\right\}.$$

Furthermore, if $(x_0', x_1') = (\lambda x_0, \lambda x_1)$, then $(\pi_q(x_0'), \pi_q(x_1')) = (\pi_q(\lambda)\pi_q(x_0), \pi_q(\lambda)\pi_q(x_1))$ for each $q \in S$. Finally, suppose for each $q \in S$ there exists λ_q such that $(\pi_q(x_0'), \pi_q(x_1')) = (\lambda_q \pi_q(x_0), \lambda_q \pi_q(x_1))$. Then it follows that $(x_0', x_1') = (\lambda x_0, \lambda x_1)$, where λ is the unique solution to the system of congruences $\pi_q(\lambda) = \lambda_q \;\forall q \in S$. Consequently, we have a well-defined bijection

$$\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \to \prod_{q\in S} \mathbb{P}^1(\mathbb{Z}/q^{e_q}\mathbb{Z}).$$

In the same manner, we see that $\exists n \in \mathbb{Z}/N\mathbb{Z}$ such that $[x_0 : x_1] = [y_0 : y_1]\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ if and only if, for each q, $\exists n_q \in \mathbb{Z}/q^{e_q}\mathbb{Z}$ such that $[x_0 : x_1] \equiv [y_0 : y_1]\left(\begin{smallmatrix} 1 & n_q \\ 0 & 1 \end{smallmatrix}\right) \pmod{q^{e_q}}$. □

Corollary 6.3 *Suppose that, for each $q \in S$, a set $\mathcal{C}_q$ of representatives for the double cosets $\Gamma_0(q^{e_q})\backslash SL(2,\mathbb{Z})/\Gamma_\infty$ has been chosen. Let $\mathcal{C}$ be a set having the property that, for any element $(\gamma_q)_{q\in S}$ of the Cartesian product $\prod_{q\in S} \mathcal{C}_q$ there is a unique element $\gamma \in \mathcal{C}$ such that*

$$\gamma \;\equiv\; \gamma_q \pmod{q^{e_q}}, \qquad (\forall q \in S).$$

Then $\mathcal{C}$ is a set of representatives for the double cosets $\Gamma_0(N)\backslash SL(2,\mathbb{Z})/\Gamma_\infty$.

Remark: A choice of representatives for $\Gamma_0(N)\backslash SL(2,\mathbb{Z})/\Gamma_\infty$ is slightly more information than a choice of representatives for the $\Gamma_0(N)$-equivalence classes of cusps: it includes also a choice of matrix $\gamma_{\mathfrak{a}}$ for each representative cusp $\mathfrak{a}$.

By corollary 6.3, we may fix a set $\mathcal{C}$ of representatives for the double cosets $\Gamma_0(N)\backslash SL(2,\mathbb{Z})/\Gamma_\infty$ such that, for each $q \in S$ and each $\gamma \in \mathcal{C}$, the matrix γ is equivalent $\pmod{q^{e_q}}$ to one of the representatives for $\Gamma_0(q^{e_q})\backslash SL(2,\mathbb{Z})/\Gamma_\infty$ fixed in §4:

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\} \cup \left\{\begin{pmatrix} 1 & 0 \\ c_1 q^l & 1 \end{pmatrix} \;\middle|\; \begin{matrix} 1 \le l < e,\ \gcd(c_1, q) = 1, \\ 1 \le c_1 < \min(q^l, q^{e-l}) \end{matrix}\right\}. \tag{6.4}$$

Such a choice determines a maximal set of inequivalent cusps for $\Gamma_0(N)$ and a choice of matrix $\gamma_{\mathfrak{a}}$ for each element $\mathfrak{a}$ of this set. Declaring that we choose our representatives $\mathfrak{a}$ and the corresponding matrices $\gamma_{\mathfrak{a}}$ in this manner does not uniquely determine $\gamma_{\mathfrak{a}}$, but it does uniquely determine the coefficients $a(\gamma_{\mathfrak{a}}, n)$, for if $\gamma_{\mathfrak{a}}$ and $\gamma'_{\mathfrak{a}}$ are equivalent $\pmod N$ then they differ by an element of the principal congruence subgroup $\Gamma(N)$ on the left, and $\Gamma(N)$ is contained in the kernel of the character $\widetilde{\chi}$.

Lemma 6.5 *Let $\mathfrak{a}$ be a cusp and $\gamma_{\mathfrak{a}}$ a matrix such that $\gamma_{\mathfrak{a}}\infty = \mathfrak{a}$ and, for each $q \in S$, $\gamma_{\mathfrak{a}}$ is equivalent $\pmod q^{e_q}$ to one of the elements of (6.4). Let $\mathfrak{a}_q$ denote the corresponding cusp. That is, $\gamma_{\mathfrak{a}_q}\infty = \mathfrak{a}_q$ with $\gamma_{\mathfrak{a}_q}$ from (6.4) and $\gamma_{\mathfrak{a}} \equiv \gamma_{\mathfrak{a}_q} \pmod{q^{e_q}}$. Let $\mu_{\mathfrak{a}}$ be the cusp parameter of $\mathfrak{a}$, defined using some character $\chi \pmod N$. The isomorphism $\mathbb{Z}/N\mathbb{Z} \to \prod_{q\in S} \mathbb{Z}/q^{e_q}\mathbb{Z}$ ensures that $\chi = \prod_{q\in S} \chi_q$ for some characters $(\chi_q)_{q\in S}$ with $\chi_q \pmod{q^{e_q}}$ for each q. For each $q \in S$, let $\mu_{\mathfrak{a}_q}$ denote the cusp parameter of $\mathfrak{a}_q$ relative to χ_q. Then*

$$m_{\mathfrak{a}} \;=\; \operatorname*{lcm}_{q\in S}\,(m_{\mathfrak{a}_q}),$$

$$\mu_{\mathfrak{a}} \;=\; \sum_{q\in S} \frac{m_{\mathfrak{a}}}{m_{\mathfrak{a}_q}}\mu_{\mathfrak{a}_q} - \left\lfloor \sum_{q\in S} \frac{m_{\mathfrak{a}}}{m_{\mathfrak{a}_q}}\mu_{\mathfrak{a}_q} \right\rfloor \qquad \textit{(greatest integer function)}.$$

Proof: The lower left entry of $\gamma_{\mathfrak{a}} \cdot \left(\begin{smallmatrix} 1 & j \\ 0 & 1 \end{smallmatrix}\right) \cdot \gamma_{\mathfrak{a}}^{-1}$ is congruent to $0 \pmod N$ if and only if it is congruent to $0 \pmod{q^{e_q}}$ for each q. This is the case if and only if j is divisible by $m_{\mathfrak{a}_q}$ for each q. The first statement follows. We see at once that for each $q \in S$,

$$\gamma_{\mathfrak{a}} \cdot \begin{pmatrix} 1 & m_{\mathfrak{a}} \\ 0 & 1 \end{pmatrix} \cdot \gamma_{\mathfrak{a}}^{-1} \equiv \gamma_{\mathfrak{a}_q} \cdot \begin{pmatrix} 1 & m_{\mathfrak{a}} \\ 0 & 1 \end{pmatrix} \cdot \gamma_{\mathfrak{a}_q}^{-1} = \gamma_{\mathfrak{a}_q} \cdot \begin{pmatrix} 1 & m_{\mathfrak{a}_q} \\ 0 & 1 \end{pmatrix}^{\frac{m_{\mathfrak{a}}}{m_{\mathfrak{a}_q}}} \cdot \gamma_{\mathfrak{a}_q}^{-1} \pmod{q^{e_q}}.$$

It follows that $d_{\mathfrak{a}} \equiv d_{\mathfrak{a}_q}^{\frac{m_{\mathfrak{a}}}{m_{\mathfrak{a}_q}}} \pmod{q^{e_q}}$ for all $q \in S$ and from this the second assertion follows immediately. □

The following lemma will be useful later on.

Lemma 6.6 *Let $\mathfrak{a}$ be a cusp and let (c, d) denote the bottom row of $\gamma_{\mathfrak{a}}$. Let a be an integer prime to N. Let $\mathfrak{a}'$ be the cusp such that $\gamma_{\mathfrak{a}'}$ represents the double coset in $\Gamma_0(N)\backslash SL(2,\mathbb{Z})/\Gamma_\infty$ corresponding to the Γ_∞-orbit of $[ac : d]$ in $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$. Then $m_{\mathfrak{a}'} = m_{\mathfrak{a}}$ and $\mu_{\mathfrak{a}'} - a \cdot \mu_{\mathfrak{a}} \in \mathbb{Z}$.*

Proof: For each q in S, the pair (c, d) is equivalent to $(0, 1), (1, 0)$ or $(c_1 q^l, 1)$ modulo q^{e_q} where c_1, l are subject to the constraints in (6.4). It follows at once that the bottom row of $\gamma_{\mathfrak{a}'}$ is equivalent to $(0, 1), (1, 0)$ of $(c_1' q^l, 1)$, respectively, where $c_1' \equiv ac_1 \pmod{q^{e_q - l}}$ and $0 < c_1' < q^{\min(l, e_q - l)}$. The value of $m_{\mathfrak{a}_q}$ is 1, except when the bottom row of $\gamma_{\mathfrak{a}}$ is equivalent to $(c_1 q^l, 1)$, with $l < e - l$, in which case it is q^{e-2l}. It follows easily that $m_{\mathfrak{a}'_q} = m_{\mathfrak{a}_q}$ for each q and thence that $m_{\mathfrak{a}'} = m_{\mathfrak{a}}$. Similarly, $d_{\mathfrak{a}_q}$ is equal to 1 if $(c, d) \equiv (0, 1)$ or $(1, 0) \pmod{q^{e_q}}$, while if it is congruent to $(c_1 q^l, 1)$, then $d_{\mathfrak{a}_q} = (1 + c_1 q^{\max(l, e-l)})$. Clearly, with c_1' as above, we have

$$(1 + c_1' q^{\max(l,e-l)}) \;\equiv\; (1 + ac_1 q^{\max(l,e-l)}) \;\equiv\; (1 + c_1' q^{\max(l,e-l)})^a \pmod{q^{e_q}}.$$

It follows that $a\mu_{\mathfrak{a}_q} - \mu_{\mathfrak{a}'_q} \in \mathbb{Z}$. The second assertion of this lemma now follows from lemma 6.5, because $m_{\mathfrak{a}}/m_{\mathfrak{a}_q}$ and $m_{\mathfrak{a}'}/m_{\mathfrak{a}'_q}$ are the same integer. $\square$

It is easy to see from the proof of lemma 6.6 that the mapping $(a, \mathfrak{a}) \to \mathfrak{a}'$ which is considered in lemma 6.6 actually defines an action of $(\mathbb{Z}/N\mathbb{Z})^\times$ on our set of cusps. Abusing notation we regard it as an "action" of the set of all elements of $\mathbb{Z}$ which are prime to N. We shall write $a \cdot \mathfrak{a}$ for the cusp $\mathfrak{a}'$ obtained from $\mathfrak{a}$ in this fashion, and $a^{-1} \cdot \mathfrak{a}$ for the unique cusp $\mathfrak{a}''$ such that $a \cdot \mathfrak{a}'' = \mathfrak{a}$.

§7. Toward a classical proof of theorem 3.8

In this section we study the problem of giving a proof of theorem 3.8 in purely classical terms, with a partial result in this direction being given in theorem 7.5 below. It is helpful to first review the classical proof of multiplicativity of Fourier coefficients at infinity. Suppose that

$$\lambda \cdot f(z) = \left(T_p^\chi f\right)(z) := \frac{1}{\sqrt{p}}\left(\chi(p) f(pz) + \sum_{b=0}^{p-1} f\left(\frac{z+i}{p}\right)\right), \tag{7.1}$$

for some $\lambda \in \mathbb{C}$ and all $z \in \mathfrak{h}$. Suppose further that

$$f(z) = \sum_{n \neq 0} a(\infty, n)\, W_{\frac{\mathrm{sgn}(n)k}{2},\, \nu - \frac{1}{2}}\Big(4\pi|n| \cdot y\Big) e^{2\pi i n x}. \tag{7.2}$$

Now plug (7.2) into (7.1), using the identity

$$\sum_{b=0}^{p-1} e^{\frac{2\pi i n b}{p}} = \begin{cases} p & \text{if } p | n, \\ 0 & \text{if } p \not| n. \end{cases}$$

By comparing coefficients of $W_{\frac{\mathrm{sgn}(n)k}{2},\, \nu - \frac{1}{2}}\Big(4\pi|n| \cdot y\Big) e^{2\pi i n x}$ in (7.1), we see that

$$\frac{\chi(p)}{\sqrt{p}} a\left(\infty, \frac{n}{p}\right) - \lambda a(\infty, n) + \sqrt{p} a(\infty, np) \;=\; 0, \tag{7.3}$$

with the understanding that $a\left(\infty, \frac{n}{p}\right) = 0$ if $p \not| n$. It follows that for all n with $\gcd(n,p) = 1$, and all $k \geq 0$, the coefficient $a(\infty, np^k) = b_k \cdot a(\infty, n)$, where $(b_k)_{k=-1}^\infty$ is the unique sequence satisfying the recurrence relation

$$\sqrt{p} b_k - \lambda b_{k-1} + \frac{\chi(p)}{\sqrt{p}} b_{k-2} = 0,$$

and the initial conditions $b_{-1} = 0,\ b_0 = 1$.

It is natural to ask whether one may prove that the Fourier coefficients at cusps other than ∞ satisfy a recurrence relation analogous to (7.3). The answer, in general, appears to be "no." In fact, what we shall prove in theorem 7.4 below is a formula which, in general, involves Fourier coefficients at *three different cusps.*

Theorem 7.4 *Let f be a Maass form of weight k level N and character χ. Assume that a maximal set of $\Gamma_0(N)$-inequivalent cusps and a matrix $\gamma_{\mathfrak{a}}$ for each cusp $\mathfrak{a}$ in this set have been chosen as in §6. Let $a(\mathfrak{a}, n)$ denote the Fourier coefficients defined using the matrices $\sigma_{\mathfrak{a}}$ determined by these choices of $\mathfrak{a}, \gamma_{\mathfrak{a}}$. Let p be a prime which does not divide N, and assume that $T_p^{\chi} f = \lambda f$. Then for each representative cusp $\mathfrak{a}$ there exist cusps $\mathfrak{a}'$ and $\mathfrak{a}''$, and integers $j', j'', \lambda', \lambda''$, such that*

$$\frac{\chi(\lambda'')}{\sqrt{p}} a\Big(\mathfrak{a}'', \frac{n+\mu_{\mathfrak{a}}}{p} - \mu_{\mathfrak{a}''}\Big) e(j''(n+\mu_{\mathfrak{a}})) \; - \; \lambda a(\mathfrak{a}, n) \\ + \; \sqrt{p}\chi(\lambda') e\left(j' \frac{n+\mu_{\mathfrak{a}}}{p}\right) a(\mathfrak{a}', \lfloor p \cdot (n+\mu_{\mathfrak{a}}) \rfloor) \; = \; 0,$$

with the caveat that $\frac{n+\mu_{\mathfrak{a}}}{p} - \mu_{\mathfrak{a}''}$ need not be integral, and the convention that if it is not then $a\left(\mathfrak{a}'', \frac{n+\mu_{\mathfrak{a}}}{p} - \mu_{\mathfrak{a}''}\right)$ is defined to be zero. The cusps $\mathfrak{a}'$, and $\mathfrak{a}''$ are $p \cdot \mathfrak{a}$ and $p^{-1} \cdot \mathfrak{a}$, respectively, in terms of the action of $\mathbb{Z}/N\mathbb{Z}$ on a set of cusp representatives satisfying our hypotheses which was defined at the end of the last section.

For prime power q^e, the values of λ', λ'', are given in the table below. The integers j', j'', are both zero unless $\mathfrak{a} = \frac{1}{c_1 q^l}$ with $l < \frac{e}{2}$. In that case, we have

$$\begin{aligned} 0 \le j', j'' &< q^{e-l}, & 0 \le c_1', c_1'' &< q^l, \\ c_1'' p &\equiv c_1 \pmod{q^l}, & c_1' &\equiv c_1 p \pmod{q^l}, \\ c_1''(p - j'' q^l c_1) &\equiv c_1 \pmod{q}^e, & (1 - q^l c_1 j' p) c_1' &\equiv c_1 p \pmod{q^e}. \end{aligned}$$

For $N = \prod_{q \in S} q^{e_q}$, the integer λ' may be taken to be the smallest positive solution to the system of congruences: $\lambda' \equiv \lambda_q' \pmod{q^{e_q}}$, $(\forall\ q \in S)$, where, for each q, the integer λ_q' is obtained by applying the result to the prime power q^{e_q} and the cusp $\mathfrak{a}_q$ for $\Gamma_0(q^{e_q})$ such that $\gamma_{\mathfrak{a}} \equiv \gamma_{\mathfrak{a}_q} \pmod{q^{e_q}}$. The values of j', j'', λ'' are obtained similarly.

$\mathfrak{a}$	λ'	λ''
∞	1	1
0	1	p
$\frac{1}{c_1 q^l}$, $l \ge \frac{e}{2}$	p	1
$\frac{1}{c_1 q^l}$, $l < \frac{e}{2}$	$(c_1')^{-1} c_1$	$(c_1'')^{-1} c_1$

(Inverses taken modulo q^e.)

Remarks: If χ is not primitive, it may not be necessary for λ' to be congruent to λ_q' modulo q^{e_q}; a lower power of q may suffice. The same applies to λ''. It will turn out that the systems of congruences for j' and j'' can always be taken modulo properly lower powers of q. (See e_q' and N' in the statement of proposition 7.5.)

Theorem 7.4 follows easily from the following proposition, by the same argument sketched in the classical case at the beginning of this section. This proposition works with the group algebra $\mathbb{C}[GL(2,\mathbb{Q})^+]$. The action of $GL(2,\mathbb{Q})^+$ on functions $\mathfrak{h} \to \mathbb{C}$ by the weight k slash operator $|_k$ extends by $\mathbb{C}$-linearity to an action of $\mathbb{C}[GL(2,\mathbb{Q})^+]$, and this identifies the Hecke operator T_p^{χ} with

$$\chi(p) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} + \sum_{b=0}^{p-1} \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix} \; \in \; \mathbb{C}[GL(2,\mathbb{Q})^+].$$

We shall also consider the right ideal in $\mathbb{C}[GL(2,\mathbb{Q})^+]$ generated by all $\gamma-\chi(\gamma)$ with $\gamma\in\Gamma_0(N)$. It is easy to see that any modular form of character χ for $\Gamma_0(N)$ is annihilated by this ideal.

Proposition 7.5 *Fix a cusp $\mathfrak{a}$ and let $\mathfrak{a}', \mathfrak{a}'', \lambda', \lambda'', j', j''$ be defined as in theorem 7.4. For each $q \in S$, let*

$$e'_q = \begin{cases} 0, & \mathfrak{a}_q = \infty, \\ e_q, & \mathfrak{a}_q = 0, \\ e_q - l, & \mathfrak{a}_q = \frac{1}{cq^l}. \end{cases}$$

Let $N' = \prod_{q\in S} q^{e'_q}$. Let $\mathcal{I}$ denote the right ideal in the group ring $\mathbb{C}[GL(2,\mathbb{Q})^+]$ generated by all $\gamma-\chi(\gamma)$, $\gamma\in\Gamma_0(N)$. Then

$$T_p^\chi\gamma_{\mathfrak{a}} \;\equiv\; \chi(\lambda'')\gamma_{\mathfrak{a}''}\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & j'' \\ 0 & 1\end{pmatrix} + \chi(\lambda')\gamma_{\mathfrak{a}'} \sum_{\substack{0\le b<N'p \\ b\equiv j' \pmod{N'}}} \begin{pmatrix} 1 & b \\ 0 & p\end{pmatrix} \pmod{\mathcal{I}}.$$

As a first step towards the proof of proposition 7.5, we show the following.

Lemma 7.6 *For each element of the set $\left\{(c,d)\in(\mathbb{Z}/N\mathbb{Z})^2 \mid \langle c,d\rangle = \mathbb{Z}/N\mathbb{Z}\right\}$, fix an element $s(c,d)$ of $SL(2,\mathbb{Z})$ such that the bottom row of $s(c,d)$ is congruent to (c,d) $\pmod N$. Take p a prime not dividing N. Then for any matrix $\gamma\in SL(2,\mathbb{Z})$ such that the bottom row of γ is congruent to (c,d) $\pmod N$, we have*

$$T_p^\chi\gamma \equiv s(c,pd)\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix} + \sum_{i=0}^{p-1} s(pc, d-ic)\begin{pmatrix} 1 & i \\ 0 & p\end{pmatrix} \pmod{\mathcal{I}},$$

where $\mathcal{I}$ is the right ideal of $\mathbb{C}[GL(2,\mathbb{Q})^+]$ defined in proposition 7.5.

Proof: Let

$$S_p = \left\{\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}\right\} \cup \left\{\begin{pmatrix} 1 & i \\ 0 & p\end{pmatrix} : 0\le i\le p\right\}.$$

Then, for all $\xi\in S_p$,

$$SL(2,\mathbb{Z})\cdot\xi\cdot SL(2,\mathbb{Z}) = \coprod_{\xi'\in S_p} SL(2,\mathbb{Z})\cdot\xi' = \coprod_{\xi'\in S_p}\xi'\cdot SL(2,\mathbb{Z}).$$

It follows that for all $\xi\in S_p, \gamma\in SL(2,\mathbb{Z})$ there exist unique $\xi'\in S_p, \gamma'\in SL(2,\mathbb{Z})$ such that $\xi\gamma=\gamma'\xi'$, and that, for fixed γ, the map $\xi\mapsto\xi'$ is a bijection $S_p\to S_p$. If ξ and γ are given, then the element ξ' may be described concretely as the unique element of S_p such that $\xi\gamma(\xi')^{-1}\in SL(2,\mathbb{Z})$.

Fix any $\gamma_0\in\Gamma_0(N)$ such that $\widetilde{\chi}(\gamma_0)=\chi(p)$, and let

$$S'_p = \left\{\gamma_0\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}\right\} \cup \left\{\begin{pmatrix} 1 & i \\ 0 & p\end{pmatrix} : 0\le i\le p\right\}.$$

Then

$$T_p^\chi = \sum_{\xi\in S'_p}\xi \pmod{\mathcal{I}}.$$

To prove that two individual matrices in $GL(2,\mathbb{Q})^+$ are equivalent $\pmod{\mathcal{I}}$ it suffices to prove that their bottom rows are congruent modulo N. It is clear that the bottom row of $\xi s(c,d)$ is (pc, pd) for all $\xi \in S_p'$, so that the bottom row of $\xi s(c,d)(\xi')^{-1}$ is the row vector $(pc,pd)\cdot(\xi')^{-1}$. As ξ ranges over S_p', this row vector ranges over the set $\{(pc, d-ic)\}\cup\{(c,pd)\}$. This completes the proof. $\square$

Proof of proposition 7.5 We assume that we have fixed a maximal set of $\Gamma_0(N)$-inequivalent cusps together with a choice of matrix $\gamma_{\mathfrak{b}}$ for each cusp $\mathfrak{b}$ in this set as in §6. Let $X = \left\{(c,d)\in(\mathbb{Z}/N\mathbb{Z})^2 \mid \langle c,d\rangle = \mathbb{Z}/N\mathbb{Z}\right\}$. For any $x=(x_0,x_1)\in X$, there exist a unique cusp $\mathfrak{b}$ from our fixed set of representatives, corresponding to the Γ_∞-orbit of $[x_0 : x_1]$ in $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z})$. If $\gamma \in SL(2,\mathbb{Z})$ has bottom row $\equiv (x_0,x_1) \pmod N$, then it is possible to choose $\gamma_0 \in \Gamma_0(N)$ and $j\in\mathbb{Z}$ such that $\gamma = \gamma_0\gamma_{\mathfrak{b}}\left(\begin{smallmatrix}1&j\\0&1\end{smallmatrix}\right)$. It follows immediately that $\gamma \equiv \widetilde{\chi}(\gamma_0)\gamma_{\mathfrak{b}}\left(\begin{smallmatrix}1&j\\0&1\end{smallmatrix}\right) \pmod{\mathcal{I}}$. It is possible to make γ_0 and j unique by requiring $0\le j\le m_{\mathfrak{b}}$, but we will not do so. It turns out that it is sometimes possible to gain a bit of control over $\widetilde{\chi}(\gamma_0)$ by allowing j to vary over a larger range, and this approach is more convenient.

In order to prove proposition 7.5, we must carry out this analysis carefully for each element of the set $\{(pc, d-ic)\}\cup\{(c,pd)\}$, as (c,d) ranges over the bottom rows of the matrices $\gamma_{\mathfrak{a}}$. We first complete the case when $N=q^e$. In this case, the pairs (c,d) considered are $(1,0),(0,1)$, and $(cq^l,1)$ for $0<l<e, 0<c<q^{e-l}$ and $q\nmid c$. Let us first look carefully at $(pc,d-ic)$ in the case when $(c,d)=(1,0)$. clearly $(pc,d-ic)=(p,-i)$ in this case. The corresponding element of $\mathbb{P}^1(\mathbb{Z}/q^e\mathbb{Z})$ is $[p:-i]$, or $[1:-i\overline{p}]$, where $\overline{p}$ denotes the inverse modulo q^e. This element of $\mathbb{P}^1(\mathbb{Z}/q^e\mathbb{Z})$ lies in the Γ_∞-orbit for which the standard representative is $[1:0]$. This tells us that $\gamma_{\mathfrak{b}}$ in this case is $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$. Furthermore, since $[1:-i\overline{p}]=[1:0]\cdot\left(\begin{smallmatrix}1&-i\cdot\overline{p}\\0&1\end{smallmatrix}\right)$, we see that j may be taken to be $-i\cdot\overline{p}$. (Here, we interpret this as taking the additive inverse and product modulo q^e, so that the answer is an integer between 0 and q^e-1, inclusive.)

In order to keep track of $\widetilde{\chi}(\gamma_0)$ as well, it is necessary to work with X rather than $\mathbb{P}^1$. The more precise statement is that $(p,-i)=p\cdot(1,0)\cdot\left(\begin{smallmatrix}1&-i\overline{p}\\0&1\end{smallmatrix}\right)$. This immediately implies that

$$\begin{pmatrix}* & *\\ p & -i\end{pmatrix} = \gamma_0\cdot\begin{pmatrix}0&-1\\1&0\end{pmatrix}\cdot\begin{pmatrix}1&-i\overline{p}\\0&1\end{pmatrix},$$

with $\gamma_0\in\Gamma_0(q^e)$ such that $\widetilde{\chi}(\gamma_0)=\chi(p)$. Hence

$$\begin{pmatrix}* & *\\ p & -i\end{pmatrix} \equiv \chi(p)\cdot\begin{pmatrix}0&-1\\1&0\end{pmatrix}\cdot\begin{pmatrix}1&-i\overline{p}\\0&1\end{pmatrix} \pmod{\mathcal{I}}$$

in this case. Now, $\left(\begin{smallmatrix}1&-i\overline{p}\\0&1\end{smallmatrix}\right)\cdot\left(\begin{smallmatrix}1&i\\0&p\end{smallmatrix}\right)=\left(\begin{smallmatrix}1&i-ip\overline{p}\\0&p\end{smallmatrix}\right)$. Clearly, for each i, the integer $i-ip\overline{p}$ is divisible by q^e. Furthermore, these integers are all distinct modulo p, and all in the range from 0 to pq^e-1. It follows that we have

$$\sum_{i=0}^{p-1}\begin{pmatrix}* & *\\ p & -i\end{pmatrix}\cdot\begin{pmatrix}1&i\\0&p\end{pmatrix} \equiv \chi(p)\cdot\begin{pmatrix}0&-1\\1&0\end{pmatrix}\cdot\sum_{\substack{0\le b\le q^e p\\ b\equiv 0 \pmod{q^e}}}\begin{pmatrix}1&b\\0&p\end{pmatrix}.$$

Having worked this case in detail to illustrate the method, we shall henceforth be more brief. To finish the case $(1,0)$ we simply note that if $(c,d)=(1,0)$ then (c,pd) is again $(1,0)$, whence a contribution of $\left(\begin{smallmatrix} * & * \\ 1 & 0 \end{smallmatrix}\right) \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \pmod{\mathcal{I}}$. This completes the proof of proposition 7.5 in the case $N = q^e$ and $\mathfrak{a} = 0$. The case $\mathfrak{a} = \infty$ is, of course, trivial.

We turn to $\mathfrak{a} = \frac{1}{c_1 q^l}$. First, consider. $(c_1 q^l, p)$. We must identify the standard representative for the orbit of this element under the action of $\mathbb{Z}/N\mathbb{Z} \times \Gamma_\infty$, where $\mathbb{Z}/N\mathbb{Z}$ acts by scalar multiplication and Γ_∞ acts by matrix multiplication on the right. Referring to §4, it is not difficult to see that the standard representative is $(c_1'' q^l, 1)$ where c_1'' is uniquely determined by the conditions that $pc_1'' \equiv c_1 \pmod{q^{e-l}}$ and $0 < c_1'' < q^{\min(l,e-l)}$. In the simpler case $l \geq e-l$, we get

$$\begin{pmatrix} * & * \\ c_1 q^l, p & \end{pmatrix} \equiv \chi(p) \begin{pmatrix} 1 & 0 \\ c_1'' q^l & 1 \end{pmatrix} \pmod{\mathcal{I}}.$$

If $l < e-l$, it is a bit more complicated. Because we deal with reduction and inversion modulo various powers of q, it is convenient to introduce a more explicit notation. For a, r integers with r positive, let $[a]_r$ denote the unique integer between 0 and $q^r - 1$, inclusive, which is congruent to $a \pmod{q^r}$ and $[a]_r^{-1}$ the unique integer in the same range such that $a \cdot [a]_r^{-1} \equiv 1 \pmod{q^r}$.

When $l < e - l$, we have $c_1'' = c_1 \left[[p]_{e-l}^{-1} \right]_l$. It is necessary to choose j'' such that $c_1 [p - j'' c_1 q^l]_{e-l}^{-1} \equiv c_1'' \pmod{q^{e-l}}$. However, it is possible, and more convenient, to choose j'' subject to the more stringent condition that $c_1 [p - j'' c_1 q^l]_e^{-1} \equiv c_1'' \pmod{q^e}$. This ensures that $\chi(p - j'' c_1 q^l) = \chi(c_1)\chi(c_1'')^{-1}$. We have

$$(c_1 q^l, p) = (c_1 q^l, p - j'' c_1 q^l) \begin{pmatrix} 1 & j'' \\ 0 & 1 \end{pmatrix} = (p - j'' c_1 q^l)(c_1'' q^l, 1) \begin{pmatrix} 1 & j'' \\ 0 & 1 \end{pmatrix},$$

and deduce

$$\begin{pmatrix} * & * \\ c_1 q^l & p \end{pmatrix} \equiv \chi(c_1)\chi(c_1'')^{-1} \begin{pmatrix} 1 & 0 \\ c_1'' q^l & 1 \end{pmatrix} \begin{pmatrix} 1 & j'' \\ 0 & 1 \end{pmatrix}.$$

Now, consider $(c_1 p q^l, 1 - i c_1 q^l)$. When $l \geq e-l$ the most expedient thing is to write this as

$$(c_1 p q^l, 1) \begin{pmatrix} 1 & [-[p]_{e-l}^{-1} \cdot i]_{e-l} \\ 0 & 1 \end{pmatrix}.$$

Much as above, we get

$$\sum_{i=0}^{p-1} \begin{pmatrix} * & * \\ c_1 p q^l & 1 - i c_1 q^l \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ c_1' q^l & 1 \end{pmatrix} \sum_{\substack{0 \leq b < pq^{e-l} \\ b \equiv 0 \pmod{q^{e-l}}}} \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}.$$

When $l < e-l$ we choose $j(i)$ so that $c_1 p [1 - i c_1 q^l - j(i) c_1 p q^l]_e^{-1} \equiv c_1' \pmod{q^e}$. It is easy to see that this has a unique solution modulo q^{e-l}, and that the quantity $i + j(i)p$ is constant, modulo q^{e-l}, as i varies. We get

$$\sum_{i=0}^{p-1} \begin{pmatrix} * & * \\ c_1 p q^l & 1 - i c_1 q^l \end{pmatrix} \equiv \sum_{\substack{0 \leq b < pq^{e-l} \\ b \equiv j(0) \pmod{q^{e-l}}}} \begin{pmatrix} 1 & b \\ 0 & p \end{pmatrix}.$$

It is easily verified that $j(0) = j'$, defined as in the statement of the theorem. This completes the proof in the prime power case.

To complete the general case, let us resume the notation $N = \prod_{q \in S} q^{e_q}$. Take (c,d) the bottom row of the matrix $\gamma_{\mathfrak{a}}$ corresponding to some cusp $\mathfrak{a}$, and consider (c, pd). For each q, we know that (c,pd) is equivalent modulo q^{e_q} to one of the row vectors for which the analysis was carried out above. Thus, for each q we obtain a cusp $\mathfrak{a}''_q$, $j''_q \in \mathbb{Z}$ and $\lambda''_q \in (\mathbb{Z}/q^{e_q}\mathbb{Z})^\times$ such that $(c,pd) \equiv \lambda''_q \cdot (c'_q, d'_q) \cdot \left(\begin{smallmatrix} 1 & j'_q \\ 0 & 1 \end{smallmatrix}\right) \pmod{q^{e_q}}$, where (c'_q, d'_q) is the bottom row of $\gamma_{\mathfrak{a}''_q}$. Solving systems of congruences, we obtain a cusp $\mathfrak{a}''$, integer j'' and element λ'' of $(\mathbb{Z}/N\mathbb{Z})^\times$ such that

$$(c, pd) \;=\; \lambda''(c', d') \begin{pmatrix} 1 & j'' \\ 0 & 1 \end{pmatrix},$$

where (c', d') is the bottom row of $\gamma_{\mathfrak{a}''}$. It follows that

$$\begin{pmatrix} * & * \\ c & pd \end{pmatrix} \;\equiv\; \chi(\lambda'')\gamma_{\mathfrak{a}''} \begin{pmatrix} 1 & j'' \\ 0 & 1 \end{pmatrix} \pmod{\mathcal{I}}.$$

Similarly, for $i = 0$ to $p-1$, by passing to the individual primes, using the prime power case, and then solving systems of congruences, we obtain

$$\begin{pmatrix} * & * \\ pc' & d - ipc' \end{pmatrix} \;\equiv\; \chi(\lambda')\gamma_{\mathfrak{a}'} \begin{pmatrix} 1 & j(i) \\ 0 & 1 \end{pmatrix} \pmod{\mathcal{I}}, \qquad (0 \le i < p)$$

for some cusp $\mathfrak{a}'$ some $\lambda' \in (\mathbb{Z}/N\mathbb{Z})^\times$, which are independent of i, and some integers $j(i), 0 \le i < p$. Furthermore, it follows from the analysis done in the prime power case above that $j(i) \cdot p + i \equiv j'_q \pmod{q^{e'_q}}$, for each q, where e'_q is defined as in the proposition and j'_q is as described (for prime powers) in the statement of theorem 7.4. It follows at once that $j(i) \cdot p + i$ is constant $\pmod{N'}$, where N' is defined as in the statement of proposition 7.5, and that the value is obtained by solving the system of congruences obtained from the various q. □

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