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ON THE NUMBER OF FOURIER COEFFICIENTS THAT DETERMINE A MODULAR FORM

DORIAN GOLDFELD* JEFFREY HOFFSTEIN**

§1. Introduction and statement of results:

The main aim of this paper is to explicitly determine how many Fourier coefficients of a modular form uniquely determine the form. Results of this type had previously been attained by Moreno **[M]** for cuspidal automorphic forms associated to $\mathrm{GL}(n)$ over a number field. In the special case of two modular forms for $\mathrm{GL}(2)$ of conductors f_1 and f_2, Moreno showed that there are effectively computable constants A and C such that if the first Af^C (where $f = \max(f_1, f_2)$) Fourier coefficients of the two forms agree then the two modular forms must be identical. Our interest focusses on modular elliptic curves. With today's technology and the assumption of the generalized Riemann hypothesis, we attempt in this paper to give for holomorphic modular forms, the sharpest possible such bounds with explicit computation of all constants. In addition, we also obtain some unconditional results. In the course of our investigations, we found the following general theorem.

A zeta function $Z(s)$ is said to satisfy a functional equation with gamma factor

$$G(s) = \prod_{i=1}^{r} \Gamma(a_i s + b_i) \qquad (a_i, b_i \in \mathbb{R}) \tag{1}$$

provided $Z(s)$ is a meromorphic function of s which satisfies the following two conditions:

First, for $\mathrm{Re}(s) > 1$, we assume the logarithmic derivative of $Z(s)$ is given by the absolutely convergent Dirichlet series

$$-\frac{Z'}{Z}(s) = \sum_{n=1}^{\infty} \lambda(n) n^{-s} \tag{2}$$

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with $\lambda(n) \in \mathbb{C}$, and that for every $\delta > 0$, there exists a constant $c_\delta > 0$ depending only on δ such that

$$\sum_{n=1}^{\infty} \lambda(n) n^{-1-\delta} < c_\delta. \tag{3}$$

Second, there exists a real number $D > 0$, called the conductor of $Z(s)$, such that

$$\Lambda(s) = s(1-s) D^s G(s) Z(s) \tag{4}$$

is an entire function of order one which satisfies the functional equation

$$\Lambda(s) = \epsilon \Lambda(1-s) \tag{5}$$

for some complex number ϵ of absolute value one.

The zeta function $Z(s)$ is said to satisfy the Riemann hypothesis if all the zeros of $\Lambda(s)$ are on the line $\mathrm{Re}(s) = \frac{1}{2}$.

Theorem 1. *Let $Z_1(s)$, $Z_2(s)$ satisfy the Riemann hypothesis and have functional equations with gamma factors $G_1(s)$, $G_2(s)$ and conductors D_1, D_2 respectively. Set*

$$-\frac{Z_i'}{Z_i}(s) = \sum_{n=1}^{\infty} \lambda_i(n) n^{-s} \qquad (i = 1, 2).$$

Assume $D_1^s G_1(s) Z_1(s)$ is analytic except for a simple pole at $s = 1$ and that $D_2^s G_2(s) Z_2(s)$ is entire. Then for every $\kappa > 1$, there exists a constant $C_\kappa > 0$ (depending only on κ) such that for all conductors D_1, $D_2 > \kappa$ as above, there exists an integer

$$n < C_\kappa (\log D_1 D_2)^2 (\log\log D_1 D_2)^4$$

for which $\lambda_1(n) \neq \lambda_2(n)$.

Theorem 1 can be applied to modular forms. In the special case of the congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \;\middle|\; c \equiv 0 \pmod{N} \right\},$$

for example, the following theorem can be obtained.

Theorem 2. *Let $f_1(z) = \sum_{n=1}^{\infty} a_1(n) e^{2\pi i n z}$, $f_2(z) = \sum_{n=1}^{\infty} a_2(n) e^{2\pi i n z}$, be two holomorphic Hecke newforms of weights w_1, w_2, associated to $\Gamma_0(N_1)$, $\Gamma_0(N_2)$, respectively. Let $N = l.c.m.(N_1, N_2)$. Assume that the Riemann hypothesis holds for the Rankin-Selberg zeta functions*

$$\sum_{n=1}^{\infty} |a_1(n)|^2 \, n^{-s-w_1+1}, \qquad \sum_{n=1}^{\infty} a_1(n) \overline{a_2(n)} n^{-s-(\frac{w_1+w_2}{2})+1}.$$

Then for every $\kappa > 1$, there exists a constant $C_\kappa > 0$ (depending only on κ, w_1, w_2) such that for all $N > \kappa$ as above, there exists an integer

$$n < C_\kappa (\log N)^2 (\log\log N)^4$$

for which $a_1(n) \neq a_2(n)$. In the special case when $w_1 = w_2 = 2$, and $N_1 = N_2 = N$, we may take $C_\kappa = 16$ for all $\kappa > e^{15}$.

If we do not assume the Riemann hypothesis, then we may still obtain unconditional results.

Theorem 3. *Let f_1, f_2 be two holomorphic Hecke newforms of weights w_1, w_2 and squarefree levels N_1, N_2, as in theorem 2. Let $N = l.c.m.(N_1, N_2)$. Then for every $\kappa > 1$, there exists a constant C_κ such that for all $N > \kappa$ as above, there exists an integer*

$$n < C_\kappa N \log N$$

for which $a_1(n) \neq a_2(n)$.

Remarks: Theorem 1 may be applied to automorphic forms associated to $\mathrm{GL}(n)$ over a number field with $n \geq 2$ (see **[J-P-S]** and **[M]**). Under the assumption of the Riemann hypothesis for Rankin-Selberg zeta functions, it is a consequence of theorem 2 that two non-isogenous modular elliptic curves of conductor $N > e^{15}$ cannot have the same number of points $\pmod p$ for all primes

$$p \leq 16(\log N)^2(\log\log N)^4.$$

Similarly, it follows from theorem 3 that there exists an absolute constant $C > 0$ such that two non-isogenous modular elliptic curves of squarefree conductors N_1, N_2 cannot have the same number of points $\pmod p$ for all primes

$$p \leq CN(\log N)$$

where $N = \mathrm{l.c.m.}(N_1, N_2)$. We do not know how to obtain such results for elliptic curves which are not assumed to be modular.

§2. Proof of Theorems:

The proof of theorem (1) is based on the following:

Lemma 4. *Fix $\kappa > 1$. Let $Z(s)$ of conductor $D > \kappa$ satisfy the Riemann hypothesis and have a functional equation (5) with gamma factor (1). Then for $x > 10\log D$, there exists a constant $B > 0$ such that*

$$\sum_{\substack{\gamma \\ \Lambda(\frac{1}{2}+i\gamma)=0}} \frac{\sin^2(\gamma \log x)}{\gamma^2} \leq B(\log D)(\log x)^2.$$

The constant B depends only on G and κ, and in the particular case where $Z(s)$ is the Rankin-Selberg zeta function associated to the convolution of two holomorphic weight two newforms for $\Gamma_0(N)$, we may choose

$$B = \begin{cases} 10.78 & \text{if } \kappa = e^3 \\ 7.19 & \text{if } \kappa = e^5 \\ 4.66 & \text{if } \kappa = e^{10} \\ 3.85 & \text{if } \kappa = e^{15} \end{cases}$$

Proof of Lemma 4: Write $\Lambda(s)$ as a product over its zeros using the Hadamard product theorem. Applying the functional equation and taking the logarithmic derivative, we may reduce to the form:

$$\log D + \frac{G'}{G}(s) + \frac{1}{s} + \frac{1}{s-1} + \frac{Z'}{Z}(s) = \sum_\gamma \frac{1}{s - \frac{1}{2} - i\gamma}. \tag{6}$$

Now if $s = \sigma + it$,

$$\text{(7)} \qquad \operatorname{Re}\left(\sum_{\gamma} \frac{1}{s - \frac{1}{2} - i\gamma}\right) = \sum_{\gamma} \frac{\sigma - \frac{1}{2}}{(\sigma - \frac{1}{2})^2 + (t-\gamma)^2}.$$

Taking $\sigma = 2$, we obtain

$$\operatorname{Re}\left(\sum_{\gamma} \frac{1}{\frac{3}{2} + i(t-\gamma)}\right) \geq \sum_{|\gamma - t| \leq 1} \frac{\frac{3}{2}}{\frac{9}{4}+1} = \frac{6}{13} N(t)$$

where

$$N(t) = \#\{\gamma \mid \; |\gamma - t| \leq 1\}.$$

Thus

$$\text{(8)} \quad N(t) \leq \frac{13}{6}\left(\log D \; + \; \operatorname{Re}\left(\frac{G'}{G}(2+it) + \frac{1}{2+it} + \frac{1}{1+it}\right) \; + \; \left|\frac{Z'}{Z}(2+it)\right|\right).$$

Now, by the hypothesis, $\left|\frac{Z'}{Z}\right| \ll 1$, and by the usual upper bound for

$$\psi(s) = \frac{\Gamma'}{\Gamma}(s)$$

we see that for $|t| \leq 1$, $D > e^3$,

$$\text{(9)} \qquad N(t) \ll \log D,$$

which for $|t| \geq 1$ gives

$$\text{(10)} \qquad N(t) \ll \log|t| \; + \; \log D.$$

In the specific case of the Rankin-Selberg zeta function associated to the convolution of two holomorphic newforms of weight two (see **[O]**, **[L]**), the gamma factor is of the form $G(s) = \Gamma(s)\Gamma(s+1)$, and it is easily checked that for $|t| \leq 1$

$$\operatorname{Re}\left[\psi(2+it) \; + \; \psi(3+it) \; + \; \frac{1}{2+it} \; + \; \frac{1}{1+it}\right] \leq 2.85,$$

and for $|t| \geq 1$

$$\operatorname{Re}\left[\psi(2+it) \; + \; \psi(3+it) \; + \; \frac{1}{2+it} \; + \; \frac{1}{1+it}\right] \leq 2.49\sqrt{|t|}.$$

The maximums are achieved at $|t| = 0, 1$ respectively. Note that we use $\sqrt{|t|}$ rather than $\log|t|$ in order to achieve a better constant at small values of t. Furthermore, the Euler product for $Z(s)$ and the fact that the coefficients satisfy the Ramanujan conjecture lead to the estimate $\frac{Z'}{Z}(2) \leq 6.33$. Combining this and the line above, we obtain from (8) that

$$N(t) \leq \frac{13}{6}\Big(\log D \;+\; 9.18\Big) \tag{11}$$

when $|t| \leq 1$ and

$$N(t) \leq \frac{13}{6}\Big((\log D \;+\; 2.49\sqrt{|t|} \;+6.33\Big) \tag{12}$$

if $|t| \geq 1$. Breaking the sum

$$\sum_{\gamma} = \sum_{|\gamma|\leq 1} + \sum_{|\gamma|>1}$$

and using the estimate $|\sin t/t| \leq 1$, we get

$$\begin{aligned} \sum_{\gamma} \frac{\sin^2(\gamma \log x)}{\gamma^2} &\leq N(0)\,(\log x)^2 + \sum_{|\gamma|>1} \frac{\sin^2(\gamma\log x)}{\gamma^2} \\ &\leq N(0)\,(\log x)^2 + 2\sum_{n=0}^{\infty} \frac{N(2n+1)}{(2n+1)^2}. \end{aligned} \tag{13}$$

Applying (11), we have

$$\begin{aligned} \sum_{\gamma} \frac{\sin^2(\gamma \log x)}{\gamma^2} &\leq \frac{13}{6}\Big[(\log x)^2(9.18 + \log D) + \frac{3}{2}\zeta(2)(6.33 + \log D) \\ &\qquad + 2(2.49)\zeta(3/2)(1-2^{-3/2})\Big] \\ &= \frac{13}{6}(\log x)^2(\log D)\left[1 + \frac{9.18}{\log D} + \frac{2.47}{(\log x)^2} + \frac{24.03}{(\log D)(\log x)^2}\right]. \end{aligned}$$

This gives the lemma in the case of the Rankin-Selberg zeta function. The general case follows from (9), (10) combined with (13).

Proof of theorem 1: Consider the integral

$$I = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \left[\frac{x^{s-\frac{1}{2}} - x^{\frac{1}{2}-s}}{s-\frac{1}{2}}\right]^2 \left(\frac{-Z_1'}{Z_1}(s)\right) ds.$$

Then

$$I = \sum_{n<x^2} \frac{\lambda_1(n)}{n^{\frac{1}{2}}} \log\Big(\frac{x^2}{n}\Big)$$

for $x \geq 1$ since

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{y^{s-\frac{1}{2}}}{(s-\frac{1}{2})^2}\,ds = \begin{cases} (\log y)^2 & \text{if } y \geq 1 \\ 0 & \text{if } y \leq 1. \end{cases}$$

On the other hand, moving the line of integration to $\mathrm{Re}(s) = \frac{1}{4}$ we pick up residues at $s = 1$ and at $s = \frac{1}{2} + i\gamma$ for each zero, obtaining

(14)
$$I = 4(x - 2 + x^{-1}) \; - \; 4\sum \frac{\sin^2(\gamma \log x)}{\gamma^2} + \frac{1}{2\pi i}\int_{\frac{1}{4}-i\infty}^{\frac{1}{4}+i\infty} \left[\frac{x^{s-\frac{1}{2}} - x^{\frac{1}{2}-s}}{s-\frac{1}{2}}\right]^2 \left(\frac{-Z_1'}{Z_1}(s)\right) ds.$$

Applying the functional equation

$$\frac{Z_1'}{Z_1}(s) = -2\log D_1 - \frac{G_1'}{G_1}(s) - \frac{G_1'}{G_1}(1-s) - \frac{Z_1'}{Z_1}(1-s),$$

and transforming $s \mapsto 1 - s$, it follows from (14) that

$$2\sum_{n<x^2} \frac{\lambda_1(n)}{n^{\frac{1}{2}}} \log\Big(\frac{x^2}{n}\Big) = 8(x - 2 + x^{-1}) \; - \; 4\sum \frac{\sin^2(\gamma \log x)}{\gamma^2} \; + \; J_1 \tag{15}$$

where

$$J_1 = \frac{1}{2\pi i}\int_{\frac{3}{4}-i\infty}^{\frac{3}{4}+i\infty} \Big[2\log D_1 + \frac{G_1'}{G_1}(s) + \frac{G_1'}{G_1}(1-s)\Big] \cdot \left[\frac{x^{s-\frac{1}{2}} - x^{\frac{1}{2}-s}}{s-\frac{1}{2}}\right]^2 ds. \tag{16}$$

On the other hand, applying the same procedure as above to $Z_2(s)$, and noting that there is no pole at $s = 1$, we obtain

$$2\sum_{n<x^2} \frac{\lambda_1(n)}{n^{\frac{1}{2}}} \log\Big(\frac{x^2}{n}\Big) = -4\sum \frac{\sin^2(\gamma' \log x)}{\gamma'^2} \; + \; J_2 \tag{17}$$

where the sum on the right now goes over zeros $\frac{1}{2} + i\gamma'$ of $Z_2(s)$, and

$$J_2 = \frac{1}{2\pi i}\int_{\frac{3}{4}-i\infty}^{\frac{3}{4}+i\infty} \Big[2\log D_2 + \frac{G_2'}{G_2}(s) + \frac{G_2'}{G_2}(1-s)\Big] \cdot \left[\frac{x^{s-\frac{1}{2}} - x^{\frac{1}{2}-s}}{s-\frac{1}{2}}\right]^2 ds. \tag{18}$$

Subtracting (17) from (15) yields

$$8(x - 2 + x^{-1}) \; - \; 4\sum \frac{\sin^2(\gamma \log x)}{\gamma^2} \; + 4\sum \frac{\sin^2(\gamma' \log x)}{\gamma'^2} \; + J_1 \; - \; J_2 = 0.$$

It follows from lemma 4 that

$$\begin{aligned} x - 2 + x^{-1} &\le \frac{1}{2}\sum \frac{\sin^2(\gamma \log x)}{\gamma^2} \; + \; J_1 \; - \; J_2 \\ &\le \frac{1}{2}B(\log D_1 D_2)(\log x)^2 \; + \; J_1 \; - \; J_2. \end{aligned} \tag{19}$$

The integrals (16) and (18) for J_1 and J_2 may be easily shown to be bounded by

$$O\big((\log D_1 D_2)(\log x)\big).$$

Thus there will be a contradiction if

$$x - 2 + x^{-1} \gg (\log D_1 D_2)(\log x)^2$$

or equivalently if

$$x \gg (\log D_1 D_2)(\log\log D_1 D_2)^2.$$

This proves the theorem.

Proof of theorem 2: Following Ogg **[O]**, let

$$f_1(z) = \sum_{n=1}^{\infty} a_1(n) e^{2\pi i n z}$$

$$f_2(z) = \sum_{n=1}^{\infty} a_2(n) e^{2\pi i n z}$$

be two holomorphic newforms of weights w and squarefree level N_1, N_2, respectively. Set $N = \text{l.c.m.}(N_1, N_2)$, $M = \text{g.c.d.}(N_1, N_2)$. For primes $p|M$, define

$$c(p) = a_1(p) a_2(p) p^{2-w} = \pm 1,$$

and

$$\Phi(s : f_1, f_2) = \left(\frac{N}{4\pi^2}\right)^s \Gamma(s)\Gamma(s + w - 1) Z(s) \tag{20}$$

where

$$Z(s) = \zeta_N(2s) \sum_{n=1}^{\infty} a_1(n)\overline{a_2(n)} n^{-s-w+1} \prod_{p|M} (1 - c(p)p^{-s})^{-1}. \tag{21}$$

Then Ogg **[O]** has shown that Φ is analytic except for a possible simple pole at $s = 1$ with residue proportional to $< f_1, f_2 >$, the Petersson inner product of f_1 with f_2. Moreover, Φ satisfies the fuctional equation

$$\Phi(s; f_1, f_2) = \Phi(1 - s; f_1, f_2). \tag{22}$$

This result was generalized by Li **[L]** to newforms of arbitrary weight. It follows that for $f_1 \neq f_2$, $\Phi(s; f_1, f_1)$ has a simple pole at $s = 1$ while $\Phi(s; f_1, f_2)$ is entire. Thus, the conditions of theorem 1 are satisfied. The proof of theorem 2 again follows from (19) which yields a contradiction if $x \gg (\log N)(\log\log N)^2$.

In the special case when $w_1 = w_2 = 2$ and $N_1 = N_2 = N$, we again apply (19), but note that in this case $J_1 = J_2$. Hence, there will be a contradiction if

$$x - 2 + x^{-1} \le \frac{B}{2}(\log N)(\log x)^2$$

where B is given as in lemma 4. If we put $x = \sqrt{C_\kappa}(\log N)(\log\log N)^2$ with $B = 3.85$ and $\kappa > e^{15}$, then a crude estimate completes the proof.

Proof of Theorem 3: We compute

$$\begin{aligned} < f_1, f_2 > &= \iint_{\Gamma_0(N)\backslash\mathfrak{h}} f_1(z)\overline{f_2(z)}\, y^{\frac{w_1+w_2}{2}} \frac{dxdy}{y^2} \\ &= \iint_{\Gamma_0(1)\backslash\mathfrak{h}} \sum_{\sigma\in\Gamma_0(N)\backslash\Gamma_0(1)} f_1(\sigma z)\overline{f_2(\sigma z)}\,(\mathrm{Im}\sigma z)^{\frac{w_1+w_2}{2}} \frac{dxdy}{y^2}, \end{aligned}$$

where $\mathfrak{h}$ denotes the upper half-plane. A set of representatives for $\Gamma_0(N)\backslash\Gamma_0(1)$ is given by

$$\left\{\begin{pmatrix} r & s \\ t & u \end{pmatrix} \;\Big|\; t|N,\ 1 \le u \le N/t\right\}.$$

Setting $M = N/t$, we have

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} Mr & s_1 \\ N & Mu_1 \end{pmatrix}\begin{pmatrix} 1/M & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix},$$

where $h, s_1, u_1 \in \mathbb{Z}$ are chosen so that $th + Mu_1 = u$ and $rh + s_1 = s$. Since f_1, f_2 are newforms, they must be eigenfunctions of the involution

$$z \mapsto \begin{pmatrix} Mr & s_1 \\ N & Mu_1 \end{pmatrix} z$$

with the same eigenvalue

$$\epsilon_M = \prod_{p|M}(-a_1(p)) = \prod_{p|M}(-a_2(p))$$

where the product goes over primes $p|M$ (see **[A-L]**). Moreover, since $|\epsilon_M|^2 = 1$, it follows that

$$\begin{aligned} < f_1, f_2 > = M^{1-\frac{w_1+w_2}{2}} \\ \times \iint_{\Gamma_0(1)\backslash\mathfrak{h}} \sum_{\substack{M|N \\ 1\le u\le M}} f_1\left(\begin{pmatrix} 1 & h \\ 0 & M \end{pmatrix} z\right)\overline{f_2\left(\begin{pmatrix} 1 & h \\ 0 & M \end{pmatrix} z\right)}\, y^{\frac{w_1+w_2}{2}} \frac{dxdy}{y^2} \end{aligned}$$

where $h \equiv (N/M)^{-1} \cdot u \pmod M$. Since we are assuming that that $a_1(n) = a_2(n)$ for all $n \ll N(\log N)$, a simple computation shows that for $M \le N$ and $z \in \Gamma_0(N)\backslash\mathfrak{h}$

$$f_1\left(\begin{pmatrix} 1 & h \\ 0 & M \end{pmatrix} z\right)\overline{f_2\left(\begin{pmatrix} 1 & h \\ 0 & M \end{pmatrix} z\right)} = \left|f_1\left(\begin{pmatrix} 1 & h \\ 0 & M \end{pmatrix} z\right)\right|^2 + O(N^{-2}).$$

Hence

$$< f_1, f_2 > \quad \geq \iint\limits_{\Gamma_0(N)\backslash\mathfrak{h}} |f_1(z)|^2 y^{\frac{w_1+w_2}{2}} \frac{dxdy}{y^2} \; + \; O(N^{-\frac{1}{2}}),$$

from which it easily follows that

$$< f_1, f_2 > \; \gg \; 1.$$

But this contradicts the fact that $< f_1, f_2 > = 0$ for newforms $f_1 \neq f_2$. Consequently, the assumption that $a_1(n) = a_2(n)$ for all $n \ll N(\log N)$ must be false. This completes the proof.

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Dorian Goldfeld, Department of Mathematics, Columbia University, NY NY 10027

Jeffrey Hoffstein, Department of Mathematics, Brown University, Providence, RI 02912