Appendix: An effective zero-free region

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All the notation in this appendix will be as in the preceding paper. Let f be a Maass form which is a newform for $\Gamma_0(N)$, with eigenvalue λ and central character χ, normalized so that $\langle f, f\rangle = 1$. We have seen that the size of $\rho(1)$, the first Fourier coefficient of f, is intimately related to the behavior of $L(s, F)$ near $s = 1$. Here $L(s, F)$ is the L-series of F, the adjoint square lift of f to $\mathrm{GL}(3)$, and the crucial question is whether or not $L(s, F)$ vanishes when s is real and close to 1. It was shown in Theorem 0.1 that if $L(s, F)$ is nonzero in a sufficiently wide neighborhood of 1 then $|\rho(1)|^2 \ll_\varepsilon \log(\lambda N + 1)$ where the implied constant is absolute and effective. The paper goes on to show that even if a zero of $L(s, F)$ does exist close to 1 for some F, such a "Siegel zero" can only happen rarely. As a consequence the bound $|\rho(1)| \ll_\varepsilon (\lambda N)^\varepsilon$ is proved, for any $\varepsilon > 0$, where the implied constant depends on ε and is ineffective.

About a year after the preceding paper was first circulated, it developed from conversations involving the above three authors that by slightly modifying the techniques introduced in that paper the possibility of a Siegel zero could be completely eliminated in many cases. In particular, Theorem 0.1 is now true unconditionally in the generic situation when f is not a lift from $\mathrm{GL}(1)$, that is to say, when the L-series of f is not equal to a Hecke L-series defined over a quadratic field. If we include all cusp forms, we can still obtain the Theorem in the λ-aspect, but must restrict ourselves to either an ineffective constant or a weaker effective constant in the N-aspect. We have:

Main Theorem. *Let f be a Maass form which is a newform for $\Gamma_0(N)$, with eigenvalue λ and central character χ, normalized so $\langle f, f\rangle = 1$. Let $\rho(1)$ denote the first Fourier coefficient of f, and F the adjoint square lift of f to* $\mathrm{GL}(3)$. *Suppose that f is not a lift from* $\mathrm{GL}(1)$. *Then there exist effective constants c_1 and c_2 such that*

$$L(1, F) \geq \frac{c_1}{\log(\lambda N + 1)}$$

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and

$$|\rho(1)|^2 \leq c_2 \log(\lambda N + 1).$$

If f is a lift from GL (1) *then there exist effective constants c_3 and c_4 such that*

$$L(1, F) \geq c_3 \min\left(\frac{1}{\sqrt{N}}, \frac{1}{\log(\lambda N + 1)}\right)$$

and

$$|\rho(1)|^2 \leq c_4 \max\left(\sqrt{N}, \log(\lambda N + 1)\right).$$

The $\sqrt{N}$ can be replaced by N^ε, for any $\varepsilon > 0$, at the cost of making c_3 and c_4 ineffective constants depending on ε.

Remark. As in the previous paper, all the arguments go through for holomorphic f or Maass forms with weight. In fact the above bounds can be made uniform in the weight, as well as the level and eigenvalue.

The Theorem breaks into two cases because although Siegel zeros that originate on GL(2) can be eliminated, in the case of a lifted form there is a possibility that $L(s, F)$ will be divisible by a quadratic Dirichlet L-series with a Siegel zero. Even here, the zero can only occur in the N-aspect. Note that in many instances there are no forms that are lifts. This is the case, for example, in $\mathrm{SL}(2, Z)$ and in $\Gamma_0(N)$ when N is prime and the central character is trivial.

The proof is as follows. We first give a slightly more general version of Lemma 3.3.

LEMMA. *Let $\varphi(s)$ be a Dirichlet series with nonnegative coefficients, absolutely convergent for* $\mathrm{Re}\,(s) > 1$. *Suppose also that $\varphi(s)$ has an Euler product, so $\varphi(s) \neq 0$ for $\mathrm{Re}\,(s) > 1$, and $\varphi'(s)/\varphi(s)$ is negative for s real and > 1. Let $\varphi(s)$ have a pole of order m at $s = 1$ and let $\Lambda(s) = s^m(1-s)^m G(s)\varphi(s)$ satisfy $\Lambda(s) = \Lambda(1-s)$, with $\Lambda(s)$ entire of order 1. Here*

$$G(s) = D^s \prod_{i=1}^{l} \Gamma(\frac{s + c_l}{2})$$

with $D > 1$ the "level" of $\varphi(s)$. Then there exists an effective constant c, depending only on l and m, such that $\varphi(s)$ has at most m real zeros in the range

$$1 - \frac{c}{\log M} < s < 1,$$

where $M = 1 + D\max\{|c_l|\}$.

Proof. Write

$$\Lambda(s) = e^{A+Bs} \prod (1 - \frac{s}{\rho}) e^{s/\rho},$$

where ρ runs over the set of zeros of $\Lambda(s)$. Taking the logarithmic derivative, and applying the functional equation, we get

$$\sum \frac{1}{s-\rho} = \frac{m}{s} + \frac{m}{s-1} + \frac{G'(s)}{G(s)} + \frac{\varphi'(s)}{\varphi(s)}.$$

Now $\varphi'(s)/\varphi(s)$ is negative for s real and greater than 1. Also, pairing conjugate roots, every term of $\sum \frac{1}{s-\rho}$ is positive, so there exists an absolute effective constant c_1 such that

$$\sum_{i=1}^{n} \frac{1}{s-\beta_i} \leq \frac{m}{s-1} + c_1 \log M,$$

where we have included all the real zeros β_i of $\varphi(s)$ with $\beta_i \geq 1 - c/\log M$. Let $s = 1 + \delta/\log M$ with $\delta < c_1^{-1}$. If c is chosen small enough, compared to δ, then a contradiction is obtained whenever $n \geq m+1$. □

Proof of Theorem. The key observation of the previous paper was that if F_1, F_2 are two distinct adjoint square lifts of Maass forms to GL(3), and $L(s, F_1 \times F_2)$ denotes the L-series of the Rankin-Selberg convolution of F_1 with F_2, then

$$\varphi(s) = \zeta(s)L(s, F_1)L(s, F_2)L(s, F_1 \times F_2)$$

is a Dirichlet series possessing positive coefficients, a functional equation as $s \to 1-s$, a simple pole at $s = 1$, and growth in the critical strip which is at most polynomial in $\mathrm{Im}(s)$ and the eigenvalues and levels of the lifted GL(2) forms as $|\mathrm{Im}(s)| \to \infty$. (See Lemma 3.1.) By Lemma 3.3 it followed that $\varphi(s)$ could have at most one real zero close to 1. Thus if one exceptional F_1 were fixed all others could be controlled.

The point of this appendix is to note that if one simply sets $F_1 = F_2 = F$, then

$$\varphi(s) = \zeta(s)L(s, F)^2 L(s, F \times F)$$

still has positive coefficients, a functional equation as $s \to 1-s$, and at most polynomial growth in $\mathrm{Im}(s)$ and the eigenvalue and level of the lifted form f as $|\mathrm{Im}(s)| \to \infty$. What changes is the order of the pole, and the fact that $L(s, F \times F)$ can now be analyzed further. Indeed, if we have the Euler product expansion

$$L(s, f) = \prod_p (1 - \xi p^{-s})^{-1}(1 - \xi' p^{-s})^{-1}$$

(where $\xi = \xi_p, \xi' = \xi'_p$ depend on p and $\xi\xi' = \chi$), then since F is the adjoint square lift of f, it follows that

$$L(s, F) = \prod_p (1 - \alpha p^{-s})^{-1}(1 - p^{-s})^{-1}(1 - \alpha^{-1} p^{-s})^{-1},$$

where $\alpha = \xi^2\bar{\chi}$. Further, one has the Euler product

$$L(s, F\times F) = \prod_p (1-\alpha^2 p^{-s})^{-1}(1-\alpha p^{-s})^{-1}(1-p^{-s})^{-1}(1-\alpha p^{-s})^{-1}$$
$$\times (1-p^{-s})^{-1}(1-\alpha^{-1}p^{-s})^{-1}(1-p^{-s})^{-1}(1-\alpha^{-1}p^{-s})^{-1}(1-\alpha^{-2}p^{-s})^{-1}.$$

Recall that the symmetric square L-series of F (cf. [1, p. 84]) is given by

$$L(s, F, \vee^2) = \prod_p (1-\alpha^2 p^{-s})^{-1}(1-\alpha p^{-s})^{-1}(1-p^{-s})^{-1}$$
$$\times (1-p^{-s})^{-1}(1-\alpha^{-1}p^{-s})^{-1}(1-\alpha^{-2}p^{-s})^{-1}.$$

Accordingly, we have the factorization (see also [2]).

$$L(s, F\times F) = L(s,F)L(s,F,\vee^2)$$

(In fact, this factors further, since

$$L(s,F,\vee^2) = \zeta(s)L(s,f,\vee^4,\bar{\chi}^2),$$

where the L-series is the twist by $\bar{\chi}^2$ of the symmetric fourth power L-series of f.) Thus

$$\varphi(s) = \zeta(s)L(s,F)^3 L(s,F,\vee^2).$$

Bump and Ginzburg [2] have shown that when f is not a lift from GL(1), $L(s,F,\vee^2)$ has a simple pole at $s=1$ and is analytic elsewhere. Thus $\varphi(s)$ has a double pole at $s=1$, and any zero of $L(s,F)$ will be a zero of $\varphi(s)$ of order at least 3. The analytic properties and functional equation of $\varphi(s)$ follow as before from the work of Gelbart and Jacquet [4] and from [5], [6] and [7]. We therefore may apply the lemma, taking $m=2$ and $M=\lambda N+1$, and we find that as $\varphi(s)$ has only a double pole at $s=1$, it cannot have a triple zero within $c/\log M$ of 1. We have thus shown that there exists an absolute effective constant c such that $L(s,F)$ has no real zeros in the interval

$$1-\frac{c}{\log(\lambda N+1)} < s < 1.$$

The Theorem in this case then follows from Theorem 0.1 of the previous paper.

If f is a lift from GL(1) then $L(s,f) = L(s,\psi,K)$ where K is some quadratic field and ψ is a Hecke character defined over K. It is easily checked then that

$$L(s,F) = L(s,\psi_K)L(s,\psi^2(\psi^{-1}\circ N_{K/Q}),K),$$

where $L(s,\psi_K)$ is the quadratic Dirichlet L-series associated to K, and in the second L-series, ψ^{-1} has been composed with the norm down to Q. As in the proof of Proposition 1.1 of the previous paper, $L(1,F)$ is bounded from below by an effective constant multiple of $1-\beta$ or $1/\log(\lambda N+1)$, where β is a possible Siegel zero. To obtain the remainder of the Theorem, first observe that if the square of the character in the second L-series is not trivial it can not have a Siegel zero. This is proved by a very minor modification of the usual argument that a Dirichlet L-series with a complex character can not have a Siegel zero [3, p. 88]. A Siegel zero can only arise from either $L(s, \psi_K)$ or a possible factorization of $L(s, \psi^2(\psi^{-1} \circ N_{K/Q}), K)$ into quadratic Dirichlet L-series, and in any of these cases, the level of the L-series will be bounded above by N. The Theorem follows after substituting either the effective trivial lower bound $1 - \beta \gg 1/\sqrt{N}$, or the ineffective bound $1 - \beta \gg N^{-\varepsilon}$ which comes from Siegel's original theorem. □

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