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# Coefficients of Maass forms and the Siegel zero

By JEFFREY HOFFSTEIN and PAUL LOCKHART*

## Introduction

In this paper we give an upper bound for the first Fourier coefficient of a Maass form for $\Gamma_0(N)$. Let $\mathfrak{H}$ be the complex upper half-plane, $\Gamma = \Gamma_0(N)$ the Hecke congruence subgroup of level $N$. Let $\chi$ be an even Dirichlet character to the modulus $N$, viewed as a character of $\Gamma$ in the usual way. We consider the set $S_0(\Gamma, \chi)$ of cusp forms for $\Gamma$ of weight zero, and character $\chi$. Thus if $f \in S_0(\Gamma, \chi)$, we have $f(\gamma z) = \chi(\gamma) f(z)$ for all $\gamma \in \Gamma$. The Laplacian

$$\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

is a self-adjoint operator on $S_0(\Gamma, \chi)$ with respect to the Petersson inner product

$$\langle f, g \rangle = \frac{1}{\mathrm{Vol}(\Gamma \backslash \mathfrak{H})} \int_{\Gamma \backslash \mathfrak{H}} f(z) \overline{g(z)}\, d\mu,$$

where $d\mu$ denotes the invariant measure $y^{-2}\, dx\, dy$.

Suppose $f(z)$ is an eigenfunction of $\Delta$ with eigenvalue $\lambda$. It is known that $\lambda \geq \frac{3}{16}$, so we may write $\lambda = \frac{1}{4} + t^2$ with $t$ either real and positive, or $0 \leq it \leq \frac{1}{4}$. Then $f(z)$ has a Fourier expansion

$$f(z) = \sum_{n \neq 0} \rho(n) |n|^{-1/2} W(nz), \tag{0.1}$$

where the Whittaker function $W(z)$ is given by

$$W(z) = (|y| \cosh \pi t)^{1/2} K_{it}(2\pi |y|) \exp(2\pi i x).$$

Here $K_{it}$ denotes the usual $K$-Bessel function.

If we further suppose that $f$ is a *newform* in $S_0(\Gamma, \chi)$, then $f$ is an eigenfunction for the Hecke algebra, as well as the involution $z \mapsto -\bar{z}$. Let $a(n)$

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denote the eigenvalue of the $n^{\text{th}}$ Hecke operator $T_n$. Then

$$\begin{aligned} \rho(n) &= \pm\rho(-n), \qquad n \in \mathbb{Z}; \\ \rho(n) &= \rho(1)a(n), \qquad n \geq 1. \end{aligned} \tag{0.2}$$

The coefficients $\rho(n)$ are thus determined up to a scalar multiple by the Hecke eigenvalues $a(n)$. Recall that $a(1) = 1$ and the $a(n)$ satisfy the relations

$$\begin{aligned} a(n) &= \chi(n)\overline{a(n)}, \qquad (n, N) = 1; \\ a(m)a(n) &= \sum_{d|(m,n)} \chi(d)a(mn/d^2), \qquad (mn, N) = 1; \\ a(p)a(n) &= a(pn), \qquad p \mid N. \end{aligned} \tag{0.3}$$

Let us normalize $f(z)$ so that $\|f\| = 1$. It follows from (0.2) that the first coefficient $\rho(1)$ is nonzero. How large can it be? We seek uniform bounds for $\rho(1)$ in terms of the eigenvalue $\lambda$ and the level $N$. We begin by relating $\rho(1)$ to the convolution $L$-function

$$L(s, f \times f) = \zeta(2s) \sum_{n=1}^{\infty} \frac{|a(n)|^2}{n^s}. \tag{0.4}$$

We use the Hecke eigenvalues $a(n)$ in this series so that its leading coefficient is 1. It is known that $L(s, f \times f)$ has a meromorphic continuation to the entire plane, with a simple pole at $s = 1$. Computing the Rankin-Selberg convolution of $f$ with itself, and taking residues, yields

$$\operatorname{Res}_{s=1} L(s, f \times f) = \frac{2\pi}{3} |\rho(1)|^{-2}. \tag{0.5}$$

Now to any newform $f$, there is an associated form $F$ on GL(3) with Fourier coefficients $a(m, n)$ and $L$-function

$$L(s, F) = \sum_{n=1}^{\infty} \frac{a(1, n)}{n^s}, \tag{0.6}$$

such that the Euler factors of $\zeta(s)L(s, F)$ agree with those of $L(s, f \times f)$ at the primes not dividing the level $N$. We thus have a relation

$$L(s, f \times f) = \zeta(s) L_N(s) L(s, F), \tag{0.7}$$

where $L_N(s)$ is a product over the primes dividing $N$ of "bad" Euler factors. The form $F$ is the so-called "adjoint square lift" of $f$, the existence of which was established by Gelbart and Jacquet in [3]. The function $L(s, F)$ is known to be entire, and $L(1, F) \neq 0$. Equation (0.7) then gives

$$\operatorname{Res}_{s=1} L(s, f \times f) = L_N(1) L(1, F). \tag{0.8}$$

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Checking cases in [3], we find that upper bounds for the Fourier coefficients at the bad primes are never worse than those for the good primes. Using these to bound the Euler factors at the bad primes, we get the estimate

$$N^{-\varepsilon} \ll_\varepsilon L_N(1) \ll_\varepsilon N^\varepsilon.$$

Combining this with (0.5) and (0.8) yields

$$N^{-\varepsilon} L(1,F)^{-1} \ll_\varepsilon |\rho(1)|^2 \ll_\varepsilon N^\varepsilon L(1,F)^{-1}. \tag{0.9}$$

Upper bounds for $\rho(1)$ then follow from corresponding lower bounds on the special value $L(1,F)$ of the associated adjoint square $L$-function. We thus formulate the problem as follows: Given a newform $f(z)$, how small can $L(1,F)$ be?

Previous attempts to estimate $\rho(1)$ from above (for example [2], Corollary 1), have invariably run into a "stone wall" at $\lambda^{1/4}$, a situation reminiscent of attempts to find effective lower bounds for $L(1,\chi_d)$ better than $|d|^{-1/2}$, where $\chi_d$ denotes the real Dirichlet character associated to the quadratic field of discriminant $d$. In fact, our situation is very much like the class-number problem for quadratic number fields, with the adjoint square $L$-function playing the role of the Dirichlet $L$-function, and the eigenvalue corresponding to the discriminant of the field. In the classical case, the estimates depend on the existence or nonexistence of a so-called "Siegel zero." We will show that this is true in our case as well.

*Remark.* The lower bound

$$|\rho(1)|^2 \gg_\varepsilon (\lambda N)^{-\varepsilon}$$

has been obtained by Iwaniec ([7], Theorem 2). In fact, Iwaniec's method is a key ingredient in our Lemma 2.1. Stated in terms of the adjoint square $L$-series, this becomes

$$L(1,F) \ll_\varepsilon (\lambda N)^\varepsilon. \tag{0.10}$$

Our main results are contained in the following theorems and corollary. Theorem 0.1 describes the case when a Siegel zero does not exist. Theorem 0.2 gives an unconditional result. The constant $c(\varepsilon)$ in Theorem 0.2 is analogous to the constant in Siegel's Theorem. It can be thought of as either ineffective, or, as in Tatuzawa's version of Siegel's Theorem, effective with at most one exception. We choose to express our result in the latter form.

THEOREM 0.1. *Suppose there exists a constant $c$ such that $L(s,F)$ has no real zeros in the range*

$$1 - \frac{c}{\log(\lambda N + 1)} < s < 1.$$

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*Then there are effective constants* $c_1$ *and* $c_2$, *depending only on* $c$, *such that*

$$L(1,F) \geq \frac{c_1}{\log(\lambda N+1)}$$

*and*

$$|\rho(1)|^2 \leq c_2 \log(\lambda N+1).$$

THEOREM 0.2. *For any* $\varepsilon > 0$, *there exists an effective constant* $c(\varepsilon)$ *so that the inequality*

$$L(1,F) \geq c(\varepsilon)(\lambda N)^{-\varepsilon}$$

*holds for all* $F$ *with at most one exception.*

In particular, Theorem 0.2 implies that $L(1,F) \gg (\lambda N)^{-\varepsilon}$ with an ineffective constant. Combining this with (0.9), we get:

COROLLARY 0.3. *Let* $f$ *be a newform for* $\Gamma_0(N)$ *with eigenvalue* $\lambda$, *normalized so that* $\|f\| = 1$. *Then for any* $\varepsilon > 0$,

$$\rho(1) \ll_\varepsilon (\lambda N)^\varepsilon.$$

The size of the coefficient $\rho(1)$ has been shown by Phillips and Sarnak [11] and Deshouillers and Iwaniec [2] to be related to the general question of the existence of cusp forms for Fuchsian groups. The connection is via a spectral mean value theorem for certain Rankin-Selberg convolutions. Corollary 0.3 has been used recently by Luo [10] to improve this result, obtaining essentially the best possible estimate.

Unfortunately, we are at present unable to make the bound in Corollary 0.3 effective with at most one exception, as in Theorem 0.2. The trouble is that whereas the exception to Theorem 0.2 is merely a single form $F$, there may be many Maass forms $f$ for which $F$ is the lift.

*Remark.* One can in fact show that if the bound of Corollary 0.3 holds, then a real zero $\beta$ of $L(s,F)$ must satisfy $1-\beta \gg_\varepsilon (\lambda N)^{-\varepsilon}$. Obtaining an upper bound for $\rho(1)$ is therefore equivalent to proving the nonexistence of a Siegel zero of $L(s,F)$. Incidentally, Vinogradov and Tahtadžjan [14] assert the nonexistence of such a zero, but do not supply a proof. The short justification which they do provide appears to be incorrect.

*Remark.* The methods used here can be easily applied to holomorphic (resp. nonholomorphic) cusp forms of weight $k$. One obtains results identical to those above, with the term $\lambda N$ replaced by $kN$ (resp. $\lambda k N$).

The plan of the paper is as follows. In Section 1 we establish Theorem 0.1. Section 2 consists of auxiliary results needed for the proof of Theorem 0.2. Finally, in Section 3 we prove Theorem 0.2, using techniques based on Goldfeld's simplified proof of Siegel's Theorem (see [4] and [5]).

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## 1. Zero-free regions and residues of $L$-series

In this section we consider $L$-series of the form

$$L(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s},$$

where the Dirichlet coefficients $b(n)$ are nonnegative and $b(1) = 1$. We assume that the above series converges for $\mathrm{Re}(s) > 1$, and that $L(s)$ has an analytic continuation to $\mathrm{Re}(s) > 0$ with a single simple pole at $s = 1$. One notion we will use throughout the paper is that of the *level* of an $L$-series. If, for some constant $D$ and product of gamma factors $G(s)$, the function $D^{s/2}G(s)L(s)$ possesses a functional equation as $s \to 1-s$, we refer to $D$ as the level of $L(s)$.

We now obtain a lower bound for the residue of such an $L$-series in terms of the width of a zero-free region. For the proof of Proposition 1.1, and several additional times throughout the paper, we will require the following integral transform (see, for example, [6]). For any positive integer $r$,

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{x^s\,ds}{s(s+1)\cdots(s+r)} = \begin{cases} \frac{1}{r!}(1-\frac{1}{x})^r, & x > 1; \\ 0, & 0 < x \le 1. \end{cases} \tag{1.1}$$

PROPOSITION 1.1. *Let $L(s)$ be an $L$-series of the above type, and set $R = \mathrm{Res}_{s=1} L(s)$. Let $M > 1$. Suppose that $L(s)$ satisfies a growth condition on the line $\mathrm{Re}\,(s) = \frac{1}{2}$ of the form*

$$|L(\tfrac{1}{2} + i\gamma)| \le M(|\gamma| + 1)^B \tag{1.2}$$

*for some constant $B$. If $L(s)$ has no real zeros in the range*

$$1 - \frac{1}{\log M} < s < 1, \tag{1.3}$$

*then there exists an effective constant $c = c(B) > 0$ such that*

$$R^{-1} \le c \log M. \tag{1.4}$$

*Proof*. Let $\frac{1}{2} < \beta < 1$ and let $r$ be a fixed integer greater than $B$. Using the integral transform (1.1) and the absolute convergence of $L(s+\beta)$ in the range of integration, we get

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{L(s+\beta)x^s\,ds}{s(s+1)\cdots(s+r)} = \frac{1}{r!}\sum_{n<x} \frac{b(n)}{n^\beta}\left(1 - \frac{n}{x}\right)^r.$$

Since the $b(n)$ are nonnegative, and $b(1) = 1$, we have for all $x \ge 2$,

$$1 \ll \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{L(s+\beta)x^s\,ds}{s(s+1)\cdots(s+r)}. \tag{1.5}$$

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Here, as throughout the proof, the implied constant is effective and depends only on $B$. From (1.2) we see that $L(s)$ has polynomial growth on the line $\mathrm{Re}(s)=\frac{1}{2}$. This is also true on the line $\mathrm{Re}(s)=3$, since $|L(3+i\gamma)|\leq L(3)$ for all $\gamma$. Hence, by the Phragmen-Lindelöf principle, we have $L(\sigma+i\gamma)=O(|\gamma|^B)$ for all $\frac{1}{2}\leq\sigma\leq 3$, $\gamma\geq 1$. Thus we may shift the line of integration to $\mathrm{Re}(s)=\frac{1}{2}-\beta<0$, picking up residues at $s=0,1-\beta$. Using the bound (1.2), we see that the right-hand side of (1.5) becomes

$$\frac{Rx^{1-\beta}}{(1-\beta)(2-\beta)\dots(r+1-\beta)}+\frac{L(\beta)}{r!}+O(Mx^{\frac{1}{2}-\beta}).$$

Taking $x=M^C$, for $C$ a sufficiently large constant, we get

$$1\ll\frac{RM^{C(1-\beta)}}{1-\beta}+L(\beta). \tag{1.6}$$

Thus, if $L(s)$ has no zeros in the interval (1.3), we choose $\beta$ equal to the left-hand endpoint, so that

$$1-\beta=\frac{1}{\log M}.$$

Since $L(s)$ has a simple pole at $s=1$, and is positive for real $s>1$, we must have $L(\beta)\leq 0$. Then (1.6) yields

$$R^{-1}\ll\log M$$

as desired. □

Now let $f$ be a newform of level $N$ and eigenvalue $\lambda$, with adjoint square lift $F$. We want to apply the previous result to the two $L$-functions $\zeta(s)L(s,F)$ and $L(s,f\times f)$. We first need to show that they satisfy the requisite bound on the critical line.

Lemma 1.2. *Let $L(s)$ denote either $\zeta(s)L(s,F)$ or $L(s,f\times f)$. There exist absolute constants $A$ and $B$ such that*

$$L(\tfrac{1}{2}+i\gamma)\leq(\lambda N+1)^A(|\gamma|+1)^B. \tag{1.7}$$

*Proof*. Let $L(s)=\zeta(s)L(s,F)$. From (0.7) we have the relation

$$L(s,f\times f)=L_N(s)L(s). \tag{1.8}$$

The function $L_N(s)$ is a product over the primes dividing $N$ of various Euler factors. These factors vary depending on the type of ramification the corresponding prime undergoes in the lifting process. Checking the possible cases given in [3], one sees that the bounds on the coefficients are never worse than

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in the "good" case. Thus, for example, Shahidi's bound $|a_p| \leq 2p^{1/5}$ (see [13]) may be used to show that for $\mathrm{Re}(s) > 2/5$,

$$N^{-\varepsilon} \ll_\varepsilon L_N(s) \ll_\varepsilon N^\varepsilon, \tag{1.9}$$

where the implied constants are effective. Combined with (1.8), this shows that the problems of bounding $L(s)$ and $L(s, f \times f)$ are essentially the same (at least for $\mathrm{Re}(s) > 2/5$).

Now from the definition (0.4) of $L(s, f \times f)$, and using Shahidi's bound on the coefficients, we see that $L(s, f \times f)$ is bounded by an absolute constant on the line $\mathrm{Re}(s) = 2$. Hence $L(s) \ll_\varepsilon N^\varepsilon$ on this line. Now by the work of Gelbart and Jacquet [3], $L(s)$ satisfies a functional equation relating $s$ and $1 - s$. Thus we get a bound for $L(s)$ on the line $\mathrm{Re}(s) = -1$. The ratio of gamma factors arising from the functional equation is bounded by certain fixed powers of the eigenvalue $\lambda$ and the imaginary part of $s$. The level of $L(s)$ is relevant, and is equal to the level of $L(s, f \times f)$. Since the level of the convolution of two GL(2) forms must divide the product of their levels squared (see [12]), the level of $L(s)$ can be at most $N^4$. Hence we get a bound of the form

$$L(-1 + i\gamma) \ll (\lambda N + 1)^A (|\gamma| + 1)^B$$

for certain absolute constants $A$ and $B$. Applying the Phragmen-Lindelöf principle in the strip $-1 \leq \mathrm{Re}(s) \leq 2$, we see that the same bound applies on the line $\mathrm{Re}(s) = \frac{1}{2}$. By the above remarks, the same goes for $L(s, f \times f)$. □

It remains to note that the coefficients of both $\zeta(s)L(s, F)$ and $L(s, f \times f)$ are nonnegative, and both are analytic except for a simple pole at $s = 1$. The nonnegativity of the coefficients, though clear for $L(s, f \times f)$, and for $\zeta(s)L(s, F)$ at generic primes, is somewhat more subtle for $\zeta(s)L(s, F)$ at bad primes. We are indebted to David Rohrlich for pointing out to us that this can be checked via the local Langlands correspondence.

Theorem 0.1 now follows from Proposition 1.1. The residues of the $L$-series $\zeta(s)L(s, F)$ and $L(s, f \times f)$ are $L(1, F)$ and $\frac{2\pi}{3}|\rho(1)|^{-2}$ respectively. If $L(s, F)$ has no zeros in an interval

$$1 - \frac{c}{\log(\lambda N + 1)} \leq s < 1,$$

then by (0.7) and (1.9) neither does $L(s, f \times f)$. Taking $M = (\lambda N + 1)^C$ in Proposition 1.1, for $C$ sufficiently large, yields Theorem 0.1.

*Remark.* Equation (1.6) also shows that if $L(s, F)$ has no zeros in the range $1 - \varepsilon < s < 1$, then

$$L(1, F)^{-1} \ll_\varepsilon (\lambda N)^\varepsilon. \tag{1.10}$$

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## 2. Estimates for sums of Hecke eigenvalues

The main result of this section is Lemma 2.3, which will be used in the argument of Section 3. Let $f$ be a newform, with adjoint square lift $F$. Let $a(m,n)$ denote the general Fourier coefficient of $F$. Throughout this section and the next, we will use a primed summation sign $\sum'$ to denote a summation over integers relatively prime to $N$. In what follows, all bounds are effective.

LEMMA 2.1.
$$\sum_{n<x} \frac{|a(n)|^4}{n} \ll_\varepsilon (x\lambda N)^\varepsilon.$$

*Proof*. Let
$$S(x) = \sum_{n<x} \frac{|a(n)|^4}{\sqrt{n}}.$$
We first show that for any $\varepsilon > 0$ there is an effective constant $c(\varepsilon)$ such that for all $x \geq 1$,

$$S(x) \leq c(\varepsilon) x^{\frac{1}{2}+\varepsilon} (\lambda N)^\varepsilon. \tag{2.1}$$

Let
$$L_4(s) = \sum_{n=1}^{\infty} \frac{|a(n)|^4}{n^s}.$$
Using the standard convexity argument (as in Lemma 1.2), we obtain an estimate

$$L_4(1+\varepsilon) \ll_\varepsilon (\lambda N)^C \tag{2.2}$$

for some absolute constant $C$. To be more precise, the Rankin-Selberg convolution of the adjoint square lift $F$ with itself (when multiplied by the appropriate zeta factors) gives a Dirichlet series $L_4^*(s)$ with positive coefficients, a subset of which are the numbers $|a(n)|^4$. Now $L_4^*(s)$ satisfies a functional equation, so the standard argument gives $L_4^*(1+\varepsilon) \ll_\varepsilon (\lambda N)^C$, and since $L_4(1+\varepsilon) \leq L_4^*(1+\varepsilon)$, we get (2.2). We now have
$$\begin{aligned} S(x) &\leq 2 \sum_{n<2x} \frac{|a(n)|^4}{\sqrt{n}} \left(1 - \frac{n}{2x}\right) \\ &= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{2L_4(s+\frac{1}{2})(2x)^s \, ds}{s(s+1)}. \end{aligned}$$
This is easily seen to be $\ll L_4(1+\varepsilon) x^{\frac{1}{2}+\varepsilon}$, by moving the line of integration to $\mathrm{Re}(s) = \frac{1}{2} + \varepsilon$. Thus

$$S(x) \leq c_1(\varepsilon) x^{\frac{1}{2}+\varepsilon} (\lambda N)^C. \tag{2.3}$$

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Following Iwaniec (see [7] and [8], Lemma 1), we use the Hecke relations (0.3) to compute

$$\begin{aligned} S(x)^2 &= \sum_{m,n<x} \frac{|a(m)|^4|a(n)|^4}{\sqrt{mn}} \\ &= \sum_{m,n<x} \frac{1}{\sqrt{mn}} \Big( \sum_{d|(m,n)} |a(mn/d^2)| \Big)^4 \\ &\leq \sum_{m,n<x} \frac{\tau((m,n))^3}{\sqrt{mn}} \sum_{d|(m,n)} |a(mn/d^2)|^4 \\ &\leq \sum_{d<x} \frac{\tau(d)^3}{d} \sum_{m,n<x/d} \tau((m,n))^3 \frac{|a(mn)|^4}{\sqrt{mn}} \\ &\leq \sum_{n<x} \frac{\tau(n)^3}{n} \sum_{n<x^2} \tau(n)^4 \frac{|a(n)|^4}{\sqrt{n}}. \end{aligned}$$

Hence we get the relation $S(x)^2 \leq c_2(\varepsilon)x^\varepsilon S(x^2)$. Iterating this, we have

$$S(x)^n \leq c_2(\varepsilon)^n x^{\varepsilon n \log n} S(x^n), \qquad n \geq 2,$$

which, after taking $n^{\text{th}}$ roots and using the estimate (2.3) above, yields

$$S(x) \leq c_3(\varepsilon) x^{\frac{1}{2}+2\varepsilon \log n} (\lambda N)^{C/n},$$

and we obtain (2.1) upon choosing $n \approx C/\varepsilon$. The estimate for $\sum_{n<x} |a(n)|^4 n^{-1}$ now follows from partial summation. □

LEMMA 2.2.

$$\sum_{n<x}{}' \frac{|a(1,n)|^2}{n} \ll_\varepsilon (x\lambda N)^\varepsilon.$$

*Proof*. The $L$-series $L(s, f \times f)$ and $\zeta(s)L(s,F)$ have the same Euler factors at the primes not dividing $N$. It follows that for $(n,N) = 1$, the coefficient of $n^{-s}$ in the Dirichlet series $\sum a(1,n)n^{-s}$ is the same as that in

$$\frac{\zeta(2s)}{\zeta(s)} \sum_{n=1}^{\infty} \frac{|a(n)|^2}{n^s}.$$

Thus

$$\begin{aligned} a(1,n) &= \sum_{m^2 d|n} \mu(d)|a(n/m^2 d)|^2 \\ &= \sum_{d|n} |a(d)|^2 \sum_{m^2|n/d} \mu(n/m^2 d). \end{aligned}$$

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Now the inner sum is always $\pm 1$, so

$$|a(1,n)| \le \sum_{d|n} |a(d)|^2, \qquad (n,N)=1.$$

Using the above estimate, we get

$$\begin{aligned}\sum_{n<x}{}' \frac{|a(1,n)|^2}{n} &\le \sum_{n<x}{}' \frac{1}{n}\Big(\sum_{d|n} |a(d)|^2\Big)^2 \\ &\le \sum_{n<x} \frac{\tau(n)}{n} \sum_{d|n} |a(d)|^4 \\ &\le \sum_{n<x} \frac{\tau(n)}{n} \sum_{n<x} \frac{\tau(n)|a(n)|^4}{n},\end{aligned}$$

which is $\ll_\varepsilon (x\lambda N)^\varepsilon$ by Lemma 2.1. $\square$

LEMMA 2.3.

$$\sum_{m^2n<x}{}' \frac{|a(m,n)|^2}{m^2n} \ll_\varepsilon (x\lambda N)^\varepsilon.$$

*Proof*. In Chapter IX of [1] we find the identity

$$\sum_{n_1,n_2=1}^{\infty} \frac{a(n_1,n_2)}{n_1^{s_1}n_2^{s_2}} = \zeta(s_1+s_2)^{-1} \sum_{n_1=1}^{\infty} \frac{a(n_1,1)}{n_1^{s_1}} \sum_{n_2=1}^{\infty} \frac{a(1,n_2)}{n_2^{s_2}}$$

for any GL(3) eigenform of level 1. For higher level, the same formula holds, in the sense that the Euler products of the two sides agree at the good primes. Equating the coefficients of $m^{-s_1}n^{-s_2}$ of the two sides yields

$$a(m,n) = \sum_{d|(m,n)} \mu(d)a(m/d,1)a(1,n/d), \qquad (mn,N)=1.$$

Using the fact that $a(n,1)=a(1,n)$ for adjoint square lifts, we obtain

$$\begin{aligned}|a(m,n)|^2 &\le \Big(\sum_{d|(m,n)} |a(1,m/d)a(1,n/d)|\Big)^2 \\ &\le \sum_{d|(m,n)} |a(1,m/d)|^2 \sum_{d|(m,n)} |a(1,n/d)|^2.\end{aligned}$$

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Hence

$$\begin{aligned}\sum_{m^2n<x}{}' \frac{|a(m,n)|^2}{m^2n} &\le \sum_{m^2n<x}{}' \frac{1}{m^2n} \sum_{d_1|(m,n)} |a(1,m/d_1)|^2 \sum_{d_2|(m,n)} |a(1,n/d_2)|^2 \\ &\le \sum_{d_1^2d_2<x} \frac{1}{d_1^2d_2} \sum_{m^2n<x}{}' \frac{|a(1,m)|^2|a(1,n)|^2}{m^2n} \\ &\le \sum_{d<x} \frac{\tau(d)}{d} \left( \sum_{n<x}{}' \frac{|a(1,n)|^2}{n} \right)^2\end{aligned}$$

and the result follows directly from Lemma 2.2. □

## 3. An analogue of Siegel's Theorem

Let $f_1$ and $f_2$ be newforms as above, with eigenvalues $\lambda_1, \lambda_2$, levels $N_1, N_2$, and associated adjoint square lifts $F_1, F_2$. Suppose that $f_1$ is not a lift from GL(1) (in other words, not monomial). Let $F_1 \times F_2$ denote the Rankin-Selberg convolution of $F_1$ with $F_2$, and let $L(s, F_1)$, $L(s, F_2)$, and $L(s, F_1 \times F_2)$ be the corresponding $L$-series. Finally, let

$$\varphi(s) = \zeta(s)L(s, F_1)L(s, F_2)L(s, F_1 \times F_2). \tag{3.1}$$

In the following argument, the function $L(s, F_1)$ will play the role of an exceptional quadratic Dirichlet $L$-series with a real zero in the original version of Siegel's Theorem, $L(s, F_2)$ will play the role of a Dirichlet $L$-series involving a different quadratic character, and $L(s, F_1 \times F_2)$ corresponds to the Dirichlet series formed with the product of the two characters. We begin with the following:

LEMMA 3.1. *Suppose $F_1$ and $F_2$ are distinct. The function $\varphi(s)$ then has a meromorphic continuation to the entire complex plane, with a single simple pole at $s = 1$. It possesses a functional equation as $s \to 1 - s$, and the coefficients in the Dirichlet series expansion of $\varphi(s)$ are nonnegative. Furthermore, on the line* $\mathrm{Re}\,(s) = \frac{1}{2}$ *we have an upper bound*

$$\phi(\tfrac{1}{2} + i\gamma) \le (\lambda_1\lambda_2N_1N_2 + 1)^A(|\gamma| + 1)^B \tag{3.2}$$

*for some absolute constants $A$ and $B$.*

*Proof.* The analytic continuation and functional equation of $\varphi(s)$ follow from the corresponding properties of the individual terms in the product (3.1). For $\zeta(s)$ these are well known, and it is from this term that the simple pole arises. The analytic continuation and functional equations for $L(s, F_1)$,

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$L(s, F_2)$, and $L(s, F_1 \times F_2)$ come from the work of Gelbart and Jacquet [3] establishing the lifting, and from the work on convolutions of automorphic forms on $\mathrm{GL}(n)$ contained in [9], [10] and [16]. The assumption that $f_1$ is not a lift implies that $F_1$ is cuspidal, and it follows that $L(s, F_1 \times F_2)$ is entire. The gamma factors contribute the $\lambda_1\lambda_2$ part of (3.2). For the level, we call upon the principle from [12] that was invoked in Lemma 1.2: The level of $L(s, F_1 \times F_2)$ divides the product of the cubes of the levels of $L(s, F_1)$ and $L(s, F_2)$, and thus is at most $(N_1N_2)^{12}$.

To see that the coefficients of the Dirichlet series for $\varphi(s)$ are positive, let $p$ denote a good prime, i.e., one which does not divide the level of $\varphi(s)$. Write the $p^{\text{th}}$ factor in the Euler product for $L(s, f_i)$ as

$$(1 - \xi_i p^{-s})^{-1}(1 - \xi_i' p^{-s})^{-1}, \qquad i = 1, 2.$$

Then the factors corresponding to the lifts $F_i$ have coefficients $1, \alpha_i, \alpha_i^{-1}$, where $\alpha_i = \xi_i^2 \bar{\chi}(p)$. Note that since $\xi_i + \xi_i' = a_i(p)$ and $\xi_i\xi_i' = \chi_i(p)$, we have either $|\alpha_i| = 1$ or $|\alpha_i| = \alpha_i$. The $p^{\text{th}}$ factor in the Euler product for $\varphi(s)$ is then

$$\prod (1 - \alpha_1^{\varepsilon_1}\alpha_2^{\varepsilon_2} p^{-s})^{-1},$$

where the product is taken over the sixteen possible pairs $(\varepsilon_1, \varepsilon_2)$ with the $\varepsilon_i$ independently running through the values $1, 0, 0, -1$. Taking the logarithm, it is easy to verify that the $p^{\text{th}}$ term in the expansion of $\log \varphi(s)$ is

$$\sum_{k=1}^{\infty} \frac{(\alpha_1^k + \alpha_1^{-k} + 2)(\alpha_2^k + \alpha_2^{-k} + 2)}{kp^{ks}}. \tag{3.3}$$

Since the $\alpha_i$ are either nonnegative real numbers, or on the unit circle, it follows that the above series has nonnegative terms, and hence so does the series for $\varphi(s)$. It then follows that $\varphi(s)$ has a pole at $s = 1$, and $L(1, F_1)$, $L(1, F_2)$, and $L(1, F_1 \times F_2)$ are all nonzero. For nongeneric primes, the positivity result can be seen by considering the local Langlands correspondence, as mentioned after Lemma 1.2. □

*Remark.* The positivity of $\varphi(s)$ is not a coincidence. There is a conjectured lifting of two automorphic forms on $\mathrm{GL}(2)$ with parameters $\alpha_1, \alpha_1^{-1}$ and $\alpha_2, \alpha_2^{-1}$, to a form on $\mathrm{GL}(4)$ with parameters $\alpha_1\alpha_2, \alpha_1\alpha_2^{-1}, \alpha_2\alpha_1^{-1}, \alpha_1^{-1}\alpha_2^{-1}$. The series for $\varphi(s)$ is simply the formal convolution of this conjectured form with its conjugate. Although we will use the positivity of $\varphi(s)$ at all primes, as it streamlines our argument, it is in fact only necessary to verify positivity at the generic primes as was done above. One simply divides $\varphi(s)$ by the Euler factors corresponding to bad primes, and uses this modified series in the arguments which follow.

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The proof of Theorem 0.2 will require the following upper bound for $L(1, F_1 \times F_2)$.

LEMMA 3.2. $L(1,F_1 \times F_2) \ll_\varepsilon (\lambda_1\lambda_2N_1N_2)^\varepsilon$.

*Proof*. Let $a_i(m,n)$ denote the general Fourier coefficient of $F_i$. Then, as a Dirichlet series,

$$L(s, F_1 \times F_2) = \zeta(3s) \sum \frac{a_1(m,n)a_2(m,n)}{(m^2n)^s}.$$

Separating out the bad primes, we have

$$L(s, F_1 \times F_2) = \zeta(3s)L_N(s, F_1 \times F_2)L^*(s, F_1 \times F_2),$$

where

$$L^*(s, F_1 \times F_2) = \sum{}' \frac{a_1(m,n)a_2(m,n)}{(m^2n)^s}$$

and $L_N(s, F_1 \times F_2)$ is a finite product of bad Euler factors (see [9]). For $\mathrm{Re}(s) > \frac{4}{5}$ we have the bound $N^{-\varepsilon} \ll_\varepsilon L_N(s, F_1 \times F_2) \ll_\varepsilon N^\varepsilon$, which follows from Shahidi's 1/5 bound [13] combined with Gelbart and Jacquet's analysis of bad primes [3]. As in Lemma 1.2, this is seen by listing the types of bad primes that can occur and verifying that the growth of the coefficients is never worse than in the generic case. Also, for $s$ in this range, $1 \ll \zeta(3s) \ll 1$. Therefore, it suffices to show that $L^*(1, F_1 \times F_2) \ll_\varepsilon (\lambda_1\lambda_2N_1N_2)^\varepsilon$. Consider the integral transform

$$I = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{L^*(s+1, F_1 \times F_2)x^s\,ds}{s(s+1)\cdots(s+r)}, \tag{3.4}$$

with $x$ and $r$ to be chosen presently. Using the identity (1.1) we get

$$I = \frac{1}{r!}\sum_{m^2n<x}{}' \frac{a_1(m,n)a_2(m,n)}{m^2n}\left(1 - \frac{m^2n}{x}\right)^r.$$

Hence

$$\begin{aligned} I &\ll \sum_{m^2n<x}{}' \frac{|a_1(m,n)a_2(m,n)|}{m^2n} \\ &\ll \left(\sum_{m^2n<x}{}' \frac{|a_1(m,n)|^2}{m^2n}\right)^{\frac{1}{2}}\left(\sum_{m^2n<x}{}' \frac{|a_2(m,n)|^2}{m^2n}\right)^{\frac{1}{2}}. \end{aligned}$$

Lemma 2.3 then gives us the bound

$$I \ll_\varepsilon (x\lambda_1\lambda_2N_1N_2)^\varepsilon. \tag{3.5}$$

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Now by the functional equation, together with the Phragmen-Lindelöf principle, $L(s, F_1 \times F_2)$ is easily shown (as in Lemma 1.2) to satisfy an upper bound

$$L(\sigma + i\gamma, F_1 \times F_2) \ll (\lambda_1 \lambda_2 N_1 N_2)^C |\gamma|^C$$

for $\sigma > 4/5$ and $\gamma$ bounded away from zero. Dividing by $\zeta(3s) L_N(s, F_1 \times F_2)$ we see that the same bound applies to $L^*(s, F_1 \times F_2)$. Moving the line of integration in (3.4) to $\mathrm{Re}(s) = -1/10$, and choosing $r = C + 1$, we get

$$I = \frac{1}{r!} L^*(1, F_1 \times F_2) + O((\lambda_1 \lambda_2 N_1 N_2)^C x^{-1/10}).$$

Combining this with (3.5) and setting $x = (\lambda_1 \lambda_2 N_1 N_2)^{10C+1}$ gives

$$L^*(1, F_1 \times F_2) \ll_\varepsilon (\lambda_1 \lambda_2 N_1 N_2)^\varepsilon$$

as desired. □

*Remark.* The product of bad Euler factors $L_N(s, F_1 \times F_2)$ is introduced to avoid the necessity for writing down explicit GL(3) Hecke relations at the bad primes.

Finally, we need the following variation on the classical result concerning the distribution of Siegel zeros.

LEMMA 3.3. *There exists an absolute constant c such that $\varphi(s)$ has at most one real zero in the range*

$$1 - \frac{c}{\log(\lambda_1 \lambda_2 N_1 N_2)} < s < 1.$$

*Proof.* Let $\Lambda(s) = s(1-s)G(s)\varphi(s)$ where $G(s)$ is the appropriate product of gamma factors so that $\Lambda(s)$ is entire and satisfies $\Lambda(s) = \Lambda(1-s)$. Write

$$\Lambda(s) = e^{A+Bs} \prod \Big(1 - \frac{s}{\rho}\Big) e^{s/\rho},$$

where $\rho$ runs over the set of zeros of $\Lambda(s)$. Taking the logarithmic derivative, we get

$$\sum \frac{1}{s-\rho} = \frac{1}{s} + \frac{1}{s-1} + \frac{G'(s)}{G(s)} + \frac{\varphi'(s)}{\varphi(s)}.$$

By (3.3), $\varphi'(s)/\varphi(s)$ is negative for $s$ real and greater than 1, hence there is an absolute constant $c_1$ such that

$$\frac{1}{s-\beta_1} + \frac{1}{s-\beta_2} \le \frac{1}{s-1} + c_1 \log(\lambda_1 \lambda_2 N_1 N_2)$$

for any two real roots $\beta_1$ and $\beta_2$ of $\varphi(s)$. Choosing $s = 1 + \delta/\log(\lambda_1 \lambda_2 N_1 N_2)$ with $\delta < c_1^{-1}$ gives the result. □

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*Proof of Theorem* 0.2. First suppose that $F$ is the lift of a form $f$ which is itself a lift from GL(1). In this case, $L(s, F)$ is a product of a quadratic Dirichlet $L$-series and a Hecke $L$-series defined over a quadratic field. It is easy to see that a Siegel zero (if it exists) can only arise from the Dirichlet $L$-series, and then $L(1, F)$ can be bounded below by Siegel's original theorem. (See the appendix for further details.) Thus we may assume for the remainder of the proof that all $F$ are lifts of nonmonomial GL(2) cusp forms.

Suppose we are given $\varepsilon > 0$. There are two cases to consider: either $L(s, F)$ never has a zero $\beta$ with $1 - \beta < \varepsilon$, or else there is an $F$ with this property. In the first case, using equation (1.6) from Section 1, we have the bound

$$L(1, F)^{-1} \ll_\varepsilon (\lambda N)^\varepsilon$$

with an effective constant, which is the desired result. In the second case, let $F_1$ be the adjoint square lift of a Maass form $f_1$ such that $L(\beta_1, F_1) = 0$ with $1 - \varepsilon < \beta_1 < 1$. We may assume that $\lambda_1 N_1$ is minimal, subject to this condition. Now let $F_2$ be arbitrary. If $\lambda_2 N_2 < \lambda_1 N_1$, then $L(s, F_2) \neq 0$ in the range $1 - \varepsilon \leq s < 1$, and again by (1.6) we are done. Suppose that $\lambda_2 N_2 \geq \lambda_1 N_1$. If $L(s, F_2)$ has no real zero within $c/\log(\lambda_1\lambda_2 N_1 N_2)$ of 1, where $c$ is the constant in Lemma 3.3, then since $\lambda_2 N_2 \geq \lambda_1 N_1$, $L(s, F_2)$ has no real zero within $\frac{1}{2}c/\log(\lambda_2 N_2)$ of 1, and thus Theorem 0.1 gives the desired result. Thus we may assume that $L(s, F_2)$ has a real zero $\beta_2$ in this interval. We now suppose that $F_2 \neq F_1$, so that we may use the function $\varphi(s)$. Then $\varphi(\beta_2) = 0$, and since $\beta_1$ is already a zero of $\varphi(s)$, it follows from Lemma 3.3 that

$$1 - \beta_1 \geq \frac{c}{\log(\lambda_1\lambda_2 N_1 N_2)}. \tag{3.6}$$

Now by Lemma 3.1, $\varphi(s)$ is an $L$-series of the type considered in Section 1, so we may apply Proposition 1.1 with $M = (\lambda_1\lambda_2 N_1 N_2)^A$, for some absolute constant $A$. Equation (1.6) then gives

$$1 \ll \frac{L(1, F_1) L(1, F_2) L(1, F_1 \times F_2)(\lambda_1\lambda_2 N_1 N_2)^{C(1-\beta_1)}}{1 - \beta_1},$$

where we have used the fact that $\varphi(\beta_1) = 0$. Now from (3.6) we get

$$\varepsilon \geq 1 - \beta_1 \geq \frac{c}{\log(\lambda_1\lambda_2 N_1 N_2)} \gg \frac{1}{\log(\lambda_2 N_2)},$$

so that

$$L(1, F_2)^{-1} \ll L(1, F_1) L(1, F_1 \times F_2)(\lambda_2 N_2)^\varepsilon. \tag{3.7}$$

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Now from Iwaniec's bound (0.10) we find $L(1, F_1) \ll_\varepsilon (\lambda_1 N_1)^\varepsilon \ll_\varepsilon (\lambda_2 N_2)^\varepsilon$, and Lemma 3.2 gives $L(1, F_1 \times F_2) \ll_\varepsilon (\lambda_1 \lambda_2 N_1 N_2)^\varepsilon \ll_\varepsilon (\lambda_2 N_2)^{2\varepsilon}$. Substituting these estimates into (3.7) yields

$$L(1, F_2)^{-1} \ll_\varepsilon (\lambda_2 N_2)^\varepsilon, \qquad F_2 \neq F_1,$$

with an effective constant. This completes the proof of Theorem 0.2. □

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