1) The ring of quaternion is the set

\[ H = \{a + bi + cj + dk | a, b, c, d \in \mathbb{Q} \} \]

with the addition by addition coordinate-wise and multiplication by extending the usual multiplication in \( \mathbb{Q} \) according to the formal rules:

\[ i^2 = j^2 = k^2 = -1, \quad ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik \]

Denote the ring

\[ M = \{a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} | a, b, c, d \in \mathbb{Q} \} \]

with the usual matrix addition and multiplication. Then, we claim that the map

\[ \varphi : M \rightarrow H \]

which is uniquely determined by

\[ i \rightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j \rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k \rightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \]

is an isomorphism between two rings. To check this, we need to check the three matrices satisfy the same relation between \( i, j, k \), i.e.:

\[ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

and so on.

The integer ring of \( M \) is given by

\[ \{ (a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}) / 2 | a, b, c, d \in \mathbb{Z}, a \equiv b \equiv c \equiv d \pmod{2} \} \]
or equivalently,
\[ \begin{pmatrix} \frac{a}{2} & 1+i & 1+i \\ -1+i & 1-i \end{pmatrix} + b \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \]

2) The conjugate of
\[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \]
is defined to be
\[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - b \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - c \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - d \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \]
and the norm is defined by
\[ \text{Norm} \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right) = a^2 + b^2 + c^2 + d^2 \]

3) The problem should be
"Assume \( n \geq 3 \). Prove that there are no polynomial (with not necessarily integer coefficients, actually we allow them with coefficients in \( \mathbb{C} \)) \((x(t), y(t), z(t))\) such that
\[ x(t)^n + y(t)^n + z(t)^n = 0 \]
except that all of them are constant polynomials."

The original problem is flawed because it would imply Fermat’s Last Theorem!

Now we assume there are such \( x, y, z \) and they are not constant polynomials.

Step 1. The condition implies
\[ 1 + \left( \frac{y(t)}{x(t)} \right)^n + \left( \frac{z(t)}{x(t)} \right)^n = 0 \]
Take derivatives with respect to \( t \),
\[ n \left( \frac{y(t)}{x(t)} \right)^{n-1} \frac{y'x - yx'}{x^2} + n \left( \frac{z(t)}{x(t)} \right)^{n-1} \frac{z'x - zx'}{x^2} = 0 \]
Clear denominators,
\[ y(t)^{n-1}(y'x - yx') + z(t)^{n-1}(z'x - zx') = 0 \]
(This differ from the hint in the Problem set by a sign since we have \( x(t)^n + y(t)^n + z(t)^n = 0 \) instead of \( x(t)^n + y(t)^n = z(t)^n \).)
Step 2. With loss of generality, we can assume that there is no common factor between any two of \(x(t), y(t), z(t)\) (notice that a common factor of any two of them must divide the third one). By Step 1, we have
\[
y(t)^{n-1}z(t)^{n-1}(z'x - zx')
\]
Since \(\mathbb{C}[t]\) is UFD and \(y, z\) have no common factor, It follows that
\[
y(t)^{n-1}|(z'x - zx')
\]
Hence
\[
\deg(y(t)^{n-1}) = (n-1)\deg(y) \leq \deg(z'x - zx')
\]
We claim that \((z'x - zx') \neq 0\) (this is crucial since, ONLY in this place, we need the assumption that \(x, y, z\) are not constant polynomials.). If \((z'x - zx') = 0\), we have \((\frac{z}{x})' = \frac{(z'x-xz')}{x^2} = 0\), thus \(\frac{z}{x}\) is constant, which implies that \(x, z\) have \(z\) (or \(x\)) as a common factor. Then \(z, x\) must be constant polynomials. Contradiction!

By the easy fact that \(\deg(zx') = \deg(z'x) = \deg(z) + \deg(x) - 1\) and \((z'x - zx') \neq 0\), we have
\[
\deg(z'x - zx') \leq \max\{\deg(z'x), \deg(zx')\} = \deg(z) + \deg(x) - 1
\]
(Note: we cannot have this inequality if \(z'x - zx' = 0!\) )

In summary, we have
\[
(n-1)\deg(y) \leq \deg(z) + \deg(x) - 1
\]

Step 3. Now by symmetry, we also have
\[
(n-1)\deg(z) \leq \deg(x) + \deg(y) - 1
\]
and
\[
(n-1)\deg(x) \leq \deg(y) + \deg(z) - 1
\]
Sum up,
\[
(n-1)(\deg(x) + \deg(y) + \deg(z)) \leq 2(\deg(x) + \deg(y) + \deg(z)) - 3
\]
This is impossible if \(n > 2!\)

4) Easy to get the formula
\[
V(n) = 2 \sum_{k=1}^{n} \phi(k) - 1
\]
where $\phi(k)$ is the Euler function.

So a little computation yields

$$V(10) = 63$$

And theorem 330 in Hardy-Wright implies that

$$\lim_{n \to \infty} \frac{\sum_{k=1}^{n} \phi(k)}{n^2} = \frac{3}{\pi^2}$$

Therefore,

$$\lim_{n \to \infty} \frac{V(n)}{n^2} = \frac{6}{\pi^2}$$

5) **Proof 1** We’ll use a well-known

FACT: A polygon with lattice points vertex has area given by the formula

$$\text{AREA} = \frac{m}{2} + n - 1$$

where: $m$ is the number of lattice points in the edges, $n$ is the number of lattice points inside the polygon.

As a corollary, we know that if a triangle with lattice points vertex has no lattice points in its interior or on its edges except its vertex, then its area is $1/2$ since $m = 3$, $n = 0$.

Note that there is a bijection between the set of Farey series $F_n$ of order $n$ and the set of visible points $V(n)$. And if $\frac{h}{k}, \frac{h'}{k'}$ are consecutive in $F_n$, then the triangle with vertex

$$(0,0), (h,k), (h',k')$$

contains no lattice points in its interior or on its edges except its vertex. Hence, its area is $1/2$. On the other hand, the area of this triangle is given by $\frac{1}{2} |hk' - h'k|$. Therefore, if $\frac{h}{k} < \frac{h'}{k'}$, we have $h'k - hk' = 1$.

**Proof 2** Induction on $n$. If the assertion holds up to $F_n$. Then take any two adjacent $\frac{h}{k} < \frac{h'}{k'}$ in $F_{n-1}$. By induction assumption, we have

$$h'k - hk' = 1$$

Suppose in $F_n$, there is a $\frac{h''}{k''}$ between $\frac{h}{k}$ and $\frac{h'}{k'}$. Then denote

$$h''k - hk'' = r, \quad h'k'' - k'h'' = s$$
where $0 < r, s ∈ \mathbb{Z}$. Then using $h'k - hk' = 1$ we can solve $h'', k''$ in terms $h, k, h', k', r, s$:

$$h'' = sh + rh', \quad k'' = sk + rk'$$

This implies $(r, s) = 1$ since $(h'', k'') = 1$.

Observe that all numbers of the form $\frac{uh + vh'}{uk + vk'}$ $(u, v > 0)$ are between $\frac{h}{k}$ and $\frac{h'}{k'}$. Since $\frac{h + h'}{k + k'}$ has no bigger denominator than $\frac{sh + rh'}{sk + rk'}$, and we have assumed $\frac{sh + rh'}{sk + rk'} ∈ F_n$, we must have $\frac{h + h'}{k + k'} ∈ F_n$. It is easy to see that there is at most one number in $F_n$ between $\frac{h}{k}$ and $\frac{h'}{k'}$ (since any such number is of the form $\frac{uh + vh'}{uk + vk'}$ and $uk + vk' = n$). Hence,

$$\frac{rh + sh'}{rk + sk'} = \frac{h + h'}{k + k'}$$

Then it follows that $(h + h')k - h(k + k') = h'(k + k') - k'(h + h') = 1$. This complete the induction.

**Proof 3** Take any adjacent pair $h/k < h'/k'$ in $F_n$. So we can assume $s/k \neq 1$.

Then we claim that $(h', k')$ is the unique pair of integers $(x, y)$ that satisfies the following

$$kx - hy = 1, \quad \text{and} \quad n - k < y \leq n$$

Since $(h, k) = 1$, such pair uniquely exists.

Now, such pair $(x, y)$ gives us a fraction $\frac{x}{y}$ in $F_n$ (since $y ≤ n$). And $\frac{h}{k} < \frac{x}{y}$. Hence, if $\frac{x}{y} \neq \frac{h'}{k'}$, we must have $\frac{x}{y} > \frac{h'}{k'}$. It follows that

$$\frac{x}{y} - \frac{h'}{k'} = \frac{xk' - yh'}{yk'} ≥ \frac{1}{yk'}$$

$$\frac{h'}{k'} - \frac{h}{k} = \frac{h'k - hk'}{kk'} ≥ \frac{1}{kk'}$$

Thus, we have

$$\frac{x}{y} - \frac{h}{k} = (\frac{x}{y} - \frac{h'}{k'}) + (\frac{h'}{k'} - \frac{h}{k}) ≥ \frac{1}{yk'} + \frac{1}{kk'} = \frac{y + k}{ykk'}$$

Since by our choice of $(x, y)$, we have $y + k > n$. Thus

$$\frac{x}{y} - \frac{h}{k} > \frac{n}{ykk'} ≥ \frac{1}{yk}$$

On the other hand, by our choice of $(x, y)$, we have

$$\frac{x}{y} - \frac{h}{k} = \frac{xk - yh}{yk} = \frac{1}{yk}$$

Contradiction!
Therefore, $\frac{x}{y} = \frac{h'}{k'}$. Since both are reduced fractions, $(x, y) = (h', k')$. And by our construction of $(x, y)$, we can conclude that

$$h'k - hk' = 1$$

6) (Sketch)

(1) Set a vote-counter to count the votes.

(2) There are 100 generators which keep producing 0 or 1 independently, and simultaneously the $i$-th generator transmits the number to the $i+1$-th and $i-1$-th person. (Convention: $-1 = 100, 101 = 1$) so that every person receives two numbers in every round.

(3) If the $i$-th person does not respond (i.e. does not vote), then he/she transmits to the vote-counter the sum mod 2 of the two numbers he/she receives.

(4) If the $i$-th person decides to vote, then he/she inverts what he/she should have sent to the vote-counter. And in the next round, he/she sends back to the vote-counter the the sum (+1) mod 2 of the two numbers if he/she votes "YES" ("NO").

(5) If one person starts to vote, then the vote-counter let other people wait.