A KRONECKER LIMIT FORMULA FOR CUBIC FIELDS

DANIEL BUMP AND DORIAN GOLDFELD

1. INTRODUCTION

The classical Kronecker limit formula evaluates the constant term in the Laurent expansion (about $s = 1$) of the zeta function of an imaginary quadratic field. This was used by Stark [6] to prove his conjectures (cf. also [7]) for a precise statement and further references). Hecke [2] and Siegel [5] developed analogous limit formulae for real quadratic fields, but Stark's conjecture is still unproved in this case, because the theory of complex multiplication, which plays a crucial role in the imaginary quadratic field case, has not been extended to real quadratic fields.

Although generalizations of Kronecker's limit formula have been obtained by [1], [3] and [4], these are all deficient in that they only evaluate the constant term in the Laurent expansion. This is in contrast to Stark's general conjectures, which predict that as the degree of the field increases, higher order terms in the Laurent expansion are also interesting, and may be expressed by simple closed formulae.

In this note, we present a new method for developing a Kroncecker limit formula for cubic fields, based on the theory of automorphic forms on GL(3). It appears that this method will generalzie to GL(n) and number fields of degree n. Much work remains to be done in this direction, and one can only begin to see a whole new world of limit formulae emerging into view.

2. CLASSICAL KRONECKER LIMIT FORMULAE

Let us recall the classical limit formula. Let

$$E(z,s) = \frac{1}{2}\sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}}, \qquad (z = x+iy,\ y>0),$$

$$E^*(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z,s) = E^*(z,1-s),$$

be the non-holomorphic Eisenstein series occurring in the continuous spectrum of the Laplace operator acting on the space of automorphic forms for $\mathrm{SL}(2,\mathbb{Z})$. Kronecker found the Laurent expansion of $E^*(z,s)$ about $s=0$:

$$E^*(z,s) = -\frac{1}{2s} + \frac{\gamma}{2} - \log\left(2\sqrt{\pi y}\right) - 2\log|\eta(z)| + \mathcal{O}(s), \tag{1}$$

where γ is Euler's constant and

$$\eta(z) = e^{\frac{\pi i z}{12}} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right)$$

is the Dedekind eta function.

Let $K = \mathbb{Q}\left(\sqrt{-D}\right)$ be an imaginary quadratic field with discriminant $-D < 0$. By a classical theorem of Dirichlet, the zeta function $\zeta_K(s)$ satisfies

$$\pi^{-s}\Gamma(s)\zeta_K(s) = \frac{1}{w}\left(\frac{4}{D}\right)^{\frac{s}{2}} \sum_{\substack{b^2-4ac=-D \\ -a<b\le a<c \\ \text{or } 0<b\le a=c}} E^*\left(\frac{-b+\sqrt{-D}}{2a},\ s\right) \tag{2}$$

where w is the number of roots of unity in K. The points $\frac{-b+\sqrt{-D}}{2a}$ on the right are the zeros of az^2+bz+c that lie in a fundamental domain for the modular group. These are known as *Heegner points.* Combining (1) and (2), one evaluates the special values of $\zeta_k(s)$ at $s=0$ as a trace (over Heegner points) of the logarithm of the eta function. This is Kronecker's limit formula.

The next substantial progress was made by Hecke [2], who developed the Kronecker limit formula for real quadratic fields. Let K denote a real quadratic field of discriminant $D > 0$, and let

$$\alpha \mapsto \alpha^{(j)}, \qquad (j = 1, 2),$$

be the distinct real embeddings of K. Let $\mathfrak{a}$ be a fixed ideal in K, and let α_1, α_2 be a $\mathbb{Z}$–basis of $\mathfrak{a}$. We will use the following notation for quadratic forms. If S is a positive definite $n \times n$ symmetric matrix, and if $\xi \in \mathbb{R}^n$, let

$$S[\xi] = {}^t\xi S \xi \in \mathbb{R},$$

so that $S[\xi]$ is a positive definite quadratic form. If $n = 2$, let

$$\zeta_S(s) = \frac{1}{2}\pi^{-s}\,\Gamma(s) \sum_{0\ne\xi\in\mathbb{Z}^2} S[\xi]^{-s},$$

being convergent for $\mathrm{Re}(s) > 1$ and having meromorphic continuation to all $s \in \mathbb{C}$ with simple poles at $s = 0, 1$. This Epstein zeta function is equal to $E^*(z,s)$ for some value

$z = z(S)$ depending on S. If now $t > 0$, let $A(t)$ be the matrix

$$A(t) = N\mathfrak{a}^{-\frac{1}{2}}\, D^{-\frac{1}{2}} \begin{pmatrix} \alpha_1^{(1)} & \alpha_1^{(2)} \\ \alpha_2^{(1)} & \alpha_2^{(2)} \end{pmatrix} \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}.$$

Let $S(t) = A(t) \cdot {}^t A(t)$. Thus S is a positive definite symmetric matrix. We let U, the group of units in K, act on t in the following manner:

$$\epsilon : t \;\mapsto\; \left|\epsilon^{(1)} t\right|, \qquad (0 < t \in \mathbb{R},\ \epsilon \in U).$$

We have

$$\int\limits_{\mathbb{R}^+/U} \zeta_{S(t)}(s)\, \frac{dt}{t} = \frac{1}{2}\pi^{-s}\,\Gamma(s) N\mathfrak{a}^s D^{\frac{s}{2}} \int\limits_{\mathbb{R}^+/U} \sum_{0\neq\xi\in\mathfrak{a}} \left[\left(\xi^{(1)} t\right)^2 + \left(\xi^{(2)} t^{-1}\right)^2\right]^{-s} \frac{dt}{t}$$

$$= \pi^{-s}\Gamma(s) N\mathfrak{a}^s D^{\frac{s}{2}} \sum_{0\neq\xi\,\in\,\mathfrak{a}/U} \int\limits_0^\infty \left[\left(\xi^{(1)} t\right)^2 + \left(\xi^{(2)} t^{-1}\right)^2\right]^{-s} \frac{dt}{t} \tag{3}$$

$$= \frac{1}{4}\pi^{-s} D^{\frac{s}{2}}\Gamma(s/2)^2 \zeta_K\left(\bar{s},\, \mathfrak{a}^{-1}\right).$$

Here $\zeta_K\left(s,\, \mathfrak{a}^{-1}\right)$ denotes the zeta function of K associated with the ideal class $\mathfrak{a}^{-1}$. This is Hecke's representation of the zeta function of a real quadratic field as an integral of the $\mathrm{SL}(2,\mathbb{Z})$ Eisenstein series over a fundamental domain for the unit group, or what we call a *Heegner cycle.* Combining (1) and (3), one obtains the constant term in the Laurent expansion of $\zeta_K\left(s,\, \mathfrak{a}^{-1}\right)$ as the integral of a Heegner cycle of the logarithm of the Dedekind eta function.

3. LIMIT FORMULAE FOR CUBIC FIELDS

The new limit formula depends on recognizing the integral Eisenstein series over a Heegner cycle as the Rankin–Selberg integral of a Hilbert modular Eisenstein series.

With S a 3×3 positive definite symmetric matrix, let

$$\zeta_S(\nu_1, \nu_2) = \pi^{-3\nu_1 - 3\nu_2 + \frac{1}{2}}\, \Gamma\left(\frac{3\nu_1}{2}\right) \Gamma\left(\frac{3\nu_2}{2}\right) \Gamma\left(\frac{3\nu_1 + 3\nu_2 - 1}{2}\right) \cdot \zeta(3\nu_1 + 3\nu_2 - 1) \sum_{\substack{0\neq\xi\,\in\,\mathbb{Z}^3 \\ 0\neq\eta\,\in\,\mathbb{Z}^3 \\ {}^t\xi\eta\,=\,0}} S[\xi]^{-\frac{3\nu_1}{2}}\, S[\eta]^{-\frac{3\nu_2}{2}}.$$

This series is absolutely convergent for $\mathrm{Re}(\nu_1), \mathrm{Re}(\nu_2) > \frac{1}{2}$, and has meromorphic continuation to all $\nu_1, \nu_2 \in \mathbb{C}$. In fact, this is an Eisenstein series for $\mathrm{SL}(3, \mathbb{Z})$.

Now, let K be a totally real cubic field with discriminant D, and fix an ideal $\mathfrak{a} \subset K$ with $\mathbb{Z}$–basis $\alpha_1, \alpha_2, \alpha_3$. Let $\beta_1, \beta_2, \beta_3$ be the dual basis with respect to the trace bilinear form of $(\mathfrak{a}\mathfrak{d})^{-1}$ where $\mathfrak{d}$ is the different of K. We consider, in analogy with the case of a real quadratic field, the matrix

$$A(t_1, t_2) = D^{-\frac{1}{6}} N\mathfrak{a}^{-\frac{1}{3}} \begin{pmatrix} \alpha_1^{(1)} & \alpha_1^{(2)} & \alpha_1^{(3)} \\ \alpha_2^{(1)} & \alpha_2^{(2)} & \alpha_2^{(3)} \\ \alpha_3^{(1)} & \alpha_3^{(2)} & \alpha_3^{(3)} \end{pmatrix} \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_1^{-1} t_2^{-1} \end{pmatrix},$$

with $t_1, t_2 > 0$. Let $S(t_1, t_2) = A(t_1, t_2) \cdot {}^t A(t_1, t_2)$. The group U of units in K acts on the parameters t_1, t_2 as follows:

$$\epsilon : (t_1, t_2) \mapsto \left(\left|\epsilon^{(1)}\right| t_1, \ \left|\epsilon^{(2)}\right| t_2 \right), \qquad (\epsilon \in U, \ 0 < t_1, t_2). \tag{4}$$

As in the quadratic case, we may consider the integral

(5)

$$\int\limits_{(\mathbb{R}^+)^2/U} \zeta_{S(t_1,t_2)}(\nu_1, \nu_2) \, \frac{dt_1}{t_1} \frac{dt_2}{t_2} = \frac{\pi^{-3\nu_1 - 3\nu_2 + \frac{1}{2}} \, D^{\frac{\nu_1 - \nu_2}{2}}}{2} \, \Gamma\left(\frac{3\nu_1}{2}\right) \Gamma\left(\frac{3\nu_2}{2}\right) \Gamma\left(\frac{3\nu_1 + 3\nu_2 - 1}{2}\right)$$

$$\cdot \, \zeta\big(3\nu_1 + 3\nu_2 - 1\big) \sum_{\substack{0 \neq \alpha \in \mathfrak{a}/U \\ 0 \neq \beta \in (\mathfrak{a}\mathfrak{d})^{-1} \\ \mathrm{tr}(\alpha\beta) = 0}} \int_0^\infty \int_0^\infty \left[\left(\alpha^{(1)} t_1\right)^2 + \left(\alpha^{(2)} t_2\right)^2 + \left(\alpha^{(3)} t_1^{-1} t_2^{-1}\right)^2 \right]^{-\frac{3\nu_1}{2}}$$

$$\cdot \left[\left(\beta^{(1)} t_1^{-1}\right)^2 + \left(\beta^{(2)} t_2^{-1}\right)^2 + \left(\beta^{(3)} t_1 t_2\right)^2 \right]^{-\frac{3\nu_2}{2}} \frac{dt_1}{t_1} \frac{dt_2}{t_2}.$$

In contrast with the real quadratic case, it is not possible to separate out the zeta function of an ideal class from the complicated expression on the right hand side – what happens here is far more interesting. We will identify this integral as a $\mathrm{GL}(2)$ convolution of an $\mathrm{SL}(2, \mathbb{Z})$ Eisenstein series with a Hilbert modular Eisenstein series. Firstly, however, let us state our final result.

For $z = x + iy, \ y > 0$, let

$$\widehat{\eta}(z) = \log |\eta(z)| + \frac{\pi y}{12},$$

where $\eta(z)$ is Dedekind's eta function. Thus $\widehat{\eta}(z)$ tends rapidly to zero as $y \to \infty$.

Let $g \in \mathrm{GL}(3,\mathbb{R})$. We may write (uniquely)

$$g = \begin{pmatrix} y_1 y_2 & y_1 x_2 & x_3 \\ 0 & y_1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} \cdot k,$$

where k is an orthogonal similitude, and $y_1, y_2 > 0$. Let $\tau_2 = x_2 + iy_2$. Define

$$\begin{aligned} c(g) \; &= \; -\frac{\gamma}{2} + \log\left(2\sqrt{\pi}\,\right) + \frac{1}{3}\log\left(y_1^2 y_2\right) - y_1\sqrt{y_2}E^*\left(\tau_2,\ \frac{3}{2}\right) \\ &\quad + \sum_{(c,d)=1} \widehat{\eta}\Big(cx_3 + dx_1 + \, i \cdot |c\tau_2 + d| \cdot y_1\Big). \end{aligned} \tag{6}$$

It may be shown that $c(g)$ is automorphic with respect to $\mathrm{SL}(3,\mathbb{Z})$, and indeed is a coefficient in the Laurent development at $\nu_1 = \nu_2 = 0$ of the $\mathrm{SL}(3,\mathbb{Z})$ Eisenstein series $\zeta_S(\nu_1,\nu_2)$ where $S = \det(g)^{-\frac{1}{6}} \cdot g \cdot {}^t g$:

$$g = D^{-\frac{1}{6}} N\mathfrak{a}^{-\frac{1}{3}} \begin{pmatrix} \alpha_1^{(1)} & \alpha_1^{(2)} & \alpha_1^{(3)} \\ \alpha_2^{(1)} & \alpha_2^{(2)} & \alpha_2^{(3)} \\ \alpha_3^{(1)} & \alpha_3^{(2)} & \alpha_3^{(3)} \end{pmatrix} \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_1^{-1}t_2^{-1} \end{pmatrix}. \tag{7}$$

We may now state our main theorem.

Theorem 3.1. *If R is the regulator of K, we have*

$$\pi^{-\frac{3s}{2}}\Gamma\left(\frac{s}{2}\right)^3 D^{\frac{s}{2}}\,\zeta_K(s,\ \mathfrak{a}) \; = \; -\frac{4R}{s} \, + \, L_A \, + \, \cdots$$

where, with $c(g)$ as in (6) and $g = g(t_1,t_2)$ as in (7), and with the action of the group U of units in K on t_1, t_2 as in (4),

$$L_A = -12 \iint\limits_{(\mathbb{R}^+)^2/U} c(g)\ \frac{dt_1}{t_1}\frac{dt_2}{t_2}.$$

This theorem is obtained from the identity (involving equation (5)) alluded to previously. We now describe this identity. If $z = x + iy,\ y > 0$, we may consider

$$E_K^*(z,s;\mathfrak{a}) = \pi^{-3s}\Gamma(s)^3 D^s N\mathfrak{a}^{2s} \sum_{\substack{(0,0)\neq(\alpha,\beta) \\ (\alpha,\beta)\,\in\,\mathfrak{a}\times\mathfrak{a}/U}} \prod_{j=1}^{3} \frac{y^s}{\left|\alpha^{(j)}z + \beta^{(j)}\right|^{2s}}.$$

This is the restriction to the diagonal of a Hilbert modular Eisenstein series. This series is convergent for $\mathrm{Re}(s) > 1$, and has meromorphic continuation to all s.

In [8], Zagier considers the convolution

$$\iint\limits_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{H}} E_K^*(z,\nu;\mathfrak{a})E^*(z,s)\,\frac{dxdy}{y^2} \tag{8}$$

and discusses its analytic properties as a function of the two complex variables s, ν. (Actually, the integral (8) is divergent; however, Zagier's theory of the Rankin-Selberg method assigns meaning to such divergent integrals.) He finds that the polar divisor of (8) consists of the six lines:

$$\begin{aligned} &s = 0, && s = 1, \\ &s = 3\nu, && s = 1-3\nu, \\ &s = 3-3\nu, && s = -2+3\nu, \end{aligned} \tag{9}$$

each with multiplicity one.

Setting

$$\begin{aligned} s &= \frac{3}{2}\Big(\nu_1+\nu_2-\frac{1}{3}\Big), & \nu_1 &= \frac{1}{3}\big(s+3\nu-1\big), \\ \nu &= \frac{1}{2}\big(\nu_1-\nu_2+1\big), & \nu_2 &= \frac{1}{3}\big(s-3\nu+2\big), \end{aligned}$$

we have the following identity for the left–hand side of equation (5):

$$\int\limits_{(\mathbb{R}^+)^2/U} \zeta_{S(t_1,t_2)}\big(\nu_1,\nu_2\big)\,\frac{dt_1}{t_1}\frac{dt_2}{t_2} = \frac{1}{12}\iint\limits_{\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{H}} E_K^*(z,\nu;\mathfrak{a})E^*(z,s)\,\frac{dxdy}{y^2}. \tag{10}$$

It is interesting to note that (10) implies, for the right–hand side, *'extra'* functional equations of $E_K^*(z,\nu;\mathfrak{a})$ and $E^*(z,s)$ for on the left–hand side, $\zeta_{S(t_1,t_2)}\big(\nu_1,\nu_2\big)$ satisfies the functional equations of Langlands. Roughly speaking, any symmetry of the hexagon comprised of the six lines (9) is a functional equation of the right–hand side of (10).

Our main theorem follows from comparing the Laurent expansions at $\nu_1 = \nu_2 = 0$ in (10).

Our proof of (10), which is purely computational, leaves the origin of this identity somewhat mysterious. S. Kudla, in private communication, has stated that (10) arises from the existence of a dual reductive *'see–saw'* pair in $Sp(6)$. It would be of considerable interest to generalize (10) in the context of $\mathrm{GL}(n)$.

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