A FURTHER IMPROVEMENT OF THE BRUN–TITCHMARSH THEOREM

DORIAN M. GOLDFELD†

1. *Introduction*

If $\pi(x; q, a)$ denotes the number of primes p not exceeding x congruent to $a \pmod q$, then Titchmarsh [6] proved using Brun's sieve that there exists a positive constant B such that

$$\pi(x; q, a) < B \frac{x}{\phi(q) \log (x/q)} \quad \text{if} \quad q \leqslant x.$$

Results of comparable quality cannot be obtained by analytic methods for $q > \sqrt{x}$ even if one assumes the so-called generalized Riemann hypothesis.

There has been considerable interest in recent years in reducing the value of B and H. L. Montgomery [4] has even obtained $B = 2$ by a very elegant form of the large sieve. The first improvement on $B = 2$ (but only an average) was obtained by Hooley, who proved that for almost all $a \pmod q$

$$\pi(x; q, a) \leqslant (2+\varepsilon) \frac{x}{\phi(q) \log (x/\sqrt{q})} \quad \text{if} \quad 1 \leqslant q \leqslant x^{2/3}$$

$$\pi(x; q, a) \leqslant (1+\varepsilon) \frac{x}{\phi(q) \log (x/q)} \quad \text{if} \quad x^{2/3} \leqslant q \leqslant x^{1-\varepsilon}$$

and for fixed a and almost all q with $Q \leqslant q < 2Q$,

$$\pi(x; q, a) \leqslant \begin{cases} \dfrac{(1+\varepsilon)\, x}{\phi(q) \log \{(x^2/Q)^{1/6}\}} & \text{if} \quad x^{1/2} \leqslant Q \leqslant x^{4/5} \\[2ex] \dfrac{(1+\varepsilon)\, x}{\phi(q) \log (x/Q)} & \text{if} \quad x^{4/5} \leqslant Q \leqslant x^{1-\varepsilon}. \end{cases}$$

More recently, Motohashi [5] has improved Hooley's first result and obtained

$$\pi(x; q, a) \leqslant 2(1+2\varepsilon) \frac{x}{\phi(q) \log (x/\sqrt{q})} \quad \text{if} \quad q \leqslant x^{1-\varepsilon}$$

except for at most $q^{1-\varepsilon/5}$ residue classes $a \pmod q$.

This result was obtained by a very ingenious combination of the Selberg sieve and of the large sieve, and it even enabled Motohashi to improve on the value $B = 2$ in

Received 1 November, 1973.

† Supported in part by National Science Foundation grant GP–36418X1.

[J. London Math. Soc. (2), 11 (1975), 434–444]

certain ranges of q uniformly for all $a \pmod q$. In particular, he proved

$$\pi(x; q, a) \leqslant \frac{x}{\phi(q)\log(x/q^{5/3})}\left(1+O\left(\frac{\log\log x}{\log x}\right)\right) \quad \text{if} \quad x^{6/17} \leqslant q \leqslant x^{3/7},$$

$$\pi(x; q, a) \leqslant 2\frac{x}{\phi(q)\log(x/\sqrt{q})}\left(1+O\left(\frac{\log\log x}{\log x}\right)\right) \quad \text{if} \quad x^{4/17} \leqslant q \leqslant x^{6/17},$$

$$\pi(x; q, a) \leqslant (1+\varepsilon)\frac{x}{\phi(q)\log(x^{7/11}/q^{73/88})} \quad \text{if} \quad x^{56/249} \leqslant q \leqslant x^{4/17},$$

and

$$\pi(x; q, a) \leqslant 2(1+\varepsilon)\frac{x}{\phi(q)\log(x/q^{26/56})} \quad \text{if} \quad 1 \leqslant q \leqslant x^{56/249}.$$

By a slightly different integration of the Selberg and large sieves and also the deep theorem of Burgess on the size of the L-functions in the critical strip (obtained by making use of the known Riemann hypothesis for function fields) we shall obtain the following improvements on Motohashi's theorems. The author would like to add, however, that he has since heard that Motohashi has already obtained the first part of Theorem (1) in the range $q \leqslant x^{1/3}$.

THEOREM 1. *Let $\varepsilon > 0$ be arbitrarily small; then there exist positive constants c_1, c_2, c_3, c_4, independent of ε, such that*

$$\pi(x; q, a) \leqslant (2+c_1\varepsilon)\frac{x}{\phi(q)\log(x/q^{3/8})} \quad \textit{if} \quad 1 \leqslant q \leqslant x^{(24/71)-c_2\varepsilon}$$

$$\pi(x; q, a) \leqslant (1+c_3\varepsilon)\frac{x}{\phi(q)\log(x/q^{\alpha(\delta)})} \quad \textit{if} \quad x^{B(\delta)-c_4\varepsilon} \leqslant q \leqslant x^{1/2},$$

where, for each $0 \leqslant \delta \leqslant \frac{1}{2}$,

$$\alpha(\delta) = \frac{19+10\delta}{8+16\delta}, \qquad B(\delta) = \frac{8+16\delta}{27+26\delta},$$

and these results hold uniformly for all $a \pmod q$. In particular, for $\delta = \frac{1}{2}$,

$$\pi(x; q, a) \leqslant (1+c_3\varepsilon)\frac{x}{\phi(q)\log(x/q^{3/2})} \quad \textit{if} \quad x^{2/5-c_4\varepsilon} \leqslant q \leqslant x^{1/2}.$$

2. *Notation*

In the proof following we shall introduce $\varepsilon > 0$ to be an arbitrarily small positive number, and the positive numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3 \ldots$ will depend at most on ε and perhaps each other and be subject to the condition $0 < \varepsilon_i < c\varepsilon$ for some absolute constant c and $i = 1, 2, \ldots$. We also let $\tau_r(n)$ denote the number of representations of n as a product of r factors. As usual, $\chi \pmod q$ represents a Dirichlet character to the modulus q.

3. *The Selberg Sieve Method*

For each integer d, let λ_d be an arbitrary real number and define for $(a, q) = 1$

$$S_k(x; q, a) = \frac{1}{k!} \sum_{\substack{n \leqslant x \\ n \equiv a \bmod q}} \left(\log \frac{x}{n}\right)^k \left(\sum_{\substack{d \mid n \\ d \mid P(z)}} \lambda_d\right)^2, \tag{1}$$

where $P(z)$ is the product of all primes less than z. Then, as noted by Selberg,

$$\pi_k(x; q, a) \leqslant \left(\frac{z}{q}+1\right)(\log x)^k + S_k(x; q, a) \tag{2}$$

as long as $\lambda_1 = 1$, where we have defined

$$\pi_k(x; q, a) = \frac{1}{k!} \sum_{\substack{p \leqslant x \\ p \equiv a \pmod q}} \left(\log \frac{x}{p}\right)^k. \tag{3}$$

In accordance with Selberg's method, the values of λ_d minimizing (1) and subject to the condition $\lambda_1 = 1$ are given by

$$\lambda_d = \begin{cases} \dfrac{1}{Y} \dfrac{\mu(d)\, d}{\phi(d)} \displaystyle\sum_{\substack{r \leqslant z/d \\ r \mid P(z) \\ (r, dq) = 1}} \dfrac{\mu^2(r)}{\phi(r)} & \text{if } \ (q, d) = 1,\ d \leqslant z, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$Y = \frac{\phi(q)}{q} \log z. \tag{4}$$

Here, as is well known,

$$|\lambda_d| \leqslant 1. \tag{5}$$

The function $S_k(x; q, a)$ can be expressed by the usual Mellin transform as

$$S_k(x; q, a) = \frac{1}{2\pi i \phi(q)} \sum_{\chi \bmod q} \bar{\chi}(a) \int_{(\alpha)} L(s, \chi)\, K(s, \chi) \frac{x^s}{s^{k+1}}\, ds, \tag{6}$$

where (α) is any vertical line with $\alpha > 1$, and

$$K(s, \chi) = \sum_{d_1, d_2 \leqslant z} \frac{\lambda_{d_1} \lambda_{d_2}}{([d_1, d_2])^s} \chi([d_1, d_2]). \tag{7}$$

Now, it follows from (6) that if the line of integration is shifted to a line (α) where $\alpha < 1$, say, then

$$S_k(x; q, a) - \frac{x}{\phi(q) \log z} = \frac{1}{2\pi i \phi(q)} \sum_{\chi \bmod q} \bar{\chi}(a) \int_{(\alpha)} L(s, \chi)\, K(s, \chi) \frac{x^s}{s^{k+1}}\, ds \tag{8}$$

since the L-function formed with the trivial character will contribute a pole with residue $\phi(q)/q$ at $s = 1$. The contribution of this pole will then be

$$1/q\, K(1,\chi_0)\, x$$

and equation (8) follows from (4) and the fact that

$$K(1, \chi_0) = 1/Y.$$

4. *Some Lemmas*

The present paper is essentially based on the following four lemmas. Lemma 2 is especially important and is based on Burgess' work on character sums, while the other lemmas are consequences of the large sieve method.

LEMMA 1. (Gallagher [2]). *Let $T > 1$; then for any complex numbers a_n and positive M, N*

$$\sum_{\chi \bmod q} \int_{-T}^{T} \Big| \sum_{M < m \leqslant M+N} a_n \chi(n) n^{it} \Big|^2 dt \ll (qT + N) \sum_{M < m \leqslant M+N} |a_n|^2.$$

LEMMA 2. (Burgess [1]). *Let* $\mathrm{Re}(s) = \tfrac{1}{2} - \delta$ *and* $0 \leqslant \delta \leqslant \tfrac{1}{2}$; *then for every* $\varepsilon > 0$

$$|L(s, \chi)| \ll q^{(3+10\delta+\varepsilon)/16}\, |s|,$$

where the constant implied by $\ll$ depends at most on ε.

The following two lemmas can be deduced from the second and fourth power mean-value theorems for Dirichlet L-functions and the functional equations for the L-functions. A proof of the fourth power mean-value theorem can be found in [4], (and see also [5]).

LEMMA 3. *Let* $0 \leqslant \delta \leqslant \tfrac{1}{2}$; *then*

$$\sum_{\chi \bmod q} \int_{-T}^{T} |L(\tfrac{1}{2} - \delta + it, \chi)|^2\, dt \ll (qT)^{1+2\delta} (\log qT)^2.$$

LEMMA 4. *With the same conditions as above,*

$$\sum_{\chi \bmod q} \int_{-T}^{T} |L(\tfrac{1}{2} - \delta + it, \chi)|^4\, dt \ll (qT)^{1+4\delta} (\log qT)^4.$$

5. *The Basic Idea of the Proof*

The basic idea in the present method can be illustrated in the following heuristic argument. Let us assume for the moment that $K(s, \chi)$ can be written in the following form

$$K(s, \chi) = \left(\sum_{d \leqslant z} \frac{\Lambda_d \chi(d)}{d^s} \right)^2 = B(s, \chi)^2$$

where $|\Lambda_d| \leqslant 1$. It would then follow from (8) and Lemma 2 that for $k = 2$

$$\left| S_k(x; q, a) - \frac{x}{\phi(q) \log z} \right| \ll \frac{x^{1/2} q^{(3+\varepsilon)/16}}{\phi(q)} \sum_{\chi \bmod q} \int_{(1/2)} |B(s, \chi)|^2 \frac{|ds|}{|s|^{k+1}}$$

$$\ll x^{1/2} q^{(3+\varepsilon)/16} (1 + z/q) \log^2 x \tag{9}$$

by Lemma 1.

Now, choosing

$$z = \left(\frac{x^{1/2}}{q^{3/16}} \right)^{1-\varepsilon}$$

assures that the right side of (9) is

$$O\left(\frac{x^{1-\varepsilon}}{\phi(q) \log z} \right)$$

as long as $q < x^{(8/19)-\varepsilon}$, so that by (2) it can be deduced that

$$\pi(x; q, a) = \pi_0(x; q, a) \leqslant (2+\varepsilon) \frac{x}{\phi(q) \log (x/q^{3/8})}$$

for all $q < x^{(8/19)-\varepsilon}$.

The transition from $\pi_k(x; q, a)$ to $\pi_0(x; q, a)$ is achieved by noting that since $S_k(x; q. a)$ is an increasing function of x, we get, for $0 < u \leqslant 1$,

$$\frac{1}{u} \int_{e^{-u}x}^{x} S_{k-1}(y; q, a) \frac{dy}{y} \leqslant S_{k-1}(x; q, a) \leqslant \frac{1}{u} \int_{x}^{e^{-u}x} S_{k-1}(y; q, a) \frac{dy}{y}$$

so that

$$(1/u) [S_k(x; q, a) - S_k(e^{-u}x; q, a)] \leqslant S_{k-1}(x; q, a) \leqslant (1/u) [S_k(e^u x; q, a) - S_k(x; q, a)];$$

this implies that

$$S_{k-1}(x; q, a) = (1 + O(u)) \frac{x}{\phi(q) \log z} + O\left(u^{-1} \frac{x^{1-t}}{\phi(q) \log z} \right),$$

from which, choosing $u = x^{-\varepsilon_1}$ for suitable ε_1, the result follows by a simple backward induction starting at $k = 2$.

Similarly, choosing lines of integration other than $\frac{1}{2}$ in the above heuristic argument leads to improvements of the Brun–Titchmarsh theorem for other ranges of q. Unfortunately, the assumption that Lemma 1 can be directly applied to $K(s, \chi)$ is not valid, and following Motohashi [5], we replace $K(s, \chi)$ by a sum of squares, an idea originating with Selberg. This sum of squares is then divided into three parts and a different line of integration and also different method applied to each. The simple upper bound (9) is then replaced by three such sums in six variables, the minimization of which leads to the final results.

6. *Proof of Theorem*

We note that $K(s, \chi)$ may be written as a sum of squares in the following way.

$$K(s, \chi) = \sum_{u \leqslant z} \frac{\chi(u)}{u^s} \sum_{v \leqslant z/u} \frac{\mu(v)}{v^{2s}} \chi(v^2) H^2\left(s, \chi, \frac{z}{uv}\right), \tag{10}$$

where

$$H\left(s, \chi, \frac{z}{d}\right) = \sum_{d_1 \leqslant z/d} \frac{\chi(d_1)}{d_1^{\,s}} \lambda_{dd_1}. \tag{11}$$

Let us now split $K(s, \chi)$ into three sums in the following way

$$K(s, \chi) = \sum_{u \leqslant z_1} (\ ) + \sum_{z_1 < u \leqslant z_2} (\ ) + \sum_{z_2 < u \leqslant z} (\ )$$

$$= K_1(s, \chi) + K_2(s, \chi) + K_3(s, \chi),$$

and for $j = 1, 2, 3$, we let (for $0 \leqslant \delta_j \leqslant \frac{1}{2}$)

$$I_j = \frac{1}{2\pi i \phi(q)} \sum_{\chi \bmod q} \int_{(1/2 - \delta_j)} L(s, \chi) K_j(s, \chi) \frac{x^s}{s^{k+1}} ds$$

so that, by (8),

$$\left| S_k(x; q, a) - \frac{x}{\phi(q) \log z} \right| \ll \sum_{j=1}^{3} |I_j|. \tag{12}$$

In all that follows we can take $k = 2$.

Case 1. It follows immediately from (10) and Lemma 2 and also (11) and Lemma 1 that

$$|I_1| \ll \frac{x^{1/2-\delta_1} q^{(3+10\delta_1+\varepsilon)/16}}{\phi(q)} \sum_{u \leqslant z_1} \frac{1}{u^{1/2-\delta_1}} \sum_{v \leqslant z/u} \frac{1}{v^{1-2\delta_1}} \sum_{\chi} \int_{(1/2-\delta_1)} |H|^2 \frac{|ds|}{|s|^{k+1}}$$

$$\ll x^{1/2-\delta_1} q^{(3+10\delta_1+\varepsilon)/16} \sum_{u \leqslant z_1} \frac{1}{u^{1/2-\delta_1}} \sum_{v \leqslant z/u} \frac{1}{v^{1-2\delta_1}} \left(1 + \frac{z}{quv}\right) \left(\frac{z}{uv}\right)^{2\delta_1} \log^2 x$$

$$\ll x^{1/2-\delta_1} (\log x)^2 z^{2\delta_1} q^{(3+10\delta_1+\varepsilon)/16} \left(z_1^{\,1/2-\delta_1} + \frac{z}{q}\right).$$

Case 2. We first of all note that

$$|H(\tfrac{1}{2} - \delta_2 + it, \chi, z/d)| \ll (z/d)^{1/2+\delta_2}.$$

Consequently, it follows that

$$\sum_{\chi} \int_{(1/2-\delta_2)} |L(s,\chi)| \left| H^2\left(s,\chi,\frac{z}{uv}\right)\right| \frac{ds}{|s|^{k+1}}$$

$$\ll \left(\frac{z}{uv}\right)^{1/4+\delta_2/2} \sum_{\chi} \int_{(1/2-\delta_2)} |L|\cdot|H|^{3/2} \frac{|ds|}{|s|^{k+1}}$$

$$\ll \left(\frac{z}{uv}\right)^{1/4+\delta_2/2} \left(\sum_{\chi} \int_{(1/2-\delta_2)} |L|^4 \frac{|ds|}{|s|^{k+1}}\right)^{1/4} \left(\sum_{\chi} \int_{(1/2-\delta_2)} |H|^2 \frac{|ds|}{|s|^{k+1}}\right)^{3/4}$$

$$\ll \left(\frac{z}{uv}\right)^{1/4+\delta_2/2} q^{1/4+\delta_2} \left[\left(q+\frac{z}{uv}\right)\left(\frac{z}{uv}\right)^{2\delta_2}\right]^{3/4}$$

by Lemma 4 and also (11) and Lemma 1. It now follows that

$$|I_2| \ll \frac{x^{1/2-\delta_2} z^{1/4+\delta_2/2}}{\phi(q)} q^{1/4+\delta_2} \sum_{z_1<u\leqslant z_2} \frac{1}{u^{3/4-\delta_2/2}} \sum_{v\leqslant z/4} \frac{1}{v^{5/4-3\delta_2/2}} \left[\left(q+\frac{z}{uv}\right)\left(\frac{z}{uv}\right)^{2\delta_2}\right]^{3/4}$$

$$\ll x^{1/2-\delta_2} z^{1/4+2\delta_2} q^{\delta_2} \sum_{z_1<u\leqslant z_2} \frac{1}{u^{3/4+\delta_2}} \sum_{v\leqslant z/u} \frac{1}{v^{5/4}} \left[1+\frac{z^{3/4}}{(quv)^{3/4}}\right]$$

$$\ll x^{1/2-\delta_2} (\log x)^2 z^{1/4+2\delta_2} q^{\delta_2} \max_{z_1,z_2} \left(z_1{}^{1/4-\delta_2}, z_2{}^{1/4-\delta_2}\right) + \left[\frac{z^{3/4}}{q^{3/4} z^{1/2+\delta_2}}\right].$$

Case 3. The estimation of I_3 is based on the fact that K_3 may be rewritten as

$$K_3(s,\chi) = \sum_{\substack{z_2<u\leqslant z \\ uv\leqslant z \\ d_1,d_2\leqslant z/(uv)}} \frac{\chi(uv^2 d_1 d_2)}{(uv^2 d_1 d_2)^s} \mu(v) \lambda_{uvd_1} \lambda_{uvd_2}$$

$$= \sum_{n\leqslant z^2/z_2} \frac{\chi(n)\, b(n)}{n^s},$$

where

$$|b(n)| \leqslant \tau_4(n).$$

Consequently,

$$|I_3| \ll \frac{x^{1/2-\delta_3}}{\phi(q)} \left(\sum_{\chi} \int_{(1/2-\delta_3)} |L|^2 \frac{|ds|}{|s|^{k+1}} \right)^{1/2} \left(\sum_{\chi} \int_{(1/2-\delta_3)} |K_3|^2 \frac{|ds|}{|s|^{k+1}} \right)^{1/2}$$

$$\ll x^{1/2-\delta_3} q^{\delta_3} \frac{z^{2\delta_3}}{z_2^{\delta_3}} \left[1 + \frac{z}{z_2^{1/2} q^{1/2}} \right] \log^{18} x.$$

If now, in accordance with (12), one combines the upper bounds obtained in the previous three cases, one gets

$$\left| S_k(x; q, a) - \frac{x}{\phi(q) \log z} \right|$$

$$\ll x^{1/2-\delta_1} q^{(3+10\delta_1+\varepsilon)/16} z^{2\delta_1} \left[z_1^{1/2-\delta_1} + \frac{z}{q} \right] \log^2 x$$

$$+ x^{1/2-\delta_2} z^{1/4+2\delta_2} q^{\delta_2} \left[\max_{z_1, z_2} (z_1^{1/4-\delta_2} z_2^{1/4-\delta_2}) + \frac{z^{3/4}}{q^{3/4} z_1^{1/2+\delta_2}} \right] \log^2 x$$

$$+ x^{1/2-\delta_3} q^{\delta_3} \frac{z^{2\delta_3}}{z_2^{\delta_3}} \left[1 + \frac{z}{z_1^{1/2} q^{1/2}} \right] \log^{18} x. \tag{13}$$

As previously explained, the completion of the proof rests on the proper choice of the variables $z, z_1, z_2, \delta_1, \delta_2, \delta_3$ so as to minimize the above upper bound while simultaneously letting $x/(\phi(q) \log z)$ be an improvement on the previous forms of the Brun–Titchmarsh theorem. This minimization problem will be dealt with in the next and final section.

7. *Minimization and Conclusion of Proof*

In regard to the six upper bound terms of equation (13), there are only two possibilities—either the term involving z/q is largest or one of the other terms is larger. Let us assume the first alternative is the case.

If we now choose

$$z = \left(\frac{x^{1/2}}{q^{(3+10\delta_1)/(16+32\delta_1)}} \right)^{1-\varepsilon_1}, \tag{14}$$

this ensures that the term involving z/q is

$$O\left(\frac{x^{1-\varepsilon_2}}{\phi(q) \log z} \right). \tag{15}$$

With this choice of z it only remains to choose the other variables so that the right side of (13) is also bounded by (15). This will indeed be the case if we choose z_1 and z_2 so. that

$$\left(\frac{q^{(1/4+\delta_2)/(1/2+\delta_2)}}{q^{(3+10\delta_1)/(8+16\delta_1)}}\right)^{1-\varepsilon_3} \leqslant z_1 \leqslant \left(\frac{x^{1/(1-2\delta_1)}}{q^{(19+42\delta_1)/\{(8+16\delta_1)(1-2\delta_1)\}}}\right)^{1-\varepsilon_4}$$

$$(q^{(5+6\delta_1)/(8+16\delta_1)})^{1+\varepsilon_5} \leqslant z_2 \leqslant \left(\frac{x^{3/\{8(1/4-\delta_2)\}}q^{J_1}}{q^{(1+\delta_2)/(1/4-\delta_2)}}\right)^{1-\varepsilon_6},$$

where

$$J_1 = \left(\frac{3+10\delta_1}{16+32\delta_1}\right)\left(\frac{1/4+2\delta_2}{1/4-\delta_2}\right),$$

and $\delta_2 \leqslant \frac{1}{4}$. The inequality leads to

$$q \leqslant (x^{(24+48\delta_1)/(71+58\delta_1)})^{1-\varepsilon_7}. \tag{16}$$

Now, if $\delta_1 = \delta_2 = \delta_3 = 0$ and

$$z_1 = q^{1/8+\varepsilon_8}$$

$$z_2 = \left(\frac{x^{3/2}}{q^{61/16}}\right)^{1-\varepsilon_9}$$

then all our conditions are satisfied and we have

$$z = \left(\frac{x^{1/2}}{g^{3/16}}\right)^{1-\varepsilon_{10}}$$

for all

$$q \leqslant x^{24/71-\varepsilon_{11}}$$

which proves the first part of Theorem 1. Moreover, the inequality (16) assures us that we cannot improve our result for any other choices of δ_1, δ_2 or δ_3. That is, of course, in the case in which the term involving z/q is the largest term on the right side of (13). We now proceed to discuss the other possibility.

To deduce the second part of Theorem 1 let us choose $z_1 = z$ so that only the term

$$x^{1/2-\delta_1}q^{(3+10\delta_1+\varepsilon)/16}z^{2\delta_1}[z^{1/2-\delta_1}+z/q]\log^2 x$$

of equation (13) enters. If we then assume that

$$z^{1/2-\delta_1} > z/q$$

this forces us to choose

$$z = \left(\frac{x}{q^{(19+10\delta_1)/\{16(1/2+\delta_1)\}}}\right)^{1-\varepsilon_{12}}$$

and with this choice of z, we get

$$q > (x^{(8+16\delta_1)/(27+26\delta_1)})^{1-\varepsilon_{13}}$$

which proves the second part of Theorem 1.

Finally, we should just like to remark that up to this point we have not used in any significant way the three variables δ_1, δ_2, δ_3 which are the lines of integrations that arise in the three cases of the previous section. Slight improvements of Theorem 1 are possible in the range

$$x^{24/71} < q < x^{2/5}$$

if one assumes that another of the six terms in (13) is largest, and in this case a proper choice of $\delta_1, \delta_2, \delta_3$ is essential.

For example, if we assume the term involving $z^{3/4}/\{q^{3/4} z_1^{1/2+\delta_2}\}$ to be largest, and $z_1 = x/q^B$ for some B, then we see that the choice

$$z = \left(\frac{x}{q^{(1/4+\delta_2)/(1+2\delta_2)+B/2}}\right)^{1-\varepsilon_{14}}$$

ensures that (15) is satisfied for this term. Moreover, upon substituting these values into the first two terms of (13), one obtains

$$B > \frac{19+10\delta_1}{8} - \frac{1+4\delta_2}{1+2\delta_2}\,\delta_1 + \varepsilon_{15},$$

$$q > x^{I_2},$$

where

$$I_2 = (\tfrac{1}{2}+\delta_1)\left[\frac{\frac{1}{4}+\delta_2}{1+2\delta_2}\,(1+2\delta_1) + B(\tfrac{1}{2}+\delta_1) - \frac{3+10\delta_1}{16}\right]^{-1} + \varepsilon_{16},$$

and consequently, choosing $\delta_1 = 0$ and $B = (19/8)+\gamma$ for some $\gamma > 0$ gives

$$q > x^{I_3},$$

where

$$I_3 = \tfrac{1}{2}\left[\frac{\frac{1}{4}+\delta_2}{1+2\delta_2} + 1 + \frac{\gamma}{2}\right]^{-1} + \varepsilon_{16}. \tag{17}$$

Also, with choice of B, we have

$$z = \left(\frac{x}{q^{I_4}}\right)^{1-\varepsilon_{14}},$$

where

$$I_4 = \frac{\frac{1}{4}+\delta_2}{1+2\delta_2} + \frac{19}{16} + \frac{\gamma}{2}. \tag{18}$$

Taking $\delta \geqslant \frac{1}{4}$, we have only to check that

$$z_1 < z/q$$

and this occurs so long as

$$\gamma > \frac{\frac{1}{2}+2\delta_2}{1+2\delta_2} - \frac{3}{8} + \varepsilon_{17}. \tag{19}$$

Finally, choosing $z_2 = z$ proves that

$$\pi(x; q, a) \leqslant \frac{x}{\phi(q) \log z},$$

where z satisfies (18), and this holds for all q satisfying (17) and all pairs δ_2, γ satisfying (19) with $\gamma > 0$, $\frac{1}{2} \geqslant \delta_2 \geqslant \frac{1}{4}$. Taking $\delta_2 = \frac{1}{4}$ and $\gamma = 7/24 + \varepsilon_{17}$ gives

$$\pi(x; q, a) \leqslant (1+\varepsilon_{18}) \frac{x}{\phi(q) \log (x/q^{5/3+\varepsilon_{19}})}$$

for all

$$x^{24/71+\varepsilon_{20}} \leqslant q \leqslant x^{1/2},$$

which is a slight improvement on Theorem 1 for a certain range of q.

Note added in proof. The author would like to thank Professor A. Selberg for many helpful discussions which led to substantial improvements in some of the results of this paper.

References

1. D. A. Burgess, " On character sums and L-series. II ", *Proc. London Math. Soc.*, 13 (1963), 524–536.
2. P. X. Gallagher, "A large sieve density estimate near $\sigma = 1$ ", *Invent. Math.*, 11 (1970), 329–339.
3. C. Hooley, " On the Brun–Titchmarsh Theorem ", *J. Reine Angew. Math.*, 255 (1972), 60–79.
4. H. L. Montgomery, *Topics in multiplicative number theory* (Springer, Berlin, 1971).
5. Y. Motohashi, " On some improvements of the Brun–Titchmarsh Theorem ", *J. Math. Soc. Japan*, 26 (1974), 306–323.
6. E. C. Titchmarsh, "A divisor problem ", *Rend. Circ. Mat. Palermo*, 54 (1930), 414–429.

The Institute for Advanced Study,
Princeton, New Jersey 08540.