ON AN ADDITIVE PRIME DIVISOR FUNCTION OF ALLADI AND ERDŐS

DORIAN GOLDFELD

This paper is dedicated to Krishna Alladi on the occasion of his 60^{th} birthday.

ABSTRACT. This paper discusses the additive prime divisor function $A(n) := \sum_{p^\alpha || n} \alpha\, p$ which was introduced by Alladi and Erdős in 1977. It is shown that $A(n)$ is uniformly distributed (mod q) for any fixed integer $q > 1$.

1. Introduction

Let $n = \prod_{i=1}^{r} p_i^{a_i}$ be the unique prime decomposition of a positive integer n. In 1977, Alladi and Erdős [1] introduced the additive function

$$A(n) := \sum_{i=1}^{r} a_i \cdot p_i.$$

Among several other things they proved that $A(n)$ is uniformly distributed modulo 2. This was obtained from the identity

$$\sum_{n=1}^{\infty} \frac{(-1)^{A(n)}}{n^s} \;=\; \frac{2^s+1}{2^s-1} \cdot \frac{\zeta(2s)}{\zeta(s)} \tag{1}$$

together with the known zero free region for the Riemann zeta function. As a consequence they proved that there exists a constant $c > 0$ such that

$$\sum_{n \le x} (-1)^{A(n)} = \mathcal{O}\left(x\, e^{-c\sqrt{\log x \log\log x}}\right),$$

for $x \to \infty$.

The main goal of this paper is to show that $A(n)$ is uniformly distributed modulo q for any integer $q \ge 2$. Unfortunately, it is not possible to obtain such a simple identity as in (1) for the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{e^{2\pi i \frac{hA(n)}{q}}}{n^s}$$

when $q > 2$ and h, q are coprime. Instead we require a representation involving a product of rational powers of Dirichlet L-functions which will have branch points at the zeros of the L-functions.

The uniform distribution of $A(n)$ is a consequence of the following theorem (1.1) which is proved in §3. To state the theorem we require some standard notation. Let μ denote the Mobius function and let ϕ denote Euler's function. For any Dirichlet character $\chi \pmod q$

This research was partially supported by NSA grant H98230-15-1-0035.

(with $q > 1$) let $\tau(\chi) = \sum_{\ell (\text{mod } q)} \chi(\ell) e^{\frac{2\pi i \ell}{q}}$ denote the associated Gauss sum and let $L(s, \chi)$ denote the Dirichlet L-function associated to χ.

Theorem 1.1. *Let h, q be fixed coprime integers with $q > 2$. Then for $x \to \infty$ we have the asymptotic formula*

$$\sum_{n \le x} e^{2\pi i \frac{hA(n)}{q}} = \begin{cases} C_{h,q} \cdot x \, (\log x)^{-1+\frac{\mu(q)}{\phi(q)}} \Big(1 + \mathcal{O}\left((\log x)^{-1}\right)\Big) & \text{if } \mu(q) \ne 0, \\ \\ \mathcal{O}\left(x \, e^{-c_0 \sqrt{\log x}}\right) & \text{if } \mu(q) = 0, \end{cases}$$

where $c_0 > 0$ is a constant depending at most on h, q,

$$C_{h,q} = \frac{V_{h,q} \cdot \sin\left(\frac{\mu(q)\,\pi}{\phi(q)}\right)}{\pi} \Gamma\left(1 - \frac{\mu(q)}{\phi(q)}\right) \prod_{\substack{\chi \pmod q \\ \chi \ne \chi_0}} L(1, \chi)^{\frac{\tau(\overline{\chi})\chi(h)}{\phi(q)}},$$

and

$$V_{h,q} := \exp\left[-\frac{\mu(q)}{\phi(q)} \sum_{p|q} \sum_{k=1}^{\infty} \frac{1}{kp^k} + \sum_{p\,|\,q} \sum_{k=1}^{\infty} \frac{e^{\frac{2\pi i h p^k}{q}}}{k\,p^k} + \sum_{p} \sum_{k=2}^{\infty} \frac{e^{\frac{2\pi i p h k}{q}} - e^{\frac{2\pi i p^k h}{q}}}{k\,p^k} \right].$$

Theorem 1.1 has the following easily proved corollary.

Corollary 1.1.1. *Let $q > 1$ and let h be an arbitrary integer. Then*

$$\sum_{n \le x} e^{2\pi i \frac{hA(n)}{q}} = \mathcal{O}\left(\frac{x}{\sqrt{\log x}}\right).$$

The above corollary can then be used to obtain the desired uniform distribution theorem.

Theorem 1.2. *Let h, q be fixed integers with $q > 2$. Then for $x \to \infty$, we have*

$$\sum_{\substack{n \le x \\ A(n) \equiv h \pmod q}} 1 = \frac{x}{q} + \mathcal{O}\left(\frac{x}{\sqrt{\log x}}\right).$$

We remark that the error term in theorem 1.2 can be replaced by a second order asymptotic term which is not uniformly distributed (mod q).

The proof of theorem (1.1) relies on explicitly constructing an L-function with coefficients of the form $e^{2\pi i \frac{hA(n)}{q}}$. It will turn out that this L-function will be a product of Dirichlet L-functions raised to complex powers. The techniques for obtaining asymptotic formulae and dealing with branch singularities arising from complex powers of ordinary L-series were first introduced by Selberg [6], and see also Hildebrand and Tenenbaum [7] for a very nice exposition with different applications. In [3], [4], [5] one finds a larger class of additive functions where these methods can also be applied yielding similar results but with different constants.

2. On the function $L(s,\psi_{h/q})$

Let h,q be coprime integers integers with $q>1$. In this paper we shall investigate the completely multiplicative function

$$\psi_{h/q}(n) := e^{\frac{2\pi i h A(n)}{q}}.$$

Then the L-function associated to $\psi_{h/q}$ is defined by the absolutely convergent series

$$L(s,\psi_{h/q}) := \sum_{n=1}^{\infty} \psi_{h/q}(n) n^{-s}, \tag{2}$$

in the region $\Re(s)>1$, and has an Euler product representation (product over rational primes) of the form

$$L(s,\psi_{h/q}) := \prod_p \left(1-\frac{e^{\frac{2\pi i h p}{q}}}{p^s}\right)^{-1}. \tag{3}$$

The Euler product (3) converges absolutely to a non-vanishing function for $\Re(s)>1$. We would like to show it has analytic continuation to a larger region.

Lemma 2.1. *Let $\Re(s)>1$. Then*

$$\log\big(L(s,\psi_{h/q})\big) = \sum_p \sum_{k=1}^{\infty} \frac{e^{\frac{2\pi i h p^k}{q}}}{k\,p^{sk}} \ + \ T_{h,q}(s)$$

where, for any $\epsilon>0$, the function

$$T_{h,q}(s) := \sum_p \sum_{k=2}^{\infty} \frac{e^{\frac{2\pi i p h k}{q}} - e^{\frac{2\pi i p^k h}{q}}}{k\,p^{sk}}$$

is holomorphic for $\Re(s) > \frac{1}{2}+\epsilon$ and satisfies $|T_{h,q}(s)| = \mathcal{O}_\epsilon(1)$ where the $\mathcal{O}_\epsilon$-constant is independent of q and depends at most on ϵ.

Proof. Taking log's, we obtain

$$\begin{aligned}\log\big(L(s,\psi_{h/q})\big) &= \sum_p \sum_{k=1}^{\infty} \frac{e^{\frac{2\pi i p h k}{q}}}{k\,p^{sk}}\\ &= \sum_p \sum_{k=1}^{\infty} \frac{e^{\frac{2\pi i h p^k}{q}}}{k\,p^{sk}} + \sum_p \sum_{k=2}^{\infty} \frac{e^{\frac{2\pi i p h k}{q}} - e^{\frac{2\pi i p^k h}{q}}}{k\,p^{sk}}.\end{aligned}$$

Hence, we may take

$$T_{h,q}(s) = \sum_p \sum_{k=2}^{\infty} \frac{e^{\frac{2\pi i p h k}{q}} - e^{\frac{2\pi i p^k h}{q}}}{k\,p^{sk}},$$

which is easily seen to converge absolutely for $\Re(s) > \frac{1}{2}$. ∎

For $q > 2$, let χ denote a Dirichlet character $\pmod q$ with associated Gauss sum $\tau(\chi)$. We also let χ_0 be the trivial character $\pmod q$.

We require the following lemma.

Lemma 2.2. *Let $h, q \in \mathbf{Z}$ with $q > 2$ and $(h,q) = 1$. Then*

$$e^{\frac{2\pi i h}{q}} = \left(\frac{1}{\phi(q)} \sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \tau(\chi) \cdot \overline{\chi(h)} \right) + \frac{\mu(q)}{\phi(q)}$$

Proof. Since $(h,q) = 1$, it follows that for $\chi \pmod q$ with $\chi \neq \chi_0$,

$$\tau(\chi)\,\overline{\chi(h)} = \sum_{\ell=1}^{q} \chi(\ell) e^{\frac{2\pi i \ell h}{q}}.$$

This implies that

$$\begin{aligned}
\sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \tau(\chi)\,\overline{\chi(h)} &= (\phi(q) - 1)\, e^{\frac{2\pi i h}{q}} \;+\; \sum_{\substack{\ell=2 \\ (\ell,q)=1}}^{q} \left(\sum_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} \chi(\ell) \right) e^{\frac{2\pi i \ell h}{q}} \\
&= (\phi(q) - 1)\, e^{\frac{2\pi i h}{q}} \;-\; \sum_{\substack{\ell=1 \\ (\ell,q)=1}}^{q} e^{\frac{2\pi i \ell h}{q}} \;+\; e^{\frac{2\pi i h}{q}}.
\end{aligned}$$

The proof is completed upon noting that the Ramanujan sum on the right side above can be evaluated as

$$\sum_{\substack{\ell=1 \\ (\ell,q)=1}}^{q} e^{\frac{2\pi i \ell h}{q}} = \sum_{d \mid (q,h)} \mu\left(\frac{q}{d}\right) d \;=\; \mu(q).$$

□

Theorem 2.3. *Let $s \in \mathbf{C}$ with $\Re(s) > 1$. Then we have the representation*

$$L(s, \psi_{h/q}) = \left(\prod_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} L(s, \overline{\chi})^{\frac{\tau(\chi)\overline{\chi(h)}}{\phi(q)}} \right) \cdot \zeta(s)^{\frac{\mu(q)}{\phi(q)}} \cdot e^{U_{h,q}(s)},$$

where

$$U_{h,q}(s) := -\frac{\mu(q)}{\phi(q)} \sum_{p \mid q} \sum_{k=1}^{\infty} \frac{1}{k p^{sk}} \;+\; \sum_{p \mid q} \sum_{k=1}^{\infty} \frac{e^{\frac{2\pi i h p^k}{q}}}{k\, p^{sk}} \;+\; \sum_{p} \sum_{k=2}^{\infty} \frac{e^{\frac{2\pi i p h k}{q}} - e^{\frac{2\pi i p^k h}{q}}}{k\, p^{sk}}$$

Proof. If we combine lemmas (2.1) and (2.2) it follows that for $\Re(s) > 1$,

$$\begin{aligned}
\log\big(L(s,\psi_{h/q})\big) &= \sum_{p}\sum_{k=1}^{\infty} \frac{e^{\frac{2\pi i h p^k}{q}}}{k\,p^{sk}} \;+\; T_{h,q}(s)\\
&= \sum_{p\nmid q}\sum_{k=1}^{\infty} \frac{e^{\frac{2\pi i h p^k}{q}}}{k\,p^{sk}} \;+\; \sum_{p\,|\,q}\sum_{k=1}^{\infty} \frac{e^{\frac{2\pi i h p^k}{q}}}{k\,p^{sk}} + \; T_{h,q}(s)\\
&= \sum_{p\nmid q}\sum_{k=1}^{\infty} \frac{\left(\frac{1}{\phi(q)} \sum\limits_{\substack{\chi \pmod q\\ \chi\neq\chi_0}} \tau(\chi)\cdot\overline{\chi(h\,p^k)} \;+\; \frac{\mu(q)}{\phi(q)}\right)}{k\,p^{sk}} \;+\; \sum_{p\,|\,q}\sum_{k=1}^{\infty} \frac{e^{\frac{2\pi i h p^k}{q}}}{k\,p^{sk}} \;+\; T_{h,q}(s).
\end{aligned}$$

Hence

$$\begin{aligned}
\log\big(L(s,\psi_{h/q})\big) = \; &\frac{1}{\phi(q)} \sum_{\substack{\chi \pmod q\\ \chi\neq\chi_0}} \tau(\chi)\overline{\chi(h)}\,\log(L(s,\overline{\chi}) \;+\; \frac{\mu(q)}{\phi(q)}\log\big(\zeta(s)\big)\\
&- \frac{\mu(q)}{\phi(q)}\sum_{p|q}\sum_{k=1}^{\infty}\frac{1}{kp^{sk}} \;+\; \sum_{p\,|\,q}\sum_{k=1}^{\infty} \frac{e^{\frac{2\pi i h p^k}{q}}}{k\,p^{sk}} \;+\; T_{h,q}(s).
\end{aligned}$$

The theorem immediately follows after taking exponentials. □

The representation of $L(s,\psi_{h/q})$ given in theorem 2.3 allows one to analytically continue the function $L(s,\psi_{h/q})$ to a larger region which lies to the left of the line $\Re(s) = 1+\varepsilon$ ($\varepsilon > 0$). This is a region which does not include the branch points of $L(s,\psi_{h/q})$ at the zeros and poles of $L(s,\chi), \zeta(s)$.

Assume that $q > 1$ and $\chi \pmod q$. It is well known (see [2]) that the Dirichlet L-functions $L(\sigma+it,\chi))$ do not vanish in the region

$$\sigma \geq \begin{cases} 1-\frac{c_1}{\log q|t|} & \text{if } |t|\geq 1,\\ 1-\frac{c_2}{\log q} & \text{if } |t|\leq 1, \end{cases} \qquad \text{(for absolute constants } c_1, c_2 > 0), \tag{4}$$

unless χ is the exceptional real character which has a simple real zero (Siegel zero) near $s = 1$.
Similarly, $\zeta(\sigma+it)$ does not vanish for

$$\sigma \geq 1 - \frac{c_3}{\log(|t|+2)}, \qquad \text{(for an absolute constant } c_3 > 0). \tag{5}$$

Assume $q > 1$ and that there is no exceptional real character (mod q). It follows from (4) and (5) that $L(s,\psi_{h/q})$ is holomorphic in the region to the right of the contour $\mathcal{C}_q$ displayed in Figure 1.

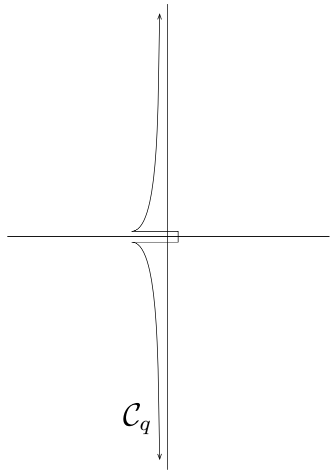

Figure 1

To construct the contour $\mathcal{C}_q$ first take a slit along the real axis from $1-\frac{c_2}{\log q}$ to 1 and construct a line just above and just below the slit. Then take two asymptotes to the line $\Re(s)=1$ with the property that if $\sigma+it$ is on the asymptote and $|t|\geq 1$, then σ satisfies (4). If $q=1$, we do a similar construction using (5).

3. Proof of theorem 1.1

The proof of theorem 1.1 is based on the following theorem.

Theorem 3.1. *Let h,q be fixed coprime integers with $q>2$ and $\mu(q)\neq 0$. Then for $x\to\infty$ there exist absolute constants $c,c'>0$ such that*

$$\sum_{n\leq x} e^{2\pi i\frac{hA(n)}{q}} = \frac{\sin\left(\frac{\mu(q)\,\pi}{\phi(q)}\right)}{\pi}\int_{1-\frac{c}{\sqrt{\log x}}}^{1}\left(\prod_{\substack{\chi \pmod q\\ \chi\neq\chi_0}} L(\sigma,\overline{\chi})^{\frac{\tau(\chi)\overline{\chi(h)}}{\phi(q)}}\right)\cdot|\zeta(\sigma)|^{\frac{\mu(q)}{\phi(q)}}\cdot e^{H_{h,q}(\sigma)}\,\frac{x^\sigma}{\sigma}\,d\sigma + \mathcal{O}\left(xe^{-c'\sqrt{\log x}}\right).$$

On the other hand if $\mu(q)=0$, then $\sum_{n\leq x} e^{2\pi i\frac{hA(n)}{q}} = \mathcal{O}\left(xe^{-c'\sqrt{\log x}}\right)$.

Proof. The proof of theorem 3.1 relies on the following lemma taken from [2].

Lemma 3.2. *Let*

$$\delta(x) := \begin{cases} 0, & \text{if } 0<x<1\\ \frac{1}{2}, & \text{if } x=1\\ 1, & \text{if } x>1,\end{cases}$$

then for $x,T>0$, we have

$$\left|\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{x^s}{s}\,ds - \delta(x)\right| < \begin{cases} x^c\cdot\min\left(1,\frac{1}{T|\log x|}\right), & \text{if } x\neq 1,\\ cT^{-1}, & \text{if } x=1.\end{cases}$$

It follows from lemma 3.2, for $x, T \gg 1$ and $c = 1 + \frac{1}{\log x}$, that

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} L\big(s, \psi_{h/q}\big) \, \frac{x^s}{s} \, ds = \sum_{n \le x} \psi_{h/q}(n) \; + \; \mathcal{O}\left(\frac{x \log x}{T}\right) \tag{6}$$

Fix large constants $c_1, c_2 > 0$. Next, shift the integral in (6) to the left and deform the line of integration to a contour

$$L^+ \; + \; \mathcal{C}_{T,x} \; + \; L^-$$

as in figure 2 below which contains two short horizontal lines:

$$L^{\pm} = \left\{ \sigma \pm iT \;\middle|\; 1 - \frac{c_1}{\log qT} \;\le\; \sigma \;\le\; 1 + \frac{1}{\log x} \right\},$$

together together with the contour $C_{T,x}$ which is similar to C_q except that the two curves asymptotic to the line $\Re(s) = 1$ go from $1 - \frac{c_1}{\sqrt{\log qT}} + iT$ to $1 - \frac{c_2}{\sqrt{\log x}} + i\varepsilon$ and $1 - \frac{c_2}{\sqrt{\log x}} - i\varepsilon$ to $1 - \frac{c_1}{\sqrt{\log qT}} - iT$, respectively, for $0 < \varepsilon \to 0$.

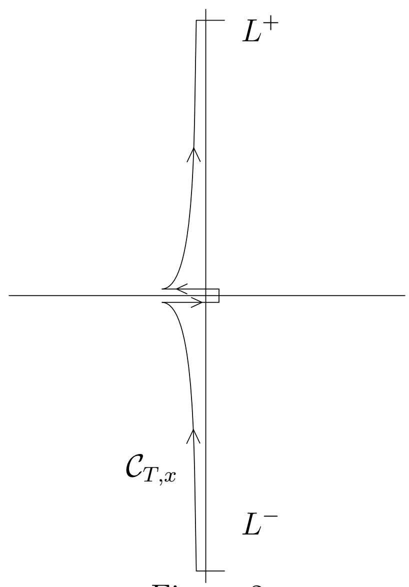

Figure 2

Now, by the zero-free regions (4), (5), the region to the right of the contour $L^+ + \mathcal{C}_{T,x} + L^-$ does not contain any branch points or poles of the L-functions $L(s, \chi)$ for any $\chi \pmod q$. It follows that

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} L\big(s, \psi_{h/q}\big) \, \frac{x^s}{s} \, ds \;=\; \frac{1}{2\pi i} \left(\int_{L^+} + \int_{\mathcal{C}_\epsilon} + \int_{L^-} \right) L\big(s, \psi_{h/q}\big) \, \frac{x^s}{s} \, ds. \tag{7}$$

The main contribution for the integral along $L^+ + \mathcal{C}_{T,x} + L^-$ in (7) comes from the integrals along the straight lines above and below the slit on the real axis $\left[1 - \frac{c_2}{\sqrt{\log x}}, \; 1\right]$. These integrals cancel if the function $L\big(s, \psi_{h/q}\big)$ has no branch points or poles on the slit. It follows from theorem 2.3 that this will be the case if $\mu(q) = 0$. The remaining integrals in 7 can then be estimated as in the proof of the prime number theorem for arithmetic progressions (see [2]), yielding an error term of the form $\mathcal{O}\left(xe^{-c'\sqrt{\log x}}\right)$. This proves the second part of theorem 3.1.

Next, assume $\mu(q) \neq 0$. In this case $L(s, \psi_{h/q})$ has a branch point at $s = 1$ coming from the Riemann zeta function, it is necessary to keep track of the change in argument. Let 0^+i denote the upper part of the slit and let 0^-i denote the lower part of the slit. Then we have $\log[\zeta(\sigma + 0^+i) = \log|\zeta(\sigma)| - i\pi$ and $\log[\zeta(\sigma + 0^-i) = \log|\zeta(\sigma)| + i\pi$.

By the standard proof of the prime number theorem for arithmetic progressions it follows that (with an error $\mathcal{O}(e^{-c'\sqrt{\log x}})$) the right hand side of (7) is asymptotic to

$$\mathcal{I}_{\text{slit}} := \frac{-1}{2\pi i} \int\limits_{1-\frac{c}{\sqrt{\log x}}}^{1} \left[\exp\Big(\log\left(L\left(\sigma + 0^+i,\ \psi_{h/q}\right)\right)\Big) - \exp\Big(\log\left(L\left(\sigma - 0^-i,\ \psi_{h/q}\right)\right)\Big)\right] \frac{x^\sigma}{\sigma}\, d\sigma. \tag{8}$$

We may evaluate I_{slit} using theorem 2.3. This gives

$$\begin{aligned}\mathcal{I}_{\text{slit}} &= \frac{-1}{2\pi i} \int\limits_{1-\frac{c}{\sqrt{\log x}}}^{1} \left(\prod_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} L(\sigma, \overline{\chi})^{\frac{\tau(\chi)\overline{\chi(h)}}{\phi(q)}}\right) \cdot e^{U_{h,q}(\sigma)} \\ &\qquad \cdot \left[\exp\left(\frac{\mu(q)}{\phi(q)}\Big(\log|\zeta(\sigma)| - i\pi\Big)\right) - \exp\left(\frac{\mu(q)}{\phi(q)}\Big(\log|\zeta(\sigma)| + i\pi\Big)\right)\right] \frac{x^\sigma}{\sigma}\, d\sigma \\ &= \frac{\sin\left(\frac{\mu(q)\,\pi}{\phi(q)}\right)}{\pi} \int\limits_{1-\frac{c}{\sqrt{\log x}}}^{1} \left(\prod_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} L(\sigma, \overline{\chi})^{\frac{\tau(\chi)\overline{\chi(h)}}{\phi(q)}}\right) \cdot |\zeta(\sigma)|^{\frac{\mu(q)}{\phi(q)}} \cdot e^{U_{h,q}(\sigma)}\ \frac{x^\sigma}{\sigma}\, d\sigma.\end{aligned}$$

As in the previous case when $\mu(q) = 0$, the remaining integrals in 7 can then be estimated as in the proof of the prime number theorem for arithmetic progressions, yielding an error term of the form $\mathcal{O}\left(xe^{-c'\sqrt{\log x}}\right)$. This completes the proof of theorem 3.1. $\square$

The proof of theorem 1.1 follows from theorem 3.1 if we can obtain an asymptotic formula for the integral

$$\mathcal{I}_{\text{slit}} = \frac{\sin\left(\frac{\mu(q)\,\pi}{\phi(q)}\right)}{\pi} \int\limits_{1-\frac{c}{\sqrt{\log x}}}^{1} \left(\prod_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} L(\sigma, \overline{\chi})^{\frac{\tau(\chi)\overline{\chi(h)}}{\phi(q)}}\right) \cdot |\zeta(\sigma)|^{\frac{\mu(q)}{\phi(q)}} \cdot e^{U_{h,q}(\sigma)}\ \frac{x^\sigma}{\sigma}\, d\sigma. \tag{9}$$

Since we have assumed q is fixed, it immediately follows that for arbitrarily large $c \gg 1$ and $x \to \infty$, we have

$$I_{\text{slit}} = \frac{\sin\left(\frac{\mu(q)\,\pi}{\phi(q)}\right)}{\pi} \int\limits_{1-\frac{c\log\log x}{\log x}}^{1} \left(\prod_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} L(\sigma,\, \overline{\chi})^{\frac{\tau(\chi)\overline{\chi(h)}}{\phi(q)}} \right) \cdot |\zeta(\sigma)|^{\frac{\mu(q)}{\phi(q)}} \cdot e^{U_{h,q}(\sigma)}\, \frac{x^\sigma}{\sigma}\, d\sigma \,+\, \mathcal{O}\left(\frac{x}{(\log x)^c}\right).$$

Now, in the region $1 - \frac{c\log\log x}{\log x} \leq \sigma \leq 1$,

$$\prod_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} L(\sigma,\, \overline{\chi})^{\frac{\tau(\chi)\overline{\chi(h)}}{\phi(q)}} \cdot \frac{e^{H_{h,q}(\sigma)}}{\sigma} = \prod_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} L(1,\, \chi)^{\frac{\tau(\overline{\chi})\chi(h)}{\phi(q)}} \cdot e^{U_{h,q}(1)} \,+\, \mathcal{O}\left(\frac{\log\log x}{\log x}\right).$$

Consequently,

$$\begin{aligned} I_{\text{slit}} \;=\; & \frac{\sin\left(\frac{\mu(q)\,\pi}{\phi(q)}\right)}{\pi} \prod_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} L(1,\, \chi)^{\frac{\tau(\overline{\chi})\chi(h)}{\phi(q)}} \cdot e^{U_{h,q}(1)} \int\limits_{1-\frac{c\log\log x}{\log x}}^{1} \zeta(\sigma)^{\frac{\mu(q)}{\phi(q)}}\, x^\sigma\, d\sigma \\ & + \mathcal{O}\left(\frac{\log\log x}{\log x} \left| \int\limits_{1-\frac{c\log\log x}{\log x}}^{1} \zeta(\sigma)^{\frac{\mu(q)}{\phi(q)}}\, x^\sigma\, d\sigma \right| \right). \end{aligned} \tag{10}$$

It remains to compute the integral of $|\zeta(\sigma)|^{\frac{\mu(q)}{\phi(q)}}$ occurring in (10). For σ very close to 1, we have

$$|\zeta(\sigma)|^{\frac{\mu(q)}{\phi(q)}} = \left(\frac{1}{|\sigma - 1|} \,+\, \mathcal{O}(1)\right)^{\frac{\mu(q)}{\phi(q)}} = \left(\frac{1}{|\sigma - 1|}\right)^{\frac{\mu(q)}{\phi(q)}} \,+\, \mathcal{O}\left(\left(\frac{1}{|\sigma - 1|}\right)^{\frac{\mu(q)}{\phi(q)}-1}\right).$$

It follows that

$$\int\limits_{1-\frac{c\log\log x}{\log x}}^{1} |\zeta(\sigma)|^{\frac{\mu(q)}{\phi(q)}}\, x^\sigma\, d\sigma \;=\; \Gamma\left(1 - \frac{\mu(q)}{\phi(q)}\right) \frac{x}{(\log x)^{1-\frac{\mu(q)}{\phi(q)}}} \,+\, \mathcal{O}\left(\frac{x}{(\log x)^{2-\frac{\mu(q)}{\phi(q)}}}\right). \tag{11}$$

Combining equations (10) and (11) we obtain

$$I_{\text{slit}} = \frac{\sin\left(\frac{\mu(q)\,\pi}{\phi(q)}\right)}{\pi} \Gamma\left(1 - \frac{\mu(q)}{\phi(q)}\right) \prod_{\substack{\chi \pmod q \\ \chi \neq \chi_0}} L(1,\, \chi)^{\frac{\tau(\overline{\chi})\chi(h)}{\phi(q)}}\, e^{U_{h,q}(1)}\, \frac{x}{(\log x)^{1-\frac{\mu(q)}{\phi(q)}}} \,+\, \mathcal{O}\left(\frac{x}{(\log x)^{2-\frac{\mu(q)}{\phi(q)}}}\right).$$

4. Examples of equidistribution (mod 3) and (mod 9)

Equidistribution (mod 3): Theorem (1.1) says that for $h = 1,\ q = 3$:

$$\sum_{n \le x} e^{\frac{2\pi i A(n)}{3}} = \frac{-V_{1,3}}{\pi}\,\Gamma\left(\frac{3}{2}\right) \prod_{\substack{\chi \pmod 3 \\ \chi \ne \chi_0}} L(1,\,\chi)^{\frac{G(\overline{\chi})}{2}} \frac{x}{(\log x)^{\frac{3}{2}}} \left(1 + \mathcal{O}\left(\frac{1}{\log x}\right)\right)$$

$$\approx (-0.503073 + 0.24042\, i)\, \frac{x}{(\log x)^{\frac{3}{2}}}.$$

We computed the above sum for $x = 10^7$ and obtained

$$\sum_{n \le 10^7} e^{\frac{2\pi i A(n)}{3}} \approx -98{,}423.00 + 55{,}650.79\, i.$$

Our theorem predicts that

$$\sum_{n \le 10^7} e^{\frac{2\pi i A(n)}{3}} \approx -88{,}870.8 + 42{,}471.7\, i.$$

Since $\log(10^7) \approx 16.1$ is very small, this explains the discrepancy between the actual and predicted results.

As $x \to \infty$, we have

$$\sum_{\substack{n \le x \\ A(n) \equiv a \pmod 3}} 1 = \frac{1}{3} \sum_{h=0}^{2} \sum_{n \le x} e^{\frac{2\pi i A(n) h}{3}} e^{-\frac{2\pi i h a}{3}}$$

$$= \frac{x}{3} + c_a \frac{x}{(\log x)^{\frac{3}{2}}} + \mathcal{O}\left(\frac{x}{(\log x)^{\frac{5}{2}}}\right)$$

where

$$c_0 = -0.335382, \qquad c_1 \approx 0.306498, \qquad c_2 \approx 0.0288842.$$

Equidistribution (mod 9):

Our theorem says that for $h \ne 3, 6$ $(1 \le h < 9)$ and $q = 9$:

$$\sum_{n \le x} e^{\frac{2\pi i h A(n)}{9}} = \mathcal{O}\left(x\, e^{-c_0\sqrt{\log x}}\right).$$

Surprisingly!! there is a huge amount of cancellation when $x = 10^7$:

$$\sum_{n \le 10^7} e^{\frac{2\pi i h A(n)}{9}} \approx \begin{cases} -315.2 - 140.4\, i & \text{if } h = 1, \\ 282.2 - 543.4\, i & \text{if } h = 2, \\ 94.5 + 321.9\, i & \text{if } h = 4, \\ 94.5 - 321.9\, i & \text{if } h = 5, \\ 282.2 + 543.4\, i & \text{if } h = 7, \\ -315.2 + 140.4\, i & \text{if } h = 8. \end{cases}$$

Acknowledgement

The author would like to thank Ada Goldfeld for creating the figures in this paper.

References

[1] Alladi, K., Erdős, P.; *On an additive arithmetic function,* Pacific Journal of Mathematics, Vol. 71, No. 2, (1977), 275–294.

[2] Davenport, H.; Multiplicative Number Theory, second edition, Graduate Texts in Math. 74, (revised by Hugh Montgomery), Springer Verlag, 1967.

[3] Ivić, A.; *On certain large additive functions, (English summary) Paul Erdős and his mathematics, I,* (Budapest, 1999), 319–331, Bolyai Soc. Math. Stud., 11, János Bolyai Math. Soc., Budapest, 2002.

[4] Ivić, Aleksandar.; Erdős, Paul.; *Estimates for sums involving the largest prime factor of an integer and certain related additive functions,* Studia Sci. Math. Hungar. 15 (1980), no. 1-3, 183–199.

[5] Ivić, Aleksandar.; Erdős, Paul.; *The distribution of quotients of small and large additive functions, II*, Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989), 83–93, Univ. Salerno, Salerno, 1992.

[6] Selberg, A,; *Note on a paper by L.G. Sathe,* J. Indian Math. Soc. (N.S.) 18, (1954), 83–87.

[7] Tenenbaum, G,: *Introduction to analytic and probabilistic number theory, Third edition*, Translated from the 2008 French edition by Patrick D. F. Ion. Graduate Studies in Mathematics, 163. American Mathematical Society, Providence, RI, 2015.

Department of Mathematics, Columbia University, New York, NY 10027
E-mail address: goldfeld@columbia.edu