Analytic Number Theory
Homework #4
(due Thursday, April 25, 2019)

Problem 1: Let $\mathfrak{h} = \big\{ z = x + iy \bigm| x \in \mathbb{R},\ y > 0 \big\}$ and let $f : \mathfrak{h} \to \mathbb{C}$ be a holomorphic modular form of weight 0 for $SL(2, \mathbb{Z})$.

- **(a)** Show that every element in $\mathfrak{h}$ is $SL(2, \mathbb{Z})$ equivalent to some $z = x + iy$ with $y \geq \frac{\sqrt{3}}{2}$. **Hint:** *Use the known fundamental domain for* $SL(2, \mathbb{Z})\backslash\mathfrak{h}$.
- **(b)** Deduce that $|f|$ attains a maximum on $\mathfrak{h}$.
- **(c)** Conclude that the only holomorphic modular forms of weight zero are the constant functions. **Hint:** *maximum modulus principle.*

Problem 2: Fix a prime p. Show that $\Gamma_0(p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \;\middle|\; c \equiv 0 \,(\mathrm{mod}\ p) \right\}$ is a subgroup of $SL(2, \mathbb{Z})$.

Problem 3: Define

$$P_k(z) := \sum_{\substack{c \in \mathbb{Z} \\ (c,d)=1}} \sum_{d \in \mathbb{Z}} \frac{e^{2\pi i \cdot \frac{az+b}{cz+d}}}{(cz+d)^k}.$$

Here, for every pair of coprime integers c, d we choose integers a, b so that $ad - bc = 1$. Show that the above series is independent of the choice of a, b. Also show that the above series converges absolutely for $k > 2$.

Problem 4: Rewrite $P_k(z)$ as a sum involving $j\,(\gamma,\ z) = cz + d$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. If $k > 2$ is an integer, prove that $P_k(z)$ is a holomorphic modular form of weight k for $SL(2, \mathbb{Z})$. **Hint:** *use the cocycle relation for j.*

Problem 5: For $s \in \mathbb{C}$ with $\Re(s) > 8$, let

$$L(s) := \sum_{n=1}^{\infty} a(n) n^{-s}, \qquad \big(a(n) \in \mathbb{C} \text{ for } n = 1, 2, \dots\big)$$

where $a(1) = 1$ and $|a(n)| \ll n^7$ (for $n = 1, 2, \dots$).

Assume that the function $\Phi(s) := (2\pi)^{-s}\Gamma(s)L(s)$ is an entire function which is bounded in any fixed vertical strip $\{s \in \mathbb{C} \mid a < \Re(s) < b\}$ and satisfies the functional equation $\Phi(s) = \Phi(12 - s)$ for all $s \in \mathbb{C}$. Using the inverse Mellin transform, prove that

$$f(z) := \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}, \qquad (z \in \mathfrak{h})$$

is the Ramanujan cusp form of weight 12. **Hint:** *It is enough to show that f satisfies $f(-1/z) = z^{12} f(z)$.*