Analytic Number Theory
Homework #3

(due Tuesday, March 20, 2018)

Problem 1: By the functional equation for

$$\xi(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

the function $s(1-s)\xi(s)$ can be regarded as an entire function of s^2-s; what is the order of this function? Use this to obtain the alternative infinite product

$$\xi(s) = \frac{\xi(1/2)}{4(s-s^2)}\prod_{\rho}\left(1-\left(\frac{s-\frac{1}{2}}{\rho-\frac{1}{2}}\right)^2\right)$$

the product extending over zeros ρ of $\xi(s)$ whose imaginary part is positive. [*This symmetrical form eliminates the exponential factors* e^{A+Bs} *and* $e^{s/\rho}$ *occurring in the usual Hadamard factorization of* $\xi(s)$.]

Problem 2: Calculate the mean value

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\zeta(\sigma+it)|^2\,dt$$

provided $\sigma > 1$.

Problem 3: Explicitly construct all Dirichlet characters (mod 15). Each such character is a completely multiplicative function $\chi : \mathbb{Z} \to \mathbb{C}$ satisfying $\chi(n+15) = \chi(n)$ for all $n \in \mathbb{Z}$.

Problem 4: Let q be an integer which has the property that every Dirichlet character χ (mod q) is real valued (takes on only the values $0, \pm 1$). Show that q must divide 24.