Analytic Number Theory
Homework #2
(due Thursday, February 22, 2018)

Problem 1: Let \( f : \mathbb{R} \to \mathbb{C} \) be in the Schwarz space of functions with rapid decay at infinity. Prove that the function

\[
F(s) := \int_1^\infty \sum_{n=1}^\infty f(ny) \frac{y^s}{y} \, dy
\]

is holomorphic for all \( s \in \mathbb{C} \).

Problem 2: Assuming the Poisson Summation Formula: \( \sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n) \)
prove that for \( y > 0 \) we have: \( \sum_{n \in \mathbb{Z}} f(ny) = y^{-1} \sum_{n \in \mathbb{Z}} \hat{f}(ny^{-1}) \).

Problem 3: Evaluate the integral

\[
\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{x^s}{s(s+1) \cdots (s+k)} \, ds
\]

for any \( \sigma > 0 \) and any integer \( k = 1, 2, 3, \ldots \) Give a sketch of the proof of your evaluation.

Problem 4: Fix \( u > 0 \). Show that the Fourier transform of \( \frac{u}{\pi(x^2+u^2)} \) is \( e^{-2\pi|x|u} \). Plug this into the Poisson summation formula to deduce that

\[
\sum_{n=-\infty}^\infty \frac{1}{n^2 + u^2} = \frac{\pi}{u} \sum_{n=-\infty}^\infty e^{-2\pi|n|u}.
\]

By letting \( u \to 0 \) prove that \( \zeta(2) = \pi^2/6 \).

Problem 5: Let \( f(x) = e^{-A|x|} \). Show that

\[
\hat{f}(y) = \frac{2A}{A^2 + 4\pi^2 y^2},
\]

\[
\hat{f}(s) = \Gamma(s) A^{-s}, \quad (\Re(s) > 0),
\]

\[
\tilde{\hat{f}}(s) = \frac{1}{2\pi} \left( \frac{2\pi}{A} \right)^{1-s} \Gamma \left( \frac{s}{2} \right) \Gamma \left( 1 - \frac{s}{2} \right), \quad (0 < \Re(s) < 2).
\]