Analytic Number Theory
Homework #2

(due Thursday, February 22, 2018)

Problem 1: Let $f : \mathbb{R} \to \mathbb{C}$ be in the Schwarz space of functions with rapid decay at infinity. Prove that the function

$$F(s) := \int_1^\infty \sum_{n=1}^\infty f(ny)\, y^s \,\frac{dy}{y}$$

is holomorphic for all $s \in \mathbb{C}$.

Problem 2: Assuming the Poisson Summation Formula: $\sum\limits_{n\in\mathbb{Z}} f(n) = \sum\limits_{n\in\mathbb{Z}} \widehat{f}(n)$ prove that for $y > 0$ we have: $\sum\limits_{n\in\mathbb{Z}} f(ny) = y^{-1} \sum\limits_{n\in\mathbb{Z}} \widehat{f}(ny^{-1})$.

Problem 3: Evaluate the integral

$$\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty} \frac{x^s}{s(s+1)\cdots(s+k)}\, ds$$

for any $\sigma > 0$ and any integer $k = 1, 2, 3, \ldots$ Give a sketch of the proof of your evaluation.

Problem 4: Fix $u > 0$. Show that the Fourier transform of $\frac{u}{\pi(x^2+u^2)}$ is $e^{-2\pi|x|u}$. Plug this into the Poisson summation formula to deduce that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+u^2} = \frac{\pi}{u}\sum_{n=-\infty}^{\infty} e^{-2\pi|n|u}.$$

By letting $u \to 0$ prove that $\zeta(2) = \pi^2/6$.

Problem 5: Let $f(x) = e^{-A|x|}$. Show that

$$\widehat{f}(y) = \frac{2A}{A^2+4\pi^2y^2},$$

$$\widetilde{f}(s) = \Gamma(s)A^{-s}, \qquad (\Re(s) > 0),$$

$$\widetilde{\widehat{f}}(s) = \frac{1}{2\pi}\left(\frac{2\pi}{A}\right)^{1-s}\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right), \qquad (0 < \Re(s) < 2).$$