Analytic Number Theory
Homework #1
(due Tuesday, February 6, 2018)

Problem 1: Let \( f : \mathbb{Z} \to \mathbb{C} \) be a completely multiplicative function, i.e., \( f(mn) = f(m)f(n) \) for all \( m, n \in \mathbb{Z} \). Assume also that \( |f(n)| \leq 1 \) for all \( n \in \mathbb{Z} \) and that \( f(1) = 1 \). Prove that for \( \Re(s) > 1 \), we have

\[
\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left( 1 - \frac{f(p)}{p^s} \right)^{-1}
\]

where the product goes over all primes \( p \).

Problem 2: Prove that the zeta function

\[
\zeta(s) := \sum_{n=1}^{\infty} n^{-s}, \quad (\Re(s) > 1),
\]

does not vanish for \( \Re(s) > 1 \).

Problem 3: Show that the Gamma function \( \Gamma(s) = \int_0^\infty e^{-u} u^{s-1} \frac{du}{u} \) has simple poles at \( s = 0, -1, -2, -3, \ldots \). Determine the residues at these poles.

Problem 4: (Uniqueness of Dirichlet series) For \( n = 1, 2, 3, \ldots \), let \( a_n, b_n \) be complex numbers with absolute values at most one. Assume that

\[
\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{n=1}^{\infty} \frac{b_n}{n^s}
\]

for all complex values of \( s \) with \( \Re(s) > 1 \). Prove that we must have \( a_n = b_n \) for all \( n = 1, 2, 3, \ldots \).