**Analytic Number Theory**
**Homework #1**

*(due Tuesday, February 6, 2018)*

**Problem 1:** Let $f : \mathbb{Z} \to \mathbb{C}$ be a completely multiplicative function, i.e., $f(mn) = f(m)f(n)$ for all $m, n \in \mathbb{Z}$. Assume also that $|f(n)| \leq 1$ for all $n \in \mathbb{Z}$ and that $f(1) = 1$. Prove that for $\Re(s) > 1$, we have

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 - \frac{f(p)}{p^s}\right)^{-1}$$

where the product goes over all primes $p$.

**Problem 2:** Prove that the zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}, \qquad (\Re(s) > 1),$$

does not vanish for $\Re(s) > 1$.

**Problem 3:** Show that the Gamma function $\Gamma(s) = \int_0^\infty e^{-u} u^s \, \frac{du}{u}$ has simple poles at $s = 0, -1, -2, -3, \ldots$ Determine the residues at these poles.

**Problem 4: (Uniqueness of Dirichlet series)** For $n = 1, 2, 3, \ldots$, let $a_n$, $b_n$ be complex numbers with absolute values at most one. Assume that

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

for all complex values of $s$ with $\Re(s) > 1$. Prove that we must have $a_n = b_n$ for all $n = 1, 2, 3, \ldots$