1) We consider the integral
\[ I = -\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{\zeta'(s+1) x^s}{\zeta(s+1) s} ds \]

Firstly, by the formula
\[ -\frac{\zeta'(s)}{\zeta(s)} = \sum_n \Lambda(n)n^{-s} \]
we get

\[ I = \sum_n \frac{\Lambda(n)}{n} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{1}{n^s} \frac{x^s}{s} ds \]
\[ = \sum_n \frac{\Lambda(n)}{n} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{(\frac{x}{n})^s}{s} ds \]
\[ = \sum_{n<x} \frac{\Lambda(n)}{n} \]

where we use the fact
\[ \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{x^s}{s} ds = 1, \quad x > 1; = 0, \quad x < 1 \]

By discarding the higher term, we easily get

\[ \sum_{n<x} \frac{\Lambda(n)}{n} = \sum_{\text{prime } p < x} \frac{\log(p)}{p} + O(1) \]

Then we estimate the integral \( I \) by shifting the integral to roughly \( Re(s) = 0 - \epsilon \) (so that we cross the pole).

\[ I = -\frac{1}{2\pi i} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{\zeta'(s+1) x^s}{\zeta(s+1) s} ds + \text{Res}_{s=0} \frac{\zeta'(s+1) x^s}{\zeta(s+1) s} \]

Recall that
\[ \frac{\zeta'(s+1)}{\zeta(s+1)} = \frac{1}{s} + h(s) \]

where \( h(s) \) is holomorphic at \( s = 0 \) and non-vanishing at \( s = 0 \).

And
\[ x^s = e^{\log(x)s} = 1 + \log(x)s + O(s^2) \]
Thus, we have Laurent expansion

\[ \frac{\zeta'(s+1) x^s}{\zeta(s+1)} = \frac{1}{s} + h(s)(1 + \log(x)s + O(s^2))/s \]

\[ = \frac{1}{s^2} + \frac{(h(0) + \log(x))}{s} + O(1) \]

Hence,

\[ I = -\frac{1}{2\pi i} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} \frac{\zeta'(s+1) x^s}{\zeta(s+1)} ds + h(0) + \log(x) \]

A standard estimate for \( \zeta \) gives us

\[ I = \log(x) + O(1) \]

Therefore,

\[ \sum_{\text{prime } p < x} \frac{\log(p)}{p} = \log(x) + O(1) \]

2) Use Abel summation (actually a sort of 'integration by parts' for discrete variable). Let \( \delta(n) = 1 \) if \( n \) is a prime, 0 otherwise (as an indicator function of primes). Then

\[ \sum_{p \leq x} \frac{1}{p} = \sum_{n \leq x} \frac{\delta(n)}{n} \]

\[ = \sum_{n \leq x} \frac{\delta(n) \log(n)}{n} \frac{1}{\log(n)} \]

\[ = \text{Abel} \sum_{n \leq x} \left( \sum_{k \leq n} \frac{\delta(k) \log(k)}{k} \right) \left( \frac{1}{\log(n)} - \frac{1}{\log(n+1)} \right) + \left( \sum_{k \leq x} \frac{\delta(k) \log(k)}{k} \right) \frac{1}{\log([x])} \]

Note that

\[ \sum_{k \leq n} \frac{\delta(k) \log(k)}{k} = \sum_{p \leq n} \frac{\log(p)}{p} \]

and Prob (1) implies that this is \( \log(n) + O(1) \). Thus,

\[ \sum_{p \leq x} \frac{1}{p} = \sum_{n \leq x} \left( (\log(n) + O(1)) \frac{\log(n+1)}{\log(n) \log(n+1)} + O(1) \right) \]
Since we have

\[ \log(n + 1) = \frac{1}{n} + O\left(\frac{1}{n^2}\right) \]

\[
\sum \frac{1}{p} \left( \sum_{n \leq x} \frac{1}{n \log(n)} + O(1) \right) = \int_2^x \frac{dt}{\log(t)} + O(1) = \log(\log(x)) + O(1)
\]

( Note that it is clearly that \( \sum_{n \leq x} \log(n + 1) \log(n + 1) \sim \sum_{n \leq x} \frac{1}{n \log(n)^2} \sim \int_2^x \frac{dt}{\log(t)^2} \sim \frac{1}{\log(x)} \) only contributes \( O(1) \).)

Remark 0.0.1. Or as some students did, we have roughly

\[
\sum \frac{1}{p} \left( \sum_{n \leq x} \frac{1}{n \log(n)} \right) \sim \sum_{n \leq x} \frac{1}{n \log(n)^2} \sim \int_2^x \frac{dt}{\log(t)^2} \sim \frac{1}{\log(x)}
\]

Then the remaining is the same.

3) Note that \((\mathbb{Z}/11\mathbb{Z})^*\) is a cyclic group with generator \( g = 2 \mod 11 \). So a character \( \chi \) is determined uniquely by its value at the generator. Therefore \( \chi \) is of order 5 if and only if \( \chi(g) \) is a fifth root of unit, i.e. \( e^{\frac{2\pi i}{5}} \) \((k = 0, 1, \ldots, 4)\). And the value of \( \chi \) at \( g^i \) \((i=1,2,\ldots,10)\) is given by \( \chi(g)^i \).

Note that the problem has something wrong in its statement. “no character of order 6” should be understood as “no character \( \chi \) with the property that the smallest integer \( n \) such that \( \chi(a)^n = 1 \) for all \( a \) is 6”. The proof is to look at the value of a character \( \chi \) at the generator \( g = 2 (\mod 11) \) if such \( \chi \) exists. Then \( \chi(g) \) has to be a primitive root of unit, i.e \( \chi(g) = e^{\frac{2\pi i}{11}} \) for some integer \( i \), \((i, 6) = 1\).
Then it follows from $\chi(g^{10}) = \chi(1 \text{ (mod 11)}) = 1$ that $e^{\frac{2\pi \sqrt{-1}}{6} \cdot 10} = 1$, i.e. $6|10k$, then $3|k$, which contracts $(6, k) = 1$!

4) We still need the fact that for a prime $p$, $(\mathbb{Z}/p\mathbb{Z})^*$ is generated by some $g = a \bmod p$. Then the character $\chi$ is real if and only if $\chi(g)$ is real. But it follows from $\chi(g)^{p-1} = \chi(g^{p-1}) = \chi(1) = 1$ that $\chi(g)$ is a root of unit. So its absolute value is 1 and hence it is either 1 or $-1$ since it is a real number. If $\chi(g) = 1$, then clearly $\chi$ is the trivial character. So the only nontrivial real character is the one determined by $\chi(g) = -1$.

5) Recall that the Mobius function is defined to be $\mu(1) = 1$, $\mu(n) = (-1)^r$ if $n$ is a product of $r$ distinct primes, and $\mu(n) = 0$ otherwise. Let $\delta$ be the function such that

$$
\delta(n) = 1, \quad n \equiv 3 \text{ (mod } q) \\
\delta(n) = 0, \quad \text{otherwise}
$$

Claim:

$$
\sum_{\chi} \chi(3)\chi(n) = \phi(q)\delta(n),
$$

for any $n > 0$, where $\chi$ run over all Dirichlet characters mod $q$ and $\phi(q)$ is the Euler function.

This is because

$$
\sum_{\chi} \chi(3)\chi(n) = \sum_{\chi} \chi(n/3)
$$

which is $\phi(q)$ if $n/3 \equiv 1 \text{ (mod } q)$, zero otherwise. (NOTE:here $n/3$ means the division in the group $(\mathbb{Z}/q\mathbb{Z})^*$. )
Therefore,

\[
\sum_{0 < n \equiv 3 \pmod{q}} \frac{\mu(n)}{n^s} = \sum_{0 < n} \frac{\mu(n)\delta(n)}{n^s} \\
= \frac{1}{\phi(q)} \sum_{n > 0} \sum_\chi \chi(3)\chi(n) \frac{\mu(n)}{n^s} \\
= \frac{1}{\phi(q)} \sum_\chi \chi(3)(\sum_{n > 0} \chi(n) \frac{\mu(n)}{n^s}) \\
= \frac{1}{\phi(q)} \sum_\chi \chi(3) \left( \sum_{n > 0, n \text{ square free}} \chi(n) \frac{\mu(n)}{n^s} \right) \\
= \frac{1}{\phi(q)} \sum_\chi \chi(3) \prod_{p \text{ prime}} (1 - \chi(p)p^{-s}) \\
= \frac{1}{\phi(q)} \sum_\chi \chi(3) \frac{1}{L(s, \chi)}
\]

**Remark 0.0.2.** In the above solution, we implicitly assumed that 

\[(3, q) = 1\]

If 3|q, the situation is worse though we still have similar expression.