Problems 2

Due: Monday, 3 October 2022 before 17:00 EDT

P2.1 Let $\psi\colon A \to B$ and $\varphi\colon B \to C$ be two morphism of R-complexes. Demonstrate that there exists an exact sequence

$$0 \longleftarrow \operatorname{Coker}(\varphi) \longleftarrow \operatorname{Coker}(\varphi\,\psi) \longleftarrow \operatorname{Coker}(\psi) \longleftarrow \operatorname{Ker}(\varphi) \longleftarrow \operatorname{Ker}(\varphi\,\psi) \longleftarrow \operatorname{Ker}(\psi) \longleftarrow 0.$$

P2.2 Consider the commutative diagram of R-complexes

$$\begin{array}{ccccccccc} E & \xleftarrow{\;\omega\;} & D & \xleftarrow{\;\theta\;} & C & \xleftarrow{\;\varphi\;} & B & \xleftarrow{\;\psi\;} & A \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} \\ E' & \xleftarrow{\;\omega'\;} & D' & \xleftarrow{\;\theta'\;} & C' & \xleftarrow{\;\varphi'\;} & B' & \xleftarrow{\;\psi'\;} & A' \end{array}$$

having exact rows.

(i) When α is an epimorphism and both β and δ are monomorphisms, prove that γ is a monomorphism.

(ii) When ε is a monomorphism and both β and δ are epimorphisms, prove that γ is an epimorphism.

(iii) When α, β, δ, and ε are isomorphisms, prove that γ is also an isomorphism.

P2.3 Consider the short exact sequence $0 \longleftarrow C \xleftarrow{\;\varphi\;} B \xleftarrow{\;\psi\;} A \longleftarrow 0$ of R-complexes.

(i) When the homology of two of R-complexes is zero, prove that the homology of the third is also zero.

(ii) Prove that the connecting morphism $\eth(\psi,\varphi)\colon \mathrm{H}(C) \to \mathrm{H}(A)[1]$ is an isomorphism if and only if $\mathrm{H}(B) = 0$.

P2.4 A *directed graph* G consists of a set $V(G)$ of vertices and a set $E(G)$ of edges formed by ordered pairs of vertices. When $e \in E(G)$ corresponds to the pair (u,v) of vertices, the vertex u is the *tail* of e and the vertex v is the *head* of e. Writing $n := |V(G)|$ and $m := |E(G)|$, the incidence matrix $\mathbf{B} := \big[b_{j,k}\big]$ of G is the $(n \times m)$-matrix defined by

$$b_{j,k} := \begin{cases} -1 & \text{if the } k\text{-th edge has the } j\text{-th vertex as its tail,} \\ 1 & \text{if the } k\text{-th edge has the } j\text{-th vertex as its head,} \\ 0 & \text{otherwise.} \end{cases}$$

The $\mathbb{Z}$-complex $C(G)$ associated to the directed graph G is

$$0 \longleftarrow \underset{0}{\mathbb{Z}^n} \xleftarrow{\;\mathbf{B}\;} \underset{1}{\mathbb{Z}^m} \longleftarrow 0\,.$$

When G has c connected components, show that the $\mathrm{H}_0\big(C(G)\big) = \mathbb{Z}^c$ and $\mathrm{H}_1\big(C(G)\big) = \mathbb{Z}^{m-n+c}$.

Hint. Find the Smith normal form of the matrix $\mathbf{B}$. First consider the case $c = 1$ and focus on the columns corresponding to a spanning tree in the underlying graph.