Problems 09
Due: Tuesday, 17 November 2020

1. A module is ***simple*** if it is not the zero module and if it has no proper submodule.

(i) Let V be a simple R-module. Show that V is cyclic.

(ii) Prove *Schur's Lemma*: If $\varphi : V \to W$ is a homomorphism of simple R-modules, then either φ is zero or an isomorphism.

(iii) Prove that the set of endomorphisms, denoted $\operatorname{End}_R(V) := \operatorname{Hom}_R(V, V)$, of a simple R-module V forms a field; multiplication is given by composition of functions and addition is defined pointwise.

2. Let R be domain and let V be an R-module. An element $v \in V$ is a ***torsion element*** if $\operatorname{Ann}(v) \neq 0$; in other words, $v \in V$ is a torsion element if and only if there is an $0 \neq r \in R$ such that $r\,v = 0$. Let $\tau(V)$ be the set of torsion elements of V. A module V is ***torsion*** if $\tau(V) = V$ and it is ***torsion-free*** if $\tau(V) = 0$.

(i) Show that $\tau(V)$ is a submodule of V.

(ii) Show that $V/\tau(V)$ is torsion-free.

(iii) For any R-module homomorphism $\varphi : V \to W$, show that $\varphi\big(\tau(V)\big) \subseteq \tau(W)$.

(iv) Give an example of an infinite abelian group that is a torsion $\mathbb{Z}$-module.

3. (i) Let $\varphi : V' \to V$ and $\psi : V \to V''$ two R-module homomorphisms. Prove that the sequence

$$(\ddagger) \qquad V' \xrightarrow{\ \varphi\ } V \xrightarrow{\ \psi\ } V'' \longrightarrow 0$$

is exact if and only if, for every R-module W, the sequence

$$(\bigstar) \qquad 0 \longrightarrow \operatorname{Hom}_R(V'', W) \xrightarrow{\operatorname{Hom}_R(\psi, W)} \operatorname{Hom}_R(V, W) \xrightarrow{\operatorname{Hom}_R(\varphi, W)} \operatorname{Hom}_R(V', W)$$

is exact.

(ii) Show that $\operatorname{Hom}_{\mathbb{Z}}\big(\mathbb{Z}/\langle m\rangle, \mathbb{Z}/\langle n\rangle\big) \cong \mathbb{Z}/\langle d\rangle$ where $d := \gcd(m, n)$.

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