Problems 06
Due: Tuesday, 20 October 2020

1. Let $R := \mathcal{C}([0,1])$ be the commutative ring of continuous real-valued functions on the closed interval $[0,1]$ in $\mathbb{R}$. For all subsets $X \subseteq [0,1]$, define
$$\mathbf{I}(X) := \{f \in R \mid f(x) = 0 \text{ for all } x \in X\}.$$
(i) Prove that $\mathbf{I}(X)$ is an ideal of R.
(ii) For any point $p \in [0,1]$, set $I_p := \mathbf{I}(\{p\})$. Prove that I_p is a maximal ideal of R and we have the isomorphism $R/I_p \cong \mathbb{R}$.
(iii) For all subsets $J \subseteq R$, set $\mathbf{V}(J) := \{x \in [0,1] \mid f(x) = 0 \text{ for all } f \in J\}$. Prove that $\mathbf{V}(J)$ is a closed subset of the interval $[0,1]$.
(iv) If I is a proper ideal of R, then show that $\mathbf{V}(I) \neq \varnothing$.
(v) Prove that any maximal ideal of R is equal to I_p for some $p \in [0,1]$.

2. Solve the following system of simultaneous congruences in $\mathbb{Q}[x]$:
$$f(x) \equiv -3 \pmod{x+1}, \quad f(x) \equiv 12x \pmod{x^2-2}, \quad f(x) \equiv -4x \pmod{x^3}.$$

3. For any field K, let $K(\!(x)\!)$ be ring of formal Laurent series with coefficients in K:
$$K(\!(x)\!) := \left\{ \sum_{j=m}^{\infty} a_n x^n \;\middle|\; a_n \in K, m \in \mathbb{Z} \text{ is arbitrary} \right\}$$
where the ring operations are defined as in the ring of formal power series $K[[x]]$. Prove that $K(\!(x)\!)$ is isomorphic to the fields of fractions for the domain $K[[x]]$.