Problems 05
Due: Tuesday, 13 October 2020

1. Define $\mathbb{F}_4$ to be all (2×2)-matrices of the form $\begin{bmatrix} a & b \\ b & a+b \end{bmatrix}$ where $a, b \in \mathbb{Z}/\langle 2 \rangle$.

(i) Prove that $\mathbb{F}_4$ is a commutative ring under the usual matrix operations.
(ii) Prove that $\mathbb{F}_4$ is a field with exactly four elements.

2. Let R be a commutative ring. An element $r \in R$ is ***nilpotent*** if $r^n = 0$ for some positive integer n.

(i) For any nilpotent element $r \in R$, then prove that $1 - r$ is a unit in R.
(ii) Prove the set of all nilpotent elements in R is an ideal.

3. (i) Let R be a commutative ring and consider two elements $f, g \in R$. Show that the canonical image of fg in the quotient ring $R/\langle f - f^2 g \rangle$ is an idempotent. Give an example where this idempotent is distinct from 0 and 1.

(ii) Let R and S be two rings and let φ and ψ be ring homomorphisms from R to S. Is the set of $f \in R$ such that $\varphi(f) = \psi(f)$ a subring of R?