Written Exam
Due: Thursday, 17 December 2020

INSTRUCTIONS

- Each question is worth 10 points.
- To receive full credit, you must explain your answers.
- Solutions are to be the result of an individual effort. For this examination, communication or collaboration with anyone other than the instructor is prohibited.
- Authorized materials are limited to course notes and problem sets (including solutions). The use of other resources is prohibited.
- Students are responsible for upholding the fundamental values of academic integrity.

PROBLEMS

1. The set $\mathrm{U}(n, K)$ consists of all unit upper triangular $(n \times n)$-matrices over the field K;

$$\mathrm{U}(n,K) := \left\{ \begin{bmatrix} 1 & * & \cdots & * & * \\ 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & * \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \;\middle|\; \text{where } * \text{ denotes an arbitrary element of } K \right\}.$$

(i) Prove that $\mathrm{U}(n,K)$ is a subgroup of $\mathrm{GL}(n,K)$.

(ii) Let p be a prime integer, let e be a positive integer, let $q := p^e$, and let $\mathbb{F}_q$ be a finite field with q elements. Find a Sylow p-subgroup of $\mathrm{GL}(n, \mathbb{F}_q)$.

(iii) Let p be a prime integer and let e be a positive integer. Show that every finite group G of order p^e is isomorphic to a subgroup of $\mathrm{U}(p^e, \mathbb{F}_p)$.

2. (i) Let I be the ideal in $\mathbb{Z}[x]$ generated by $x - 7$ and 15. Prove that the quotient ring $\mathbb{Z}[x]/I$ is isomorphic to $\mathbb{Z}/\langle 15 \rangle$.

(ii) Consider the factorization $3x^3 + 4x^2 + 3 = (x+2)^2(3x+2) = (x+2)(x+4)(3x+1)$ in $\mathbb{F}_5[x]$. Explain why this equation does not contradict the fact that $\mathbb{F}_5[x]$ is a unique factorization domain.

(iii) Let p be a prime integer. Show that the polynomial $x^p + px + (p-1)$ is irreducible in $\mathbb{Q}[x]$ if and only if p is greater than 2.

3. For an abelian group G, the ***Pontrjagin dual*** is the group $D(G) := \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$.

(i) For any $\mathbb{Z}$-module homomorphism $\varphi : G \to H$, show that post-composition with φ induces a $\mathbb{Z}$-module homomorphism $D(\varphi) : D(H) \to D(G)$.

(ii) Fixed element $g \in G$. Show that evaluation at g defines a $\mathbb{Z}$-module homomorphism $\psi_g : D(G) \to \mathbb{Q}/\mathbb{Z}$.

(iii) Prove the map $\Psi : G \to D\big(D(G)\big)$ defined by $\Psi(g) = \psi_g$, for all $g \in G$, is a $\mathbb{Z}$-module homomorphism.

(iv) For any *finite* abelian group G, prove that $\Psi : G \to D\big(D(G)\big)$ is an isomorphism.