Problems 5

Due: Friday, 24 March 2023 before 17:00 EST

Students registered in MATH 413 should submit solutions to any three problems, whereas students in MATH 813 should submit solutions to all five.

P5.1. **(i)** For any univariate polynomial $f := a_m\, x^m + a_{m-1}\, x^{m-1} + \cdots + a_1\, x + a_0$ of degree m in the ring $\mathbb{K}[x]$, define its ***homogenization*** in the ring $\mathbb{K}[x,y]$ to be

$$f^{\mathsf{h}} := a_m\, x^m + a_{m-1}\, x^{m-1}\, y + \cdots + a_1\, x\, y^{m-1} + a_0\, y^m\,.$$

Prove that the polynomial f has a root in $\mathbb{K}$ if and only if there exists a point (b,c) in $\mathbb{A}^2(\mathbb{K})$ such that $(b,c) \neq (0,0)$ and $f^{\mathsf{h}}(b,c) = 0$.

(ii) Let $\mathbb{K}$ be a field that is not algebraically closed. Exhibit a bivariate polynomial h in the ring $\mathbb{K}[x,y]$ such that the affine subvariety $\mathrm{V}(h)$ in $\mathbb{A}^2(\mathbb{K})$ is just the origin $(0,0)$.

(iii) Let $\mathbb{K}$ be a field that is not algebraically closed. For any positive integer n, demonstrate that there exists a polynomial f in the ring $\mathbb{K}[x_1, x_2, \dots, x_n]$ such that the affine subvariety $\mathrm{V}(f)$ in $\mathbb{A}^n(\mathbb{K})$ is the origin $(0,0,\dots,0)$.

(iv) Let $\mathbb{K}$ be a field that is not algebraically closed. Prove that any $X = \mathrm{V}(g_1, g_2, \dots, g_r)$ in $\mathbb{A}^n(\mathbb{K})$ can be defined by a single equation.

P5.2. For any ideal I in the ring $S := \mathbb{K}[x_1, x_2, \dots, x_n]$ and any polynomial f in S, the *saturation* of I with respect to f is the set

$$(I : f^\infty) := \{ g \in S \mid \text{there exists a positive integer } m \text{ such that } f^m\, g \in I \}\,.$$

(i) Prove that $(I : f^\infty)$ is an ideal in the ring S.

(ii) Prove that there is an ascending chain of ideals $(I : f) \subseteq (I : f^2) \subseteq (I : f^3) \subseteq \cdots$.

(iii) For any positive integer ℓ, prove that we have the equality $(I : f^\infty) = (I : f^\ell)$ if and only if we have the equality $(I : f^\ell) = (I : f^{\ell+1})$.

P5.3. Two ideals I and J in the ring $S := \mathbb{K}[x_1, x_2, \dots, x_n]$ are ***comaximal*** if $I + J = S$.

(i) Over an algebraically closed field, show that the ideals I and J are comaximal if and only if we have $\mathrm{V}(I) \cap \mathrm{V}(J) = \varnothing$. Without the algebraically closed hypothesis, show that this can be false.

(ii) When the ideals I and J are comaximal, show that $I\,J = I \cap J$.

(iii) When the ideals I and J are comaximal, show that, for all positive integers i and j, the ideals I^i and J^j are comaximal.

P5.4. **(i)** Consider the affine subvariety $X := \mathrm{V}(x^2 - y\,z, x\,z - x)$ in $\mathbb{A}^3$. Demonstrate that X is a union of 3 irreducible components. Describe them and find their prime ideals.

(ii) Show that the set of real points on the irreducible complex surface

$$\mathrm{V}\left((x^2 + y^2)\,z - x^3\right) \subset \mathbb{A}^3$$

is connected but is not equidimensional; it is the union of a closed curve and a closed surface in the induced Euclidean topology.

P5.5. Let I be a monomial ideal in the ring $S := \mathbb{K}[x_1, x_2, \dots, x_n]$. For any monomial ideal J generated by pure powers of a subset of the variables, every zerodivisor in the quotient ring S/J is nilpotent, so the ideal J is primary.

(i) Suppose that $x^{\mathbf{u}}$ is a minimal generator of the monomial ideal I such that $x^{\mathbf{u}} = x^{\mathbf{v}_1}\,x^{\mathbf{v}_2}$ where the monomials $x^{\mathbf{v}_1}$ and $x^{\mathbf{v}_2}$ are relatively prime. Show that

$$I = \left(I + \langle x^{\mathbf{v}_1}\rangle\right) \cap \left(I + \langle x^{\mathbf{v}_2}\rangle\right).$$

(ii) Using part (i), find an irredundant primary decomposition of the monomial ideal $\left\langle x^3y, x^3z, xy^3, y^3z, xz^3, yz^3 \right\rangle$.