10 Geometric Applications

What are the benefits of projective geometry? We highlight two: the image of projective subvariety under a morphism is always closed, and the number of common zeros equals the product of the degrees of the polynomials.

10.0 Projective Elimination

What distinguishes projective subvarieties from affine subvarieties? The extra points in the ambient projective space make imagines easier to understand.

10.0.0 Example. The image of a morphism of affine subvarieties is not necessarily an affine subvariety. Consider the affine subvariety $X := \mathrm{V}(xy-1) \subset \mathbb{A}^2$. Under the projection map $\pi_2 \colon \mathbb{A}^2 \to \mathbb{A}^1$ defined by $(a,b) \mapsto b$, we see that $\pi_2(X) = \{ b \in \mathbb{A}^1 \mid b \neq 0 \}$. To take advantage of projective geometry, regard $X \subseteq \mathbb{A}^1 \times \mathbb{A}^1$ as a subset in $\mathbb{P}^1 \times \mathbb{A}^1$ by identifying the first affine line with an affine open subset in the projective line. The Zariski closure $\overline{X} \subseteq \mathbb{P}^1 \times \mathbb{A}^1$ is $\overline{X} = \{ ([a_0 : a_1], b) \mid a_1 \, b = a_0 \}$. We still have the projection map $\pi_2 \colon \mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ given by $([a_0 : a_1], b) \to y$, but now $\pi_2(\overline{X}) = \mathbb{A}^1$. The new point $([0:1], 0)$ is mapped to the origin. ⋄

10.0.1 Remark (Families of projective subvarieties)**.** A polynomial f in $\mathbb{K}[x_0, x_1, \dots, x_n, y_1, y_2, \dots, y_m] = (\mathbb{K}[y_1, y_2, \dots, y_m])[x_0, x_1, \dots, x_n]$ is homogeneous of degree d in the variables $x_0, x_1, \dots, x_n$ if there are $h_u \in \mathbb{K}[y_1, y_2, \dots, y_m]$ such that $f = \sum_{|u|=d} x^u \, h_u$. For each such polynomial f, the hypersurface

$$\mathrm{V}(f) = \big\{ \big([a_0 : a_1 : \cdots : a_n], (b_1, b_2, \dots, b_m) \big) \in \mathbb{P}^n \times \mathbb{A}^m \mid f(a_0, a_1, \dots, a_n, b_1, b_2, \dots, b_m) = 0 \big\}$$

is well-defined. By intersecting hypersurfaces, we see that any ideal I in the ring $\mathbb{K}[x_0, x_1, \dots, x_n, y_1, y_2, \dots, y_m]$ with generators that are homogeneous in the variables $x_0, x_1, \dots, x_n$ determines a subvariety $\mathrm{V}(I) \subseteq \mathbb{P}^n \times \mathbb{A}^m$.

Do all closed subsets come from homogeneous ideals? When $m = 0$, we already know that each projective subvariety $X \subseteq \mathbb{P}^n$ corresponds to a homogeneous ideal. A similar argument produces the following mild generalization.

10.0.2 Proposition. *For any subvariety $X \subseteq \mathbb{P}^n \times \mathbb{A}^m$, the ideal* $\mathrm{I}(X)$ *in the ring* $\mathbb{K}[x_0, x_1, \dots, x_n, y_1, y_2, \dots, y_m]$ *that is homogeneous the variables $x_0, x_1, \dots, x_n$ and vanishes on X, is the intersection of the homogenizations of $I_i = I(U_i \cap X)$ where $U_i \subset \mathbb{P}^n \times \mathbb{A}^m$ is the distinguished open subset defined by $x_i \neq 0$ for all $0 \leqslant i \leqslant n$,.* ■

10.0.3 Definition. The *projective elimination ideal* for an ideal I in $\mathbb{K}[x_0, x_1, \dots, x_n, y_1, y_2, \dots, y_m]$, that is homogeneous in the variables $x_0, x_1, \dots, x_n$, is the ideal $\widehat{I} := (I : \mathfrak{m}^\infty) \cap \mathbb{K}[y_1, y_2, \dots, y_n]$.

10.0.4 Theorem (Projective elimination)**.** *Let $\pi_2 \colon \mathbb{P}^n \times \mathbb{A}^m \to \mathbb{A}^m$ be the projection defined by $\big([a_0 : a_1 : \cdots : a_n], (b_1, b_2, \dots, b_m)\big) \mapsto (b_1, b_2, \dots, b_m)$. For any ideal I in $\mathbb{K}[x_0, x_1, \dots, x_n, y_1, y_2, \dots, y_m]$ that is homogeneous in the variables $x_0, x_1, \dots, x_n$, we have $\pi_2\big(\mathrm{V}(I)\big) \subseteq \mathrm{V}(\widehat{I})$. When the field $\mathbb{K}$ is algebraically closed, we also have $\pi_2\big(\mathrm{V}(I)\big) = \mathrm{V}(\widehat{I})$.*

Proof. Suppose that the point $(b_1, b_2, \dots, b_m) \in \pi_2\big(\mathrm{V}(I)\big)$ is the image of a point $\big([a_0 : a_1 : \cdots : a_n], (b_1, b_2, \dots, b_m)\big) \in \mathrm{V}(I)$. For each polynomial $f \in I$ that is homogeneous in the variables $x_0, x_1, \dots, x_n$, we have $f(a_0, a_1, \dots, a_n, b_1, b_2, \dots, b_m) = 0$. There exists $0 \leqslant i \leqslant n$ such that $a_i \neq 0$. For any $h \in \widehat{I}$, it follows that $x_i^k \, h \in I$ for some $k \gg 0$, so we have $a_i^k \, h(b_1, b_2, \dots, b_m) = 0$. We deduce that $h(b_1, b_2, \dots, b_m) = 0$ and $(b_1, b_2, \dots, b_m) \in \mathrm{V}(\widehat{I})$.

It remains to prove the inclusion $\mathrm{V}(\widehat{I}) \subseteq \pi_2\big(\mathrm{V}(I)\big)$. Suppose that there is a point $c := (c_1, c_2, \dots, c_m) \in \mathrm{V}(\widehat{I})$ such that $c \notin \pi_2\big(\mathrm{V}(I)\big)$. Let $f_1, f_2, \dots, f_r$ be generators for the ideal I that are homogeneous in the variables $x_0, x_1, \dots, x_n$. Since the homogeneous polynomials

$$f_1(x_0, x_1, \dots, x_n, c_1, c_2, \dots, c_m), f_2(x_0, x_1, \dots, x_n, c_1, c_2, \dots, c_m), \dots, f_r(x_0, x_1, \dots, x_n, c_1, c_2, \dots, c_m)$$

in $\mathbb{K}[x_0, x_1, \dots, x_n]$ define the empty subvariety in $\mathbb{P}^n$, the projective weak nullstellensatz implies that $\mathfrak{m}^k \subseteq \langle f_1(x,c), f_2(x,c), \dots, f_r(x,c) \rangle$ for some $k \gg 0$. Hence, for each x^u with $|u| = k$, there exists an expression $x^u = \sum_{i=1}^{r} f_i(x,c) \, p_{i,u}(x)$, where $p_{i,u} \in \mathbb{K}[x_0, x_1, \dots, x_n]$ are homogeneous. For all $1 \leqslant j \leqslant \binom{k+n}{k}$, there exists $1 \leqslant i_j \leqslant r$ and $v_j \in \mathbb{N}^{n+1}$ such that $g_j := x^{v_j} f_{i_j}$ and the polynomials $g_j(x,c)$ form a $\mathbb{K}$-vector space basis for the homogeneous polynomials in $\mathbb{K}[x_0, x_1, \dots, x_n]$ having degree k. Setting $g_j = \sum_{|u|=k} x^u \, q_{j,u}$, we see that $\mathbf{Q} := [q_{j,u}]$ is an $\big(\binom{k+n}{k} \times \binom{k+n}{k}\big)$-matrix of polynomials in the ring $\mathbb{K}[y_1, y_2, \dots, y_m]$. Hence, we have $D := \det(\mathbf{Q}) \in \mathbb{K}[y_1, y_2, \dots, y_m]$ and $D(c_1, c_2, \dots, c_m) \neq 0$. By Cramer's rule, we obtain

$$D \, x^u = \sum_{j=1}^{\binom{k+n}{k}} \ell_{j,u} \, g_j$$

for a suitable matrix $\mathbf{L} = [\ell_{j,u}]$ with entries in $\mathbb{K}[y_1, y_2, \dots, y_m]$. It follows that $D \, x^u \in \langle f_1, f_2, \dots, f_r \rangle = I$ and $D \in \widehat{I}$. However, this contradicts our assumption that $c \in \mathrm{V}(\widehat{I})$. □

10.0.5 Definition. An algebraic variety X is *complete* if for all varieties Y, the projection morphism $\pi_2 \colon X \times Y \to Y$ is a closed map (sends subvarieties to subvarieties).

The analogous property for topological spaces characterizes compact spaces X.

10.0.6 Theorem. *For any algebraically closed field $\mathbb{K}$ and any nonnegative integer n, the variety $\mathbb{P}^n$ is complete.*

This theorem is true over any field; Grothendieck gives a prove via Nakayama's Lemma and Chevalley gives a valuation-theoretic prove. Nagata exhibited the first example of a nonprojective complete variety. Chow showed that every complete variety is dominated by a projective variety with the same function field.

Sketch of Proof. We must demonstrate that, for all varieties Y, the map $\pi_2 \colon \mathbb{P}^n \times Y \to Y$ is closed. The problem is "local" on Y so we may assume that Y is affine. Since the projective elimination theorem shows that $\pi_2 \colon \mathbb{P}^n \times \mathbb{A}^m \to \mathbb{A}^m$, the claim follows. □

10.1 Hilbert Functions

How do we obtain numerical invariants of projective subvarieties? Working with homogeneous rings and ideals provides new mechanisms for associating integers to projective subvarieties.

10.1.0 Definition. A *graded $\mathbb{K}$-algebra R* is a ring with a direct-sum decomposition $R = \bigoplus_{j \in \mathbb{Z}} R_j$ as $\mathbb{K}$-vector spaces that is compatible with multiplication: for all integer j and k, we have $R_j \cdot R_k \subseteq R_{j+k}$.

The adjective "direct-sum" means that every element f in R can be expressed uniquely as $f = \sum_i f_i$ where $f_i \in R_i$.

10.1.1 Examples. The polynomial ring $S := \mathbb{K}[x_0, x_1, \dots, x_n]$ is graded where S_j is $\mathbb{K}$-vector space spanned of all monomials of degree j.

When I is a homogeneous ideal in S, the quotient ring S/I is graded; the $\mathbb{K}$-vector space $(S/I)_j$ is spanned by the image of the monomials of degree j under the canonical map $S \to S/J$. ◇

10.1.2 Definition. For any graded $\mathbb{K}$-algebra R, the *Hilbert function* $h_R \colon \mathbb{Z} \to \mathbb{N}$ is defined, for all integers j, by $h_R(j) := \dim_{\mathbb{K}} R_j$.

10.1.3 Examples. Counting the monomials of degree j in S via stars-and-bars, it follows that $h_S(j) = \binom{j+n}{n}$.

When f is a homogenous polynomial of degree m and $I := \langle f \rangle$, we have $h_{S/I}(j) = \binom{j+n}{n} - \binom{j-m+n}{n}$ because the elements in the ideal I of degree j are of the form $f\,g$ where g is a homogeneous polynomial of degree $j - m$. ◇

Gröbner bases reduce the computation of Hilbert functions to monomial ideals.

10.1.4 Proposition (Macaulay). *For any homogeneous ideal I is S, we have $h_{S/I}(j) = h_{S/\operatorname{LT}(I)}(j)$ for all integers j.*

Proof. It suffices to show that the set $\mathcal{B}$ of all monomials not in the leading term ideal $\operatorname{LT}(I)$ forms a $\mathbb{K}$-vector space basis for S/I. We first establish that $\mathcal{B}$ is linearly independent. If there were a relation $g = c_1\, x^{u_1} + c_2\, x^{u_2} + \cdots + c_\ell\, x^{u_\ell} \in I$ with $x^{u_j} \in \mathcal{B}$ and $0 \neq c_j \in \mathbb{K}$, then we would have $\operatorname{LT}(g) \in \operatorname{LT}(I)$. Since $\operatorname{LT}(g)$ is $c_j\, x^{u_j}$ for some $1 \leqslant j \leqslant \ell$ which are not in $\mathcal{B}$, this is a contradiction.

Suppose that $\mathcal{B}$ does not span the quotient S/I. Among the set of elements of S that are not in the span of I and $\mathcal{B}$, we may take f to be one with minimal leading term. If $\operatorname{LT}(f)$ were in $\mathcal{B}$, we could subtract it, getting an element with still smaller leading term. It follows that $\operatorname{LT}(f) \in \operatorname{LT}(I)$. Subtracting an element of I with the same leading term as f results in a similar contradiction. □

10.1.5 Example. Consider the ring $S := \mathbb{Q}[w,x,y,z]$ equipped with a graded reverse lexicographic monomial order. The generators of the ideal $I := \langle \underline{y^2} + xz, \underline{xy} - wz, \underline{x^2} - wy \rangle$ are a Gröbner basis. It follows that the monomials in

$$\left\{ \begin{matrix} 1, w, z, w^2, w\,z, z^2, w^3, w^2\,z, w\,z^2, z^3, \dots \\ x, w\,x, x\,z, w^2\,x, w\,x\,z, x\,z^2, \dots \\ y, w\,y, y\,z, w^2\,y, w\,y\,z, y\,z^2, \dots \end{matrix} \right\} = \mathbb{Q}[w,z] \sqcup \mathbb{Q}[w,z]\,x \sqcup \mathbb{Q}[w,z]\,y$$

are a $\mathbb{Q}$-vector space basis for the quotient S/I. Thus, we have $h_{S/J}(0,1,2,3,\dots) = (1,4,7,10,\dots)$ or $h_{S/J}(J) = 3j{+}1$ for all $j \in \mathbb{N}$. ◇

10.1.6 Theorem. *For any homogeneous ideal I in S, there exists a unique $p_{S/I}(t) \in \mathbb{Q}[t]$, called* Hilbert polynomial *of the quotient S/I, such that $h_{S/I}(j) = p_{S/I}(j)$ for all $i \gg 0$.*

Sketch of Proof. We proceed by induction on n. When $n = -1$, we have $S = \mathbb{K}$ and $h_{S/J}(j) = 0$ for all positive integers j. Assume that $n \geqslant 0$ and each monomial ideal in $S' := \mathbb{K}[x_0, x_1, \dots, x_{n-1}]$ has a Hilbert polynomial. Since $\dim_{\mathbb{K}} I_j = \binom{j+n}{j} - h_{S/I}(j)$, it suffices to show that the function $j \mapsto \dim_{\mathbb{K}} I_j$ agrees with a polynomial for all sufficiently large j.

For any nonnegative integers k, consider the auxiliary ideal

$$I[k] := \left\{ f \in S' \;\middle|\; f\,x_n^k \in I \right\}.$$

It follows that chain $I[0] \subseteq I[1] \subseteq I[2] \subseteq \cdots$ of ideals is eventually stationary: $I[m] = I[m+1] = \cdots$ for some nonnegative integer m. The monomials in I of degree j are the disjoint union of the monomials in $I[k]_{j-k}\, x_n^k$ for all $0 \leqslant k \leqslant j$. Hence, we have

$$\dim_{\mathbb{K}} I_j = \sum_{k=0}^{j} \dim_{\mathbb{K}} \bigl(I[k]_{j-k}\, x_n^k\bigr) = \sum_{k=0}^{j} \dim_{\mathbb{K}} I[k]_{j-k} = \sum_{k=m}^{j} \dim_{\mathbb{K}} I[m]_{j-k} + \sum_{k=0}^{m-1} \dim_{\mathbb{K}} I[k]_{j-k}\,.$$

The first part is a finite sum of polynomials and the second part is constant. □

10.1.7 Definition. For a projective subvariety X in $\mathbb{P}^n$, the *Hilbert polynomial* p_X is defined to be $p_{S/I} \in \mathbb{Q}[t]$ where I is a homogeneous ideal in S such that $X = \operatorname{V}(I)$. The *dimension* of X is the degree of

One verifies that the Hilbert polynomial of X is independent of choice of homogeneous ideal satisfying $X = \operatorname{V}(I)$.

its Hilbert polynomial, the *degree* of X is $(\dim X)!$ times the leading coefficient of its Hilbert polynomial, and the *arithmetic genus* of X is $(-1)^{\dim X}\big(p_X(0) - 1\big)$.

10.1.8 Example (Hypersurfaces)**.** A hypersurface in $\mathbb{P}^n$ is determined by a homogeneous polynomial f in S of degree m. Since

$$h_{S/\langle f \rangle}(j) = \binom{j+n}{n} - \binom{j-m+n}{n} = \frac{m}{(n-1)!}\, j^{n-1} + \cdots ,$$

this projective subvariety has dimension $n-1$, degree m, and arithmetic genus 0. ◇

10.1.9 Example (Rational normal curves)**.** For a positive integer m, the Veronese map $\nu_m \colon \mathbb{P}^1 \to \mathbb{P}^m$ is defined by

$$[x_0 : x_1] \mapsto [x_0^m : x_0^{m-1} x_1 : \cdots : x_1^m] .$$

It follows that $h_{\nu_m(\mathbb{P}^1)}(j) = mj + 1$, so the projective variety $\nu_m(\mathbb{P}^1)$ is 1-dimensional, degree m, and arithmetic genus 0. ◇

10.1.10 Example (Veronese embedding)**.** For the map $\nu_m \colon \mathbb{P}^n \to \mathbb{P}^N$ where $N = \binom{n+m}{m}$ given by

$$[x_0 : x_1 : \cdots : x_n] \mapsto [x_0^m : x_0^{m-1}\, x_1 : \cdots : x_n^m]$$

we have $h_{\nu_m(\mathbb{A}^n)}(t) = \binom{mt+n}{n}$. ◇

10.2 Intersection Multiplicities

How do we count the number of points where two varieties meet? We want a method of counting that is well-defined even as varieties vary in families—it should satisfy a continuity principle.

10.2.0 Example. Set $\mathbb{K}[\mathbb{A}^2] = \mathbb{K}[x, y]$. Consider the plane curves $C_t := \mathrm{V}(y + x^2 - t) \subseteq \mathbb{A}^2$ and $D := \mathrm{V}(y) \subseteq \mathbb{A}^2$. The intersection $C_t \cap D = \{(\pm\sqrt{t}, 0)\}$ is two distinct points for $t \neq 0$ and one point for $t = 0$. The curve C_t is tangent to D if and only if $t = 0$. ◇

10.2.1 Example. Consider the plane curves $C_t := \mathrm{V}(y - x^3 - tx) \subseteq \mathbb{A}^2$ and $D := \mathrm{V}(y) \subseteq \mathbb{A}^2$. The intersection $C_t \cap D = \{(0,0), (\pm\sqrt{t}, 0)\}$ which is three distinct points for $t \neq 0$ and one point for $t = 0$. ◇

10.2.2 Remark. Let $a := (a_1, a_2, \dots, a_n)$ be a point in $\mathbb{A}^n$ and let

$$M_a = \langle x_1 - a_1, x_2 - a_2, \dots, x_n - a_n \rangle$$

be the associated maximal ideal in the ring $\mathbb{K}[x_1, x_2, \dots, x_n]$. Recall that any M_a-primary ideal Q satisfies $M_a^k \subseteq Q \subseteq M_a$ for some $k \gg 0$. The induced quotient map

$$\frac{\mathbb{K}[x_1, x_2, \dots, x_n]}{M_a^k} \to \frac{\mathbb{K}[x_1, x_2, \dots, x_n]}{Q}$$

is surjective and $\dim_{\mathbb{K}}\bigl(\mathbb{K}[x_1, x_2, \dots, x_n]/M_a^k\bigr) = \binom{n+k-1}{n}$. Hence, the quotient $\mathbb{K}[x_1, x_2, \dots, x_n]/Q$ is a finite-dimensional $\mathbb{K}$-vector space.

10.2.3 Definition. Let I be an ideal in $\mathbb{K}[x_1, x_2, \dots, x_n]$. Assume that M_a is a minimal associated prime of I, so that the corresponding primary ideal Q is uniquely determined. Assuming that the field $\mathbb{K}$ is algebraically closed, the ideal M_a is a minimal associated prime of I if and only if the point a is an irreducible component of the affine subvariety $\mathrm{V}(I)$. The *multiplicity* of ideal I at the point a is

$$\operatorname{mult}(I, a) = \dim_{\mathbb{K}}\left(\frac{\mathbb{K}[x_1, x_2, \dots, x_n]}{Q}\right).$$

When $M_a \not\supseteq I$, we have $\operatorname{mult}(I, a) = 0$.

10.2.4 Example. Consider

$$\langle y, y - t + x^2\rangle = \langle y, x^2 - t\rangle = \begin{cases}\langle y, x - \sqrt{t}\rangle \cap \langle y, x + \sqrt{t}\rangle & t \neq 0,\\ \langle y, x^2\rangle & t = 0.\end{cases}$$

When $t \neq 0$, each primary component has multiplicity 1 because $\{1\}$ is a $\mathbb{K}$-vector space basis for the quotient. When $t = 0$, there is just one primary component with multiplicity 2 because $\{1, x\}$ is a $\mathbb{K}$-vector space basis for the quotient. ◇

10.2.5 Example. Consider

$$\langle y, y - x^3 + t\,x\rangle = \langle y, x^3 - t\,x\rangle = \begin{cases}\langle y, x\rangle \cap \langle y, x - \sqrt{t}\rangle \cap \langle y, x + \sqrt{t}\rangle & t \neq 0,\\ \langle y, x^3\rangle & t = 0.\end{cases}$$

When $t \neq 0$, each primary component has multiplicity 1 because $\{1\}$ is a $\mathbb{K}$-vector space basis for the quotient. When $t = 0$, there is just one primary component with multiplicity 3 because $\{1, x, x^2\}$ is a $\mathbb{K}$-vector space basis for the quotient. ◇

10.2.6 Example. Consider $I = \bigl\langle yx, (x-2)^2 x\bigr\rangle = \langle x\rangle \cap \bigl\langle y, (x-2)^2\bigr\rangle$. The second component is associated to M_a where $a = (2, 0)$ and $\operatorname{mult}(I, a) = 2$. The multiplicity at $(1, 3)$ is zero and the multiplicity at $(0, 0)$ is not defined. ◇

10.2.7 Example. Consider the ideal $I = \bigl\langle y, y - x^2 + x^3\bigr\rangle$ in $\mathbb{A}^2$. It follows that $\mathrm{V}(I) = \{(0,0), (1,0)\}$. We can compute $\operatorname{mult}\bigl(I, (0,0)\bigr)$ using colon ideals. Since $\bigl(I : (I : \langle x, y\rangle^\infty)\bigr) = \langle y, x^2\rangle$ and the monomials $1, x$ are not in this ideal, we deduce that $\operatorname{mult}\bigl(I, (0,0)\bigr) = 2$. Similarly, $\bigl(I : (I : \langle x-1, y\rangle^\infty)\bigr) = \langle y, x-1\rangle$ and the monomial 1 are not in this ideal, we deduce that $\operatorname{mult}\bigl(I, (1,0)\bigr) = 1$. ◇

10.2.8 Proposition. *For any ideal I in the ring $\mathbb{K}[x_1, x_2, \dots, x_n]$ whose associated primes are all of the form M_a for some point $a \in \mathbb{A}^n$, we have*

$$\dim_{\mathbb{K}} \frac{\mathbb{K}[x_1, x_2, \dots, x_n]}{I} = \sum_{a \in \mathrm{V}(I)} \operatorname{mult}(I, a).$$

Sketch of Proof. Choose an irredundant primary decomposition of the ideal $I = Q_1 \cap Q_2 \cap \cdots \cap Q_r$ such that $\sqrt{Q_i} = M_{a_i}$ for all $1 \leqslant i \leqslant r$. There is a linear map $\varphi\colon \mathbb{K}[x_1, x_2, \dots, x_n] \to \bigoplus_{j=1}^{r} \mathbb{K}[x_1, x_2, \dots, x_n]/Q_j$ defined by $f \mapsto (f + Q_1, f + Q_2, \dots, f + Q_r)$. Since $\operatorname{Ker}(\varphi) = I$, it suffices to show that φ is surjective.

For all $1 \leqslant i \leqslant r$ and any sufficiently large integer k, we have $Q_i \supseteq M_{a_i}^k$, so the quotients

$$\frac{\mathbb{K}[x_1, x_2, \dots, x_n]}{M_{a_j}^k} \to \frac{\mathbb{K}[x_1, x_2, \dots, x_n]}{Q_j}$$

are surjective. Hence, the map φ is surjective provided that the map $\psi\colon \mathbb{K}[x_1, x_2, \dots, x_n] \to \bigoplus_{j=1}^{r} \mathbb{K}[x_1, x_2 \dots, x_n]/M_{a_j}^k$ is surjective.

Surjectivity of ψ means that there exists a polynomial with prescribed Taylor series of order k at the points $a_1, a_2, \dots, a_r$.

We proceed by induction on r. The case $r = 1$ is straightforward, because we may assume that a_1 is the origin. For the inductive step, consider the polynomials mapping to zero in $\mathbb{K}[x_1, x_2, \dots, x_n]/M_{a_j}^k$ for all $1 \leqslant j < r$ which form an ideal I'. It is enough to show that the induced map $\psi_r\colon I' \to \mathbb{K}[x_1, x_2, \dots, x_n]/M_{a_r}^k$ is surjective. The image of ψ_r is an ideal, so it suffices to check it contains a unit—an element that does not vanish at a_r. For all $1 \leqslant i < r$, let L_i be a linear form with $L_i(a_i) = 0$ but $L_i(a_r) \neq 0$. The polynomial $f = \prod_{i=1}^{r-1} L_i^k \in I'$ but $f(a_r) \neq 0$. □

10.3 The Bézout Theorem

What happens when two plane curves intersect? Using intersection multiplicities, we obtain a beautiful uniform result.

10.3.0 Lemma. *For any homogeneous ideal I in $S := \mathbb{K}[x_0, x_1, \dots, x_n]$ whose Hilbert polynomial $p_{S/I}$ has degree zero, the projective subvariety $\mathrm{V}(I)$ in $\mathbb{P}^n$ is a finite set of points.*

Sketch of Proof. Suppose $\mathrm{V}(I)$ is contains infinity many points. One of the distinguished affine open sets $U_i \subseteq \mathbb{P}^n$ contains infinitely many points of $\mathrm{V}(I)$. Without loss of generality, we may assume that $i = 0$. Let J be the dehomogenization of I with respect to the variable x_0. It follows that there are surjections

$$R := \frac{\mathbb{K}[x_0, x_1, \dots, x_n]}{I} \xrightarrow{\mu_0} \frac{\mathbb{K}[y_1, y_2, \dots, y_n]}{J} \longrightarrow \mathbb{K}[U_0 \cap \mathrm{V}(J)]\,.$$

For any $j \in \mathbb{N}$, set $W_j := \operatorname{im}\big(R_j \to \mathbb{K}[U_0 \cap \mathrm{V}(J)]\big)$; this is the set of functions on $U_0 \cap \mathrm{V}(J)$ that can be realized as polynomials of degree at most j. We have $\dim_{\mathbb{K}} W_j \leqslant \dim_{\mathbb{K}} R_j$. Since $U_0 \cap \mathrm{V}(J)$ contains infinitely many points, we deduce that $\dim_{\mathbb{K}} \mathbb{K}[U_0 \cap \mathrm{V}(J)] = \infty$ and W_j is unbounded for $j \gg 0$. On the other hand, $\dim_{\mathbb{K}} R_j$ is bounded because p_R is constant which is a contradiction. □

10.3.1 Lemma. *Assume that the field* $\mathbb{K}$ *is algebraically closed. For any saturated homogeneous ideal* I *in* $S := \mathbb{K}[x_0, x_1, \dots, x_n]$ *whose Hilbert polynomial* $p_{S/I}$ *is a nonzero constant, the associated primes of* I *are the ideals* $\langle a_1\, x_0 - a_0\, x_1, a_2\, x_0 - a_0\, x_2, \dots, a_n\, x_{n-1} - a_{n-1}\, x_n \rangle$ *for some point* $a \in \mathrm{V}(I) \subseteq \mathbb{P}^n(\mathbb{K})$.

Sketch of Proof. Since $p_{S/I}$ is nonzero, the projective Nullstellensatz establishes that $\mathrm{V}(I)$ is nonempty. The associated primes of any homogeneous ideal are also homogeneous. The only possible embedded prime is $\mathfrak{m} = \langle x_0, x_1, \dots, x_n \rangle$ which would correspond to an irrelevant primary component. In the saturated case, these do not appear. □

10.3.2 Proposition. *Assume that the field* $\mathbb{K}$ *is algebraically closed. For any homogeneous ideal* I *in the ring* $S := \mathbb{K}[x_0, x_1, \dots, x_n]$ *whose Hilbert polynomial* $p_{S/I}$ *is a nonzero constant, we have*

$$\sum_{a \in \mathrm{V}(I)} \operatorname{mult}(I, a) = p_{S/I} \,. \qquad \blacksquare$$

10.3.3 Theorem (Bézout 1779). *Assume that the field* $\mathbb{K}$ *is algebraically closed. For any two projective curves* C *and* D *in* $\mathbb{P}^n$ *having no common components, we have* $\sum_{a \in C \cap D} \operatorname{mult}\big(\mathrm{I}(C \cap D), a\big) = \deg(C)\, \deg(D)$. ■

10.3.4 Examples. Two quadric curves intersect in four points, some of which may coincide. To properly account for all intersections, we may need to consider complex coordinates or points at infinity.

- Since the intersection of the homogeneous ideals $\langle x^2 + y^2 - z^2 \rangle$ and $\langle x^2 + 3y^2 - 2z^2 \rangle$ is
$$\langle x - y, \sqrt{2}\, y - z \rangle \cap \langle x - y, \sqrt{2}\, y + z \rangle \cap \langle x + y, \sqrt{2}\, y - z \rangle \cap \langle x + y, \sqrt{2}\, y + z \rangle \,,$$
two quadrics can intersect in four distinct points. In this case, the intersection multiplicity at each point is 1.
- Since the intersection of the homogeneous ideals $\langle x^2 + y^2 - z^2 \rangle$ and $\big\langle (x - z)^2 + y^2 \big\rangle$ is $\langle x - \mathrm{i}\, y, z \rangle \cap \langle x + \mathrm{i}\, y, z \rangle \cap \big\langle x - z, y^2 \big\rangle$, two quadrics can intersect in three distinct points (two at infinity). In this case, the intersection multiplicity at the point $[1 : 0 : 1]$ is 2.
- Since the intersection of the homogeneous ideals $\big\langle x^2 + y^2 - z^2 \big\rangle$ and $\big\langle (x - z)^2 + 4y^2 - 4z^2 \big\rangle$ is
$$\langle x + z, y^2 \rangle \cap \langle 3x - z, 3y - 2\sqrt{2}\, z \rangle \cap \langle 3x - z, 3y + 2\sqrt{2}\, z \rangle \,,$$
two quadrics can intersect in three distinct points. In this case, the intersection multiplicity at the point $[-1 : 0 : 1]$ is 2.
- Since $\big\langle x^2 + y^2 - z^2, x^2 + 4y^2 - z^2 \big\rangle = \big\langle x - z, y^2 \big\rangle \cap \big\langle x + z, y^2 \big\rangle$, two quadrics can intersect in two distinct points. The intersection multiplicity at both of the points $[\pm 1 : 0 : 1]$ is 2.

- Since the intersection of the homogeneous ideals $\langle x^2 + y^2 - z^2 \rangle$ and $\langle 5x^2 + 6xy + 5y^2 + 6yz - 5z^2 \rangle$ is

$$\langle x - z, y \rangle \cap \langle y^2 - 2z(x+z), y(x+z), (x+z)^2 \rangle \ ,$$

  two quadrics can intersect in two distinct points. In this case, the intersection multiplicity at the point $[-1:0:1]$ is 3.
- Since

$$\langle x^2 + y^2 - z^2, 4x^2 + y^2 + 6xz + 2z^2 \rangle = \langle y^2 - 2x(x+z), (x+z)^2 \rangle \ ,$$

  two quadrics can intersect at a unique point. In this case, the intersection multiplicity at the point $[-1:0:1]$ is 4. ◇

Extending this result to higher-dimensional varieties is an important problem in algebraic geometry. What is the "right" notion of multiplicity?

10.3.5 Example. Consider the subvarieties $X := \mathrm{V}(x_1, x_2) \cup \mathrm{V}(x_3, x_4)$ and $Y := \mathrm{V}(x_1 - x_3, x_2 - x_4)$ in $\mathbb{P}^4$. Since

$$\frac{\mathbb{K}[x_0, x_1, \dots, x_4]}{I + J} \cong \frac{\mathbb{K}[x_0, x_3, x_4]}{\langle x_4^2, x_3 x_4, x_3^2 \rangle} = \operatorname{Span}_{\mathbb{K}}(1) \oplus \bigoplus_{i \geqslant 1} \operatorname{Span}_{\mathbb{K}}(x_0^i, x_0^{i-1} x_3, x_0^{i-1} x_4) \, ,$$

we see that $p_{X \cap Y} = 3$. However, we also have $p_X = t^2 + 3t + 1$ and $p_Y = \frac{1}{2}t^2 + \frac{3}{2}t + 1$, so $\deg(X) \ \deg(Y) = 2 \cdot 1 = 2 < 3$. ◇

Fixing these problems leads to intersection theory:
- geometric approach: Fulton
- algebraic approach: Vogel
- intersection homology, standard homological conjectures.