9 *Projective Geometry*

To increase symmetry and obtain more uniform results, we change our ambient space adding points "at infinity" and compactifying it. For this projective geometry, we leave the realm of affine subvarieties. Fortunately, there is an elegant approach to projective geometry relying on homogeneous polynomials.

9.0 Projective Varieties

How do we describe projective subvarieties? As with affine subvarieties, these subsets arise as the vanishing sets of some polynomials.

9.0.0 Remark. A polynomial f in the ring $R := \mathbb{K}[x_0, x_1, \dots, x_n]$ is *homogeneous* if all its terms have the same (total) degree. When f is homogeneous of degree d, we have

$$f(\lambda\, x_0, \lambda\, x_1, \dots, \lambda\, x_n) = \lambda^d\, f(x_0, x_1, \dots, x_n)$$

for all $\lambda \in \mathbb{K}$. Each polynomial f in R can be decomposed into *homogeneous pieces* $f = f_0 + f_1 + \cdots + f_d$ where each $f_i \in R$ is homogeneous polynomial of degree i and $\deg(f) = d$. An ideal I in R is *homogeneous* if it admits a collection of homogeneous generators. Equivalently, a polynomial belongs to a homogeneous ideal if and only if each of its homogeneous pieces is also in the ideal.

9.0.1 Proposition. *For any homogeneous ideal I in R, the set*

$$\mathrm{V}(I) := \big\{ [a_0 : a_1 : \cdots : a_n] \;\big|\; f(a_0, a_1 \dots, a_n) = 0 \text{ for all } f \in I \big\}$$

is a projective subvariety in $\mathbb{P}^n(\mathbb{K})$.

Proof. Suppose that f is a homogeneous polynomial of degree d in the ideal I. On the distinguished open subset U_i for some $0 \leqslant i \leqslant n$, we have $x_i \neq 0$, so $f = 0$ if and only if $x_i^{-d} f = 0$. Homogeneity implies that $\widehat{f} := x_i^{-d} f = f\big(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}\big)$ is a well-defined polynomial on affine open subset $U_i \cong \mathbb{A}^n(\mathbb{K})$. Hence, for each index i, the subset $U_i \cap \mathrm{V}(f) = \mathrm{V}(\widehat{f}) \subseteq U_i \cong \mathbb{A}^n(\mathbb{K})$ is an affine subvariety. Intersecting the projective subvarieties $\mathrm{V}(f)$, for all homogeneous $f \in I$, establishes the assertion. $\square$

9.0.2 Remark. A homogeneous polynomial in the ring R does not define a function from $\mathbb{P}^n(\mathbb{K})$ to $\mathbb{K}$ because evaluation depends on the choice of representative for a point in $\mathbb{P}^n$. However, for any homogeneous polynomial f of degree d and any $\lambda \in \mathbb{K}$, it follows that $f(\lambda\, a_0, \lambda\, a_1, \dots, \lambda\, a_n) = \lambda^d f(a_0, a_1, \dots, a_n)$, so the vanishing of f depends only on the equivalence class $a := [a_0 : a_1 : \cdots : a_n]$. Thus, the polynomial f determines a function from $\mathbb{P}^n$ to $\mathbb{F}_2$ by setting $f(a) = 0$ if $f(a_0, a_1, \dots, a_n) = 0$ and $f(a) = 1$ if $f(a_0, a_1, \dots, a_n) \neq 0$.

9.0.3 Definition. For any index i satisfying $0 \leqslant i \leqslant n$, the *dehomogenization relative to x_i* is the $\mathbb{K}$-algebra homomorphism

$$\mu_i \colon \mathbb{K}[x_0, x_1, \dots, x_n] \to \mathbb{K}[y_0, y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n]$$

defined by

$$x_j \mapsto \begin{cases} y_j & \text{if } j \neq i \\ 1 & \text{if } j = i. \end{cases}$$

For any polynomial f in the ring $\mathbb{K}[y_0, y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n]$, the preimage $\mu_i^{-1}(f)$ contains the set

$$\left\{ x_i^d\, f\big(\tfrac{x_0}{x_i}, \tfrac{x_1}{x_i}, \dots, \tfrac{x_{i-1}}{x_i}, \tfrac{x_{i+1}}{x_i}, \dots, \tfrac{x_n}{x_i}\big) \;\middle|\; d \geqslant \deg(f) \right\}.$$

Conversely, the *homogenization of f respect to x_i* is defined to be

$$\widetilde{f}(x_0, x_1, \dots, x_n) := x_i^{\deg(f)}\, f\big(\tfrac{x_0}{x_i}, \tfrac{x_1}{x_i}, \dots, \tfrac{x_{i-1}}{x_i}, \tfrac{x_{i+1}}{x_i}, \dots, \tfrac{x_n}{x_i}\big).$$

The homogenization of an ideal $I \subset \mathbb{K}[y_0, \dots, y_{i-1}, y_{i+1}, \dots, y_n]$ is the ideal generated by the homogenization of each $f \in I$.

9.0.4 Remark. The homogenization of an ideal may not be generated by the homogenization of its generators. The homogenization the generators of the ideal $I := \langle y_2 - y_1^2, y_3 - y_1\, y_2 \rangle$ in the ring $\mathbb{K}[y_1, y_2, y_3]$ relative to x_0 gives $J := \langle x_0\, x_2 - x_1^2, x_3\, x_0 - x_1\, x_2 \rangle$. Since it is not a $\mathbb{K}$-linear combination of the generators, the polynomial $x_2^2 - x_1\, x_3$ does not belong to J. However, this polynomial does belong to the homogenization of I, because

$$y_2^2 - y_1\, y_3 = y_2\, (y_2 - y_1^2) - y_1\, (y_3 - y_1\, y_2).$$

Hence, the ideal J is a proper subset of the homogenization of J.

9.0.5 Definition. A monomial order on the ring $\mathbb{K}[y_1, y_2, \dots, y_n]$ is *graded* if it is compatible with the partial order induced by degree; the relation $y^u = y_1^{u_1}\, y_2^{u_2} \cdots y_n^{u_n} > y_1^{v_1}\, y_2^{v_2} \cdots y_n^{v_n} = y^v$ holds whenever we have $|u| = u_0 + u_1 + \cdots + u_n > v_0 + v_1 + \cdots + v_n = |v|$.

9.0.6 Proposition. *Let I be an ideal in the ring $S := \mathbb{K}[x_1, x_2, \dots, x_n]$ and let J be its homogenization in the ring $R := \mathbb{K}[x_0, x_1, \dots, x_n]$ relative to x_0. For any Gröbner basis $g_1, g_2, \dots, g_r$ of the ideal I with respect to some graded monomial order $>_S$, the homogenizations $\widetilde{g}_1, \widetilde{g}_1, \dots, \widetilde{g}_r$ of the polynomials $g_1, g_2, \dots, g_r$ relative to x_0 generate the ideal J.*

Sketch of Proof. Consider the monomial order $>_R$ on R defined by

$$x_0^{u_0}\, x_1^{u_1} \cdots x_n^{u_n} > x_0^{v_0}\, x_1^{v_1} \cdots x_n^{v_n} \Leftrightarrow \begin{cases} \text{if } y_1^{u_1}\, y_2^{u_2} \cdots y_n^{u_n} >_S y_1^{v_1}\, y_2^{v_2} \cdots y_n^{v_n} \\ \text{or } y_1^{u_1}\, y_2^{u_2} \cdots y_n^{u_n} = y_1^{v_1}\, y_2^{v_2} \cdots y_n^{v_n} \text{ and } u_0 > v_0. \end{cases}$$

It suffices to show that $\widetilde{g}_1, \widetilde{g}_1, \dots, \widetilde{g}_r$ is a Gröbner basis with respect to $>_R$ of the ideal J. For any homogeneous polynomial $\tilde{f} \in R$ such that $f := \mu_0(\tilde{f})$ and $\operatorname{LT}(f) = c\, y_1^{u_1}\, y_2^{u_2} \cdots y_n^{u_n}$ for some $c \in \mathbb{K}$ and some $u \in \mathbb{N}^n$, observe that $\operatorname{LT}(\tilde{f}) = c\, x_0^{\deg(\tilde{f}) - \deg(f)}\, x_1^{u_1}\, x_2^{u_2} \cdots x_n^{u_n}$, so $\operatorname{LT}(f) = \mu_0\big(\operatorname{LT}(\tilde{f})\big)$. We also have $\mu_0(J) \subseteq I$.

Suppose that $\widetilde{h}$ is a homogeneous polynomial in J. It suffices to prove that $\operatorname{LT}(\widetilde{h})$ is divisible by $\operatorname{LT}(\widetilde{g}_j)$ for some $1 \leqslant j \leqslant r$. We have $h := \mu_0(\widetilde{h}) \in I$. Since $g_1, g_2, \dots, g_r$ are a Gröbner basis for I, we see that $\operatorname{LT}(h)$ is divisible by some $\operatorname{LT}(g_j)$. Applying the first observation twice, we conclude that $\operatorname{LT}(\widetilde{h})$ is divisible by $\operatorname{LT}(\widetilde{g}_j)$. □

9.1 Projective Closure

How are homogeneous ideals related to projective varieties? We seek a dictionary between homogeneous ideals and projective subvarieties.

9.1.0 Definition. For any subset $W \subseteq \mathbb{P}^n(\mathbb{K})$, the *homogeneous ideal* vanishing on W is defined to be

$$\mathrm{I}(W) := \big\langle f \in R := \mathbb{K}[x_0, x_1, \dots, x_n] \;\big|\; f \text{ is homogeneous and } f(a) = 0 \text{ for all } a \in W \big\rangle\,.$$

As in the affine dictionary, this homogeneous ideal has a geometric interpretation.

9.1.1 Proposition. *For any subset W in $\mathbb{P}^n(\mathbb{K})$, the smallest projective subvariety $\overline{W}$ containing W, also known as its* projective closure*, is defined by the homogeneous ideal vanishing on W, so $\overline{W} = \mathrm{V}\big(\mathrm{I}(W)\big)$.*

Proof. Since W lies in the projective subvariety $\mathrm{V}\big(\mathrm{I}(W)\big)$, it follows that $\overline{W} \subseteq \mathrm{V}\big(\mathrm{I}(W)\big)$. Consider a point $a \notin \overline{W}$. There is an open subset U_i in $\mathbb{P}^n(\mathbb{K})$ such that $a \in U_i$ and $x_i(a) = a_i \neq 0$. Since $U_i \cap \overline{W}$ is closed, there exists a polynomial $f \in \mathrm{I}(U_i \cap \overline{W})$ that does not vanish at a. Let $\widetilde{f}$ be the homogenization of f; we still have $\widetilde{f}(a) \neq 0$. The polynomial $\widetilde{f}$ vanishes at all the points of $\overline{W} \cap U_i$ and x_i vanishes at each point of $\overline{W}$ not contained in U_i. It follows that $x_i\, \widetilde{f} \in \mathrm{I}(W)$ and $(x_i\, \widetilde{f})(a) \neq 0$, so $a \notin \mathrm{V}\big(\mathrm{I}(W)\big)$. □

Given a projective subvariety, we want to find a homogeneous ideal that vanishes on it.

9.1.2 Lemma. *For any affine subvariety X in $\mathbb{A}^n \cong U_0 \subset \mathbb{P}^n$ given by the ideal I in $\mathbb{K}[y_1, y_2, \dots, y_n]$, the homogeneous ideal $\mathrm{I}(X)$ vanishing on X is the homogenization J of the ideal I in $\mathbb{K}[x_0, x_1, \dots, x_n]$ and the projective closure of X is $\overline{X} = \mathrm{V}(J)$.*

Proof. Provided $\mathrm{I}(X) = J$, Proposition 9.1.1 shows that $\mathrm{V}(J) = \overline{X}$.

$\supseteq$: For each homogeneous $f \in J$, set $\widehat{f} := \mu_0(f) \in I$. It follows that f vanishes on the point $[1 : a_1 : \cdots : a_n] \in \mathbb{P}^n$ whenever we have $(a_1, a_2, \ldots, a_n) \in X$, so we deduce that $f \in \mathrm{I}(X)$.

$\subseteq$: Given a homogeneous $f \in \mathrm{I}(X)$, we have $f(1, b_1, b_2, \ldots, b_n) = 0$ for all $(b_1, b_2, \ldots, b_n) \in X$. Setting $\widehat{f} := \mu_0(f)$, we have $\widehat{f} \in I$. Since $f = x_0^{\deg(\widehat{f})} f\big(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0}\big) = x_0^{\deg(\widehat{f}) - \deg(f)} \widehat{f}^{\mathsf{h}}$ where $\widehat{f}^{\mathsf{h}}$ is the homogenization of $\widehat{f}$, we conclude that $f \in J$. □

9.1.3 Proposition. *Let $X \subseteq \mathbb{P}^n$ be a projective variety. For all $0 \leqslant i \leqslant n$, consider the ideal $I_i := \mathrm{I}(U_i \cap X)$ in $\mathbb{K}[y_0, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n]$ and let J_i denote the homogenization of I_i relative to x_i. The homogeneous ideal $\mathrm{I}(X)$ in $\mathbb{K}[x_0, x_1, \ldots, x_n]$ vanishing on X equals $J_0 \cap J_1 \cap \cdots \cap J_n$.*

Proof. We prove containment in both directions.

$\subseteq$: For all $0 \leqslant i \leqslant n$, we have $X \supseteq \overline{U_i \cap X}$ so $\mathrm{I}(X) \subseteq \mathrm{I}(U_i \cap X) = J_i$ by Lemma 9.1.2.

$\supseteq$: As each $a \in X$ is contained in an open set U_i for some $0 \leqslant i \leqslant n$, we have $X \subseteq \bigcup_i (X \cap U_i)$ and $\mathrm{I}(X) \supseteq \bigcap_i \mathrm{I}(X \cap U_i) = \bigcap_i J_i$. □

9.1.4 Corollary. *For any projective subvariety $X \subseteq \mathbb{P}^n$, there exists a homogeneous ideal J in $\mathbb{K}[x_0, x_1, \ldots, x_n]$ such that $X = \mathrm{V}(J)$.* □

As in affine algebraic geometry, a version of the Nullstellensatz is needed to identify all the homogeneous ideals that vanish on a projective subvariety.

9.1.5 Theorem (Projective Weak Nullstellensatz)**.** *Assume that $\mathbb{K}$ is an algebraically closed field. For any homogeneous ideal I in the polynomial ring $R := \mathbb{K}[x_0, x_1, \ldots, x_n]$, the following are equivalent.*

(a) *The affine subvariety $\mathrm{V}(I)$ in $\mathbb{A}^{n+1}$ is a finite set.*
(b) *For each $0 \leqslant i \leqslant n$, we have $x_i^{m_i} \in I$ for some nonnegative integer m_i.*
(c) *For each $0 \leqslant i \leqslant n$, there exists a polynomial g in the reduced Gröbner basis of the ideal I such that $\mathrm{LT}(g)$ is a nonnegative power of x_i.*
(d) *The projective subvariety $\mathrm{V}(I)$ in $\mathbb{P}^n$ is empty.*
(e) *The radical of ideal I is either $\langle x_0, x_1, \ldots, x_n \rangle$ or $\langle 1 \rangle$.*
(f) *There exists a positive integer d such that every polynomial of degree greater than d is contained in the ideal I.*

Proof.

(a) $\Rightarrow$ (b): When $\mathrm{V}(I) = \varnothing$ in $\mathbb{A}^n$, the Weak Nullstellensatz 6.0.1 establishes that 1 belongs to I, so $m_i = 0$ for all i suffices. Assume that $\mathrm{V}(I) \neq \varnothing$ in $\mathbb{A}^n$. For any $0 \leqslant i \leqslant n$, let $a_1, a_2, \ldots, a_\ell \in \mathbb{K}$ be i-th coordinates of the points in $\mathrm{V}(I)$. It follows that the polynomial $f_i := \prod_{j=1}^{\ell} (x_i - a_j)$ vanishes on $\mathrm{V}(I)$ and $f_i \in \mathrm{I}\big(\mathrm{V}(I)\big)$. By the Hilbert Nullstellensatz 6.0.2, there is a positive integer m_i such that $f_i^{m_i} \in I$.

The homogeneous piece of $f_i^{m_i}$ having degree $\ell\, m_i$, namely the monomial $x_i^{\ell\, m_i}$, belongs to the homogeneous ideal I.

(b) $\Rightarrow$ (c): For all $0 \leqslant i \leqslant n$, suppose that $x_i^{m_i} \in I$. Since $x_i^{m_i} \in \mathrm{LT}(I)$, there exists a polynomial g in the reduced Gröbner basis of the ideal I such that $\mathrm{LT}(g)$ divides $x_i^{m_i}$.

(c) $\Rightarrow$ (a): For all $0 \leqslant i \leqslant n$, suppose that $x_i^{m_i} \in \mathrm{LT}(I)$. When we have $u_i \geqslant m_i$ for all $0 \leqslant i \leqslant n$, the monomial $x_0^{u_0}\, x_1^{u_1} \cdots x_n^{u_n}$ lies in the ideal $\mathrm{LT}(I)$. Hence, the number of monomials not in $\mathrm{LT}(I)$ is at most $m_0\, m_1 \cdots m_n$. The monomials not in $\mathrm{LT}(I)$ form a $\mathbb{K}$-vector space basis for the quotient R/I. It suffices to show that, for any $0 \leqslant i \leqslant n$, there can be only finitely many distinct i-th coordinates for points in $\mathrm{V}(I)$. Since $\mathbb{K}$-vector space R/I is finite-dimensional, there exists a nonnegative integer m and $c_0, c_1, \dots, c_m \in \mathbb{K}$, not all zero, such that $c_m\, [x_i^m] + c_{m-1}\, [x_i^{m-1}] + \cdots + c_0\, [x_i^0] = 0$. It follows that $c_m\, x_i^m + c_{m-1}\, x_i^{m-1} + \cdots + c_0\, x_i^0 \in I$. Since a nonzero polynomial in one variable can have only finitely many zeros, the points of $\mathrm{V}(I)$ have only finitely many different i-th coordinates.

(a) $\Rightarrow$ (d): Suppose that $[a_0 : a_1 : \cdots : a_n] \in \mathrm{V}(I) \subseteq \mathbb{P}^n$. The line passing through the origin and the point $(a_0, a_1, \dots, a_n)$ lies in $\mathrm{V}(I) \subseteq \mathbb{A}^{n+1}$. Since $\mathbb{K}$ is infinite, this is an infinite set.

(d) $\Rightarrow$ (e): Suppose that $\mathrm{V}(I) = \varnothing$ in $\mathbb{P}^n$. It follows that $\mathrm{V}(I)$ in $\mathbb{A}^n$ is contained in $\{(0, 0, \dots, 0)\}$. Since $\langle x_0, x_1, \dots, x_n \rangle \subseteq \mathrm{I}\big(\mathrm{V}(I)\big)$ and the Strong Nullstellensatz 6.0.5 yields $\mathrm{I}\big(\mathrm{V}(I)\big) = \sqrt{I}$, the radical ideal $\sqrt{I}$ is either $\langle x_0, x_1, \dots, x_n \rangle$ or $\langle 1 \rangle$.

(e) $\Rightarrow$ (f) and (f) $\Rightarrow$ (b): Both implications are tautological. $\square$

9.2 Projective Dictionary

Which ideals corresponds to projective subvarieties?

9.2.0 Definition. The monomial ideal $\mathfrak{m} = \langle x_0, x_1, \dots, x_n \rangle$ in the ring $R := \mathbb{K}[x_0, x_1, \dots, x_n]$ is the *irrelevant ideal* because $\mathrm{V}(\mathfrak{m}) = \varnothing$ in $\mathbb{P}^n$.

9.2.1 Definition. The *saturation* of an ideal I in R with respect to the irrelevant ideal $\mathfrak{m}$ is the set

$$(I : \mathfrak{m}^\infty) := \big\{ f \in R \;\big|\; \text{for all } g \in \mathfrak{m} \text{ there exists a nonnegative integer } m \text{ such that } f\, g^m \in I \big\}\,.$$

An ideal I is *saturated* if $I = (I : \mathfrak{m}^\infty)$.

9.2.2 Lemma. *For any ideal I in the polynomial ring R, the saturation $(I : \mathfrak{m}^\infty)$ is an ideal. We also have the inclusions $I \subseteq (I : \mathfrak{m}) \subseteq (I : \mathfrak{m}^\infty)$, the equality $(I : \mathfrak{m}^\ell) = (I : \mathfrak{m}^\infty)$ for all sufficiently large integers ℓ, and the equality $\sqrt{I : \mathfrak{m}^\infty} = (\sqrt{I} : \mathfrak{m})$.*

Sketch of Proof. The first parts are essentially the same as the special case on the problem set. We establish that $\sqrt{I : \mathfrak{m}^\infty} = \sqrt{I} : \mathfrak{m}$.

- Suppose that $f \in \sqrt{I : \mathfrak{m}^\infty}$. There exists some positive integer m such that $f^m \in (I : \mathfrak{m}^\infty)$. Given $g \in \mathfrak{m}$, we see that $f^m g^\ell \in I$ for some positive integer ℓ. It follow that $(f\,g)^{\max(m,\ell)} \in I$, so we have $f\,g \in \sqrt{I}$. Since this holds for all $g \in \mathfrak{m}$, we deduce that $f \in (\sqrt{I} : \mathfrak{m})$.
- Suppose that $f \in (\sqrt{I} : \mathfrak{m})$. For all $0 \leqslant i \leqslant n$, we have $f\,x_i \in \sqrt{I}$. Thus, there exists a positive integer m such that $(f\,x_i)^m \in I$. It follows that $f^m\,\mathfrak{m}^{(n+1)m} \subseteq I$, so $f^m \in (I : \mathfrak{m}^{(n+1)m}) \subseteq (I : \mathfrak{m}^\infty)$. We conclude that $f \in \sqrt{I : \mathfrak{m}^\infty}$. ☐

9.2.3 Lemma. *For any ideal I in the ring R, we have the inclusion of affine subvarieties $\overline{\mathrm{V}(I) \setminus \mathrm{V}(\mathfrak{m})} \subseteq \mathrm{V}(I : \mathfrak{m}^\infty)$ in $\mathbb{A}^{n+1}$. When $\mathbb{K}$ is an algebraically closed field, we also have $\overline{\mathrm{V}(I) \setminus \mathrm{V}(\mathfrak{m})} = \mathrm{V}(I : \mathfrak{m}^\infty)$ in $\mathbb{A}^{n+1}$.*

Proof. For any two ideal I and J in R, Theorem 7.0.6 establishes the inclusion of affine subvarieties $\overline{\mathrm{V}(I) \setminus \mathrm{V}(J)} \subseteq \mathrm{V}(I : J)$ in $\mathbb{A}^{n+1}$, where equality holds when $\mathbb{K}$ is algebraically closed. Lemma 9.2.2 shows that $(I : \mathfrak{m}^\ell) = (I : \mathfrak{m}^\infty)$, for all sufficiently large integers ℓ. ☐

9.2.4 Proposition. *Assume that the field $\mathbb{K}$ is algebraically closed. For any homogeneous ideal I in R, the projective subvariety $\mathrm{V}(I)$ in $\mathbb{P}^n$ is empty if and only if $(I : \mathfrak{m}^\infty) = R$.*

Proof. The affine subvariety $\mathrm{V}(I)$ in $\mathbb{A}^n$ is contained in $\{0\}$ if and only if $\varnothing = \overline{\mathrm{V}(I) \setminus \mathrm{V}(\mathfrak{m})} = \mathrm{V}(I : \mathfrak{m}^\infty)$. By the weak nullstellensatz, these equivalent conditions are the same as $I : \mathfrak{m}^\infty = R$. ☐

9.2.5 Lemma. *The radical of any homogeneous ideal is homogeneous.*

Proof. Consider the polynomial $f = f_0 + f_1 + \cdots + f_d \in \sqrt{I}$ where f_i is a homogeneous polynomial of degree i and $\deg(f) = d$. We must show that each homogeneous piece f_i belongs to $\sqrt{I}$. We proceed by induction on the number of pieces. The assertion is trivial when this number is 0 or 1. If one proves that $f_d \in \sqrt{I}$, then the induction hypothesis applied to $f - f_d$ will establish the claim. Since we have $f \in \sqrt{I}$, there exists a positive integer m such that $f^m \in I$. Expanding $(f_0 + f_1 + \cdots + f_d)^m$, we see that the top degree piece is f_d^m. As I is a homogeneous ideal, $f_d^m \in I$ and $f_d \in \sqrt{I}$. ☐

9.2.6 Theorem (Projective Strong Nullstellensatz)**.** *Assume that $\mathbb{K}$ is an algebraically closed field. For any homogeneous ideal I in the polynomial ring R such that $(I : \mathfrak{m}^\infty) \neq \langle 1 \rangle = R$, the homogenous ideal vanishing on the nonempty projective variety $\mathrm{V}(I)$ is $\mathrm{I}\big(\mathrm{V}(I)\big) = \sqrt{I}$.*

Proof. Let C be the affine subvariety in $\mathbb{A}^{n+1}$ defined by the ideal I and let X be the projective subvariety in $\mathbb{P}^n$ defined by the same ideal I. We first claim that ideal $\mathrm{I}(C)$ in R vanishing on C is equal to the homogeneous ideal $\mathrm{I}(X)$ in R vanishing on X.

$\subseteq$: Suppose that the homogeneous polynomial f belongs to $\mathrm{I}(C)$. Given a point $[a_0 : a_1 : \cdots : a_n] \in X$, the entire line through the origin and the point $(a_0, a_1, \ldots, a_n) \in \mathbb{A}^{n+1}$ lies in the affine subvariety C. Since f vanishes at all points on this line, it follows that $f \in \mathrm{I}(X)$ and $\mathrm{I}(C) \subseteq \mathrm{I}(X)$.

$\supseteq$: Suppose that the polynomial f belongs to $\mathrm{I}(X)$. Since any nonzero points in C gives the homogeneous coordinates for a point in X, it follows that f vanishes on $C \setminus \{0\}$. It remains to show that f vanishes at the origin. Since the ideal $\mathrm{I}(X)$ is homogeneous, we know that the homogeneous pieces f_i of f, where $f = f_0 + f_1 + \cdots + f_d$ and $\deg(f) = d$, vanish on X. Hence, the constant term f_0 vanishes on X. Since $X \neq \varnothing$, we have $f_0 = 0$ and f vanishes at the origin.

The Strong Nullstellensatz 6.0.5 implies that $\sqrt{I} = \mathrm{I}(C) = \mathrm{I}(X)$. □

9.2.7 Theorem. *For any algebraically closed field $\mathbb{K}$, we have*

$$\left\{ \begin{array}{c} \textit{projective subvarieties} \\ \textit{in } \mathbb{P}^n(\mathbb{K}) \end{array} \right\} \xrightarrow{\mathrm{I}} \left\{ \begin{array}{c} \textit{saturated radical} \\ \textit{homogeneous ideals in } R \end{array} \right\}$$

$$\left\{ \begin{array}{c} \textit{projective subvarieties} \\ \textit{in } \mathbb{P}^n(\mathbb{K}) \end{array} \right\} \xleftarrow{\mathrm{V}} \left\{ \begin{array}{c} \textit{saturated radical} \\ \textit{homogeneous ideals in } R \end{array} \right\}$$

are inclusion-reversing bijections. Furthermore, the irreducible projective subvarieties correspond to homogeneous prime ideals.

Sketch of Proof. Combine Proposition 9.2.4, the Projective Strong Nullstellensatz, and the affine case. □