6 *Affine Dictionary*

We develop the fundamental relationship between geometry and algebra. Over an algebraically closed field, there is a remarkably thorough correspondence between affine subvarieties and ideals in a polynomial ring.

6.0 Nullstellensatz

'Nullstellensatz' is a German word that means 'zero places theorem'. This family of theorems resolve the following question: given an ideal I in the polynomial ring $S := \mathbb{K}[x_1, x_2, \dots, x_n]$ defining the affine subvariety $\mathrm{V}(I)$, which polynomials vanish on $\mathrm{V}(I)$?

What is the ideal $\mathrm{I}\big(\mathrm{V}(I)\big)$?

6.0.0 Example. Given the polynomial $f = (x-3)^4(x-2)^2$ in $\mathbb{Q}[x]$, we have $\mathrm{V}(f) = \{2, 3\}$ in $\mathbb{A}^1(\mathbb{Q})$ and $\mathrm{I}\big(\mathrm{V}(f)\big) = \langle (x-3)(x-2) \rangle \supset \langle f \rangle$. ⋄

6.0.1 Theorem (Weak Nullstellensatz). *Let $\mathbb{K}$ be an algebraically closed field. For any ideal I in S such that $\mathrm{V}(I) = \varnothing$ in $\mathbb{A}^n(\mathbb{K})$, we have $I = \langle 1 \rangle$.*

The weak Nullstellensatz is false over $\mathbb{R}$: $\mathrm{V}_{\mathbb{R}}(x^2+1) = \varnothing$ but $\langle x^2+1 \rangle \neq \mathbb{R}[x]$.

Proof. We proceed by induction on the number n of variables in the ring S. The case $n = 0$ is trivial because $\mathbb{K}$ is a field. Since the ring $\mathbb{K}[x_1]$ is a principal ideal domain and every non-contant polynomial over an algebraically closed field has a root, the base case holds.

The weak Nullstellensatz generalizes the fundamental theorem of algebra; every system of polynomials that generates an ideal smaller than $\mathbb{C}[x_1, x_2, \dots, x_n]$ has a zero in $\mathbb{A}^n(\mathbb{C})$.

Fix an ideal $I = \langle f_1, f_2, \dots, f_r \rangle$ in $\mathbb{K}[x_1, x_2, \dots, x_n]$. We may assume that $f_1 \notin \mathbb{K}$. Suppose that $\deg(f_1) = \ell$. Consider the invertible linear change of coordinates:

$$x_1 = \widetilde{x}_1 \qquad x_2 = \widetilde{x}_2 + a_2\,\widetilde{x}_1 \qquad \cdots \qquad x_n = \widetilde{x}_n + a_n\,\widetilde{x}_1\,,$$

where the coefficients a_i are to-be-determined elements of $\mathbb{K}$. Under this linear change of coordinates, the generator f_1 has the form

$$\begin{aligned} f_1(x_1, x_2, \dots, x_n) &= f_1(\widetilde{x}_1, \widetilde{x}_2 + a_2\,\widetilde{x}_1, \dots, \widetilde{x}_n + a_n\,\widetilde{x}_1) \\ &= g(a_2, a_3, \dots, a_n)\,\widetilde{x}_1^{\ell} + \text{terms in which } \widetilde{x}_1 \text{ has degree less than } \ell\,. \end{aligned}$$

As no algebraic closed field is finite, we may choose the coefficients $a_2, a_2, \dots, a_n$ so that $g(a_2, a_3, \dots, a_n) \neq 0$. It suffices to prove that $1 \in \widetilde{I}$. With this coordinate change, we can apply the Extension Theorem 5.2.1. Setting $J := \widetilde{I} \cap \mathbb{K}[\widetilde{x}_2, \widetilde{x}_3, \dots, \widetilde{x}_n]$, a partial solution in $\mathbb{A}^{n-1}(\mathbb{K})$ always extends: $\mathrm{V}(J) = \pi_2\big(\mathrm{V}(\widetilde{I})\big) = \varnothing$. By the induction hypothesis, we see that $1 \in J \subseteq \widetilde{I}$. □

Having understood the empty affine subvariety, we next examine the functions vanishing on a nonempty affine subvariety.

6.0.2 Theorem (Hilbert Nullstellensatz)**.** *Let $\mathbb{K}$ be an algebraically closed field. For any polynomials $g_1, g_2, \dots, g_r$ in the ring $S := \mathbb{K}[x_1, x_2, \dots, x_n]$, the polynomial f belongs to the ideal $\mathrm{I}\big(\mathrm{V}(g_1, g_2, \dots, g_r)\big)$ if and only if there is a positive integer m such that f^m belongs to the ideal $\langle g_1, g_2, \dots, g_r \rangle$.*

David Hilbert proved this famous theorem in a 1893 paper on invariant theory.

Proof. Given that f vanishes at the common zeros of $g_1, g_2, \dots, g_r$, we must show that there exist a positive integer m and polynomials h_i in the ring S, for all $1 \leqslant i \leqslant r$, such that $f^m = h_1\, g_1 + h_2\, g_2 + \cdots + h_r\, g_r$. Set $\widetilde{I} := \langle g_1, g_2, \dots, g_r, 1 - y\, f \rangle$ in $\mathbb{K}[x_1, x_2, \dots, x_n, y] = S[y]$. We claim that $\mathrm{V}(\widetilde{I}) = \varnothing$ in $\mathbb{A}^{n+1}$. For any point $(a_1, a_2, \dots, a_{n+1})$ in $\mathbb{A}^{n+1}$, there are two possibilities:

- Suppose that $(a_1, a_2, \dots, a_n) \in \mathrm{V}(g_1, g_2, \dots, g_r)$. It follows that $f(a_1, a_2, \dots, a_n) = 0$. Hence, the polynomial $1 - y\, f$ does not vanish at the point $(a_1, a_2, \dots, a_{n+1})$.
- Suppose that $(a_1, a_2, \dots, a_n) \notin \mathrm{V}(g_1, g_2, \dots, g_r)$. There exists an index $1 \leqslant i \leqslant r$ such that $g_i(a_1, a_2, \dots, a_n) \neq 0$, which implies that $\pi_1^*(g_i)(a_1, a_2, \dots, a_{n+1}) \neq 0$.

The Weak Nullstellensatz establishes that $1 \in \widetilde{I}$. Hence, there exist polynomials $h_1, h_2, \dots, h_{r+1} \in S[y]$ such that

$$1 = h_1\, g_1 + h_2\, g_2 + \cdots + h_r\, g_r + h_{r+1}\, (1 - y\, f)\,.$$

Setting $y = 1/f(x_1, x_2, \dots, x_n)$ gives

$$\begin{aligned} 1 &= h_1(x_1, x_2, \dots, x_n, 1/f)\, g_1 + h_2(x_1, x_2, \dots, x_n, 1/f)\, g_2 \\ &\quad + \cdots + h_r(x_1, x_2, \dots, x_n, 1/f)\, g_r \end{aligned}$$

and clearing denominators yields the required relation. □

Can the Nullstellensatz be made effective? Consider $I := \langle g_1, g_2, \dots, g_r \rangle$. For any $f \in \mathrm{I}\big(\mathrm{V}(I)\big)$, we have $f^m \in I$ for some $m > 0$. Can we bound m in terms of, say, the degrees of the polynomials g_i? Janos Kollár shows that, if I is generated by r homogenous polynomials g_i of degree greater than 2, then $f \in \sqrt{I}$ implies that $f^m \in I$ for some $m \leqslant \prod_{i=1}^r \deg(g_i)$. If $r < n$, this result is sharp (no smaller value of m will work in general). Kollár also finds sharp bounds for m when $r \geqslant n$.

6.0.3 Definition. The *radical* of an ideal I is the set of all elements f in the ring S such that some power of f lies in I;

$$\sqrt{I} := \big\{ f \mid f^m \in I \text{ for some positive integer } m \big\}\,.$$

By construction, we have $I \subseteq \sqrt{I}$.

For any $f = a_\ell \prod_{i=1}^{\ell} (x - \alpha_i)^{m_i}$, we have $\sqrt{\langle f \rangle} = \Big\langle a_\ell \prod_{i=1}^{\ell} (x - \alpha_i) \Big\rangle$, so the radical is generated by the square-free part of the univariate polynomial f.

6.0.4 Lemma. *For any ideal I, $\sqrt{I}$ is also an ideal and $\sqrt{\sqrt{I}} = \sqrt{I}$.*

Proof. For any two elements f and g in the radical $\sqrt{I}$, there are positive integers p and q such that $f^p \in I$ and $g^q \in I$. For any r and s in the ring S, every term in the binomial expansion of $(r\, f + s\, g)^{p+q-1}$ has a factor $f^i\, g^j$ with $i + j = p + q - 1$. Since either $i \geqslant p$ or $j \geqslant q$, it follows that $(r\, f + s\, g)^{p+q-1}$ lies in the ideal I and the element $r\, f + s\, g$ lies in the radical ideal $\sqrt{I}$. For the second assertion, we have

$$\begin{aligned} \sqrt{\sqrt{I}} &= \big\{ f \mid f^m \in \sqrt{I} \text{ for some positive integer } m \big\} \\ &= \big\{ f \mid f^{m\ell} \in I \text{ for positive integers } \ell \text{ and } m \big\} = \sqrt{I}\,. \end{aligned}$$ □

6.0.5 Theorem (Strong Nullstellensatz). *Assume that the coefficient field* $\mathbb{K}$ *is algebraically closed. For any ideal* I *in* S*, we have* $\operatorname{I}\big(\operatorname{V}(I)\big) = \sqrt{I}$.

Proof.
$\supseteq$: The membership $f \in \sqrt{I}$ implies that $f^m \in I$ for some positive integer m. Hence, the power f^m vanishes on the affine subvariety $\operatorname{V}(I)$, so f vanishes on $\operatorname{V}(I)$ and $f \in \operatorname{I}\big(\operatorname{V}(I)\big)$.
$\subseteq$: Suppose that f vanishes on $\operatorname{V}(I)$. By the Hilbert Nullstellensatz, there exists a positive integer m such that $f^m \in I$, so $f \in \sqrt{I}$. □

6.0.6 Remark. Over an algebraically closed field, the inclusion-reversing bijections

$$\begin{aligned} &\operatorname{I}\colon \{\text{affine subvarieties}\} \to \{\text{radical ideals}\} \\ &\operatorname{V}\colon \{\text{radical ideals}\} \to \{\text{affine subvarieties}\} \end{aligned}$$

are mutual inverses.

6.1 Maximal and Prime Ideals

Given an ideal I in the polynomial ring $S := \mathbb{K}[x_1, x_2, \dots, x_n]$, can we find generators for its radical ideal $\sqrt{I}$? More modestly, we can determine if a given polynomial belongs to a radical ideal.

Currently, there are three algorithms for compute the radical of an ideal; Eisenbud–Huneke–Vasconcelos, Shimoyama–Yokoyama, and Gianni–Trager–Zacharias. None of these method is best and all three algorithms are computationally expensive.

6.1.0 Proposition (Radical membership). *Let* $\mathbb{K}$ *be an arbitrary field. For any ideal* $I = \langle g_1, g_2, \dots, g_r\rangle$ *in* S*, we have* $f \in \sqrt{I}$ *if and only if the ideal* $\langle g_1, g_2, \dots, g_r, 1 - y\,f\rangle$ *equals the ring* $S[y] = \mathbb{K}[x_1, x_2, \dots, x_n, y]$.

Proof. Set $\widetilde{I} := \langle g_1, g_2, \dots, g_r, 1 - y\,f\rangle$.
$\Rightarrow$: Suppose that $f \in \sqrt{I}$. There exists a positive integer m such that $f^m \in I$ and $f^m = \pi_1^*(f^m) \in \widetilde{I}$. Moreover, we have

$$\begin{aligned} 1 &= y^m f^m + (1 - y^m f^m) \\ &= y^m f^m + (1 + yf + \cdots + y^{m-1} f^{m-1})(1 - yf) \in \widetilde{I}\,. \end{aligned}$$

$\Leftarrow$: Suppose that $1 \in \widetilde{I}$. There exists polynomials $h_1, h_2, \dots, h_{r+1}$ in the ring $S[y]$ such that $1 = h_1\,g_1 + h_2\,g_2 + \cdots + h_r\,g_r + h_{r+1}\,(1 - y\,f)$. By setting $y = 1/f$, we obtain

$$1 = h_1(x_1, x_2, \dots, x_n, 1/f)\,g_1 + \cdots + h_r(x_1, x_2, \dots, x_n, 1/f)\,g_r$$

and clearing denominators yields the required relationship. □

6.1.1 Definition. An ideal I in S is *maximal* if $1 \notin I$ and I is maximal with respect to inclusion. Equivalently, I is maximal if and only if the quotient S/I is a field; the only ideals in a field are 0 and $\langle 1\rangle = S$.

6.1.2 Proposition. *For any point $(a_1, a_2, \dots, a_n)$ in $\mathbb{A}^n(\mathbb{K})$, the corresponding ideal $I := \langle x_1 - a_1, x_2 - a_2, \dots, x_n - a_n \rangle$ in the ring S is maximal. When the field $\mathbb{K}$ is algebraically closed, every maximal ideal in the ring S corresponds to some point $(a_1, a_2, \dots, a_n)$ in $\mathbb{A}^n(\mathbb{K})$.*

There is a bijection between points of the affine space $\mathbb{A}^n(\overline{\mathbb{K}})$ and maximal ideals in the ring $\overline{\mathbb{K}}[x_1, \dots, x_n]$.

Proof. Suppose that the ideal J properly contains I. There exists an element $f \in J$ such that $f \notin I$. Using the division algorithm, there exists $q_1, q_2, \dots, q_n$ in S and r in $\mathbb{K}$ such that

$$f = q_1(x_1 - a_1) + q_2(x_2 - a_2) + \cdots + q_n(x_n - a_n) + r\,.$$

Since $f \notin I$, we have $r \neq 0$. However, the memberships $f \in J$ and $q_1(x_1 - a_1) + \cdots + q_n(x_n - a_n) \in I \subset J$ imply that $r \in J$, so $J = S$.

Assume that $\mathbb{K}$ is algebraically closed. Let I be a maximal ideal. Since $1 \notin I$, the Weak Nullstellensatz shows that $\mathrm{V}(I) \neq \varnothing$. There is a point $(a_1, a_2, \dots, a_n) \in \mathrm{V}(I)$, so $\mathrm{I}\big(\mathrm{V}(I)\big) \subseteq \mathrm{I}\big(\{(a_1, a_2, \dots, a_n)\}\big)$. Clearly, we have $\langle x_1 - a_1, x_2 - a_2, \dots, x_n - a_n \rangle \subseteq \mathrm{I}\big(\{(a_1, a_2, \dots, a_n)\}\big)$. For any $f \in \mathrm{I}\big(\{(a_1, a_2, \dots, a_n)\}\big)$, the division algorithm proves that there arepolynomials $q_1, q_2, \dots, q_r$ in the ring S and a constant r in the field $\mathbb{K}$ such that $f = q_1(x_1 - a_1) + q_2(x_2 - a_2) + \cdots + q_n(x_n - a_n) + r$. Evaluating at the point $(a_1, a_2, \dots, a_n)$ in $\mathrm{V}(I)$, we see that $r = 0$. It follows that $\mathrm{I}\big(\{(a_1, a_2, \dots, a_n)\}\big) = \langle x_1 - a_1, x_2 - a_2, \dots, x_n - a_n \rangle$. The maximality of the ideal I yields the required equality. □

6.1.3 Definition. An ideal I is *prime* if $I \neq \langle 1 \rangle$ and, for any two elements f and g in the ring S, the membership $f\,g \in I$ implies that $f \in I$ or $g \in I$. Equivalently, the ideal I is prime if and only if the quotient ring S/I is a domain (no zerodivisors). It follows that every prime ideal is a radical ideal.

6.1.4 Examples. A principal ideal $\langle f \rangle$ in S is prime if and only if f is an irreducible polynomial. Any maximal ideal I is prime because the field S/I has no zerodivisors. ◇

6.1.5 Proposition. *An affine subvariety X in $\mathbb{A}^n$ is irreducible if and only if its ideal $\mathrm{I}(X)$ is prime.*

There is a bijection between the irreducible affine subvarieties in $\mathbb{A}^n(\mathbb{K})$ and the prime ideals in the ring $\mathbb{K}[x_1, x_2, \dots, x_n]$.

Proof. Suppose that the affine subvariety X is irreducible. For any two polynomials f and g in the ring S such that $f\,g \in \mathrm{I}(X)$, we have $X \subseteq \mathrm{V}(f\,g) = \mathrm{V}(f) \cup \mathrm{V}(g)$ and $X = \big(X \cap \mathrm{V}(f)\big) \cup \big(X \cap \mathrm{V}(g)\big)$. Since X is irreducible, we have either $X = X \cap \mathrm{V}(f)$ or $X = X \cap \mathrm{V}(g)$, so $X \subseteq \mathrm{V}(f)$ or $X \subseteq \mathrm{V}(g)$. Thus, either $f \in \mathrm{I}(X)$ or $g \in \mathrm{I}(X)$.

Conversely, let I be a prime ideal and suppose $\mathrm{V}(I) = X \cup Y$ with $\mathrm{V}(I) \neq X$ and $\mathrm{V}(I) \neq Y$. It follows that $I = \mathrm{I}(X) \cap \mathrm{I}(Y)$ where $I \neq \mathrm{I}(X)$ and $I \neq \mathrm{I}(Y)$. Hence, there exists $f \in \mathrm{I}(X)$ and $g \in \mathrm{I}(Y)$ such that $f \notin I$ and $g \notin I$. However, we have $f\,g \in \mathrm{I}(X) \cap \mathrm{I}(Y) = I$ which is a contradiction. □

6.1.6 Example. A hypersurface $\mathrm{V}(f)$ in $\mathbb{A}^n$ is irreducible if and only if f is an irreducible polynomial. ◇

6.1.7 Proposition. *Let $\mathbb{K}$ be an infinite field. For any rational map $\rho\colon \mathbb{A}^n(\mathbb{K}) \dashrightarrow \mathbb{A}^m(\mathbb{K})$, the affine subvariety $X = \overline{\rho(\mathbb{A}^n)}$ is irreducible.*

Proof. Let $\rho_j = f_j/g_j$ for all $1 \leqslant j \leqslant m$ and set $W := \mathrm{V}(g_1\, g_2 \cdots g_m)$. Since the affine subvariety X is the Zariski closure of the image $\rho(\mathbb{A}^n \setminus W)$, the ideal $\mathrm{I}(X)$ is the set of $h \in \mathbb{K}[y_1, y_2, \dots, y_m]$ such that the function $h \circ \rho$ is zero for any point $(a_1, a_2, \dots, a_n) \in \mathbb{A}^n \setminus W$. The product $g_1\, g_2 \cdots g_m$ does not vanish at any $(a_1, a_2, \dots, a_n) \in \mathbb{A}^n \setminus W$, so the function $(g_1\, g_2 \cdots g_m)^N\,(h \circ \rho)$ is equal to zero at the values of $(x_1, x_2, \dots, x_n) \in \mathbb{A}^n \setminus W$ for which $h \circ \rho$ is equal to zero. Setting ℓ to be the total degree of h, the product $(g_1\, g_2 \cdots g_m)^\ell\,(h \circ \rho)$ is a polynomial in $\mathbb{K}[x_1, x_2, \dots, x_n]$. We deduce that $h \in \mathrm{I}(X)$ if and only if $(g_1\, g_2 \cdots g_m)^\ell\,(h \circ \rho)$ is zero for all $(a_1, a_2, \dots, a_n) \in \mathbb{A}^n \setminus W$ which means that $(g_1\, g_2 \cdots g_m)^\ell\,(h \circ \rho)$ is the zero polynomial. We conclude that $h \in \mathrm{I}(X)$ if and only if $(g_1\, g_2 \cdots g_m)^\ell\,(h \circ \rho) = 0$.

Suppose that polynomials f and g in the ring $\mathbb{K}[y_1, y_2, \dots, y_m]$ satisfy $f\,g \in \mathrm{I}(X)$. Let ℓ and m denote the total degree of f and g respectively. The total degree of the product $f\,g$ in $\ell + m$, so $(g_1\, g_2 \cdots g_m)^{\ell+m}\,(f\,g \circ \rho) = 0$. Because we have

$$(g_1\, g_2 \cdots g_m)^{\ell+m}\,(f\,g \circ \rho) = (g_1\, g_2 \cdots g_m)^{\ell}\,(f \circ \rho)(g_1\, g_2 \cdots g_m)^{m}(g \circ \rho)\,,$$

we see that $f \in \mathrm{I}(X)$ or $g \in \mathrm{I}(X)$, so the ideal $\mathrm{I}(X)$ is prime. □

6.1.8 Example. Every toric ideal is prime. ◇

6.2 Operations on Ideals

For sums, intersections, and products of ideals, how do we find their generators? What is the geometric interpretation for these binary operations on ideals?

6.2.0 Definition. For any two ideals I and J in $S := \mathbb{K}[x_1, x_2, \dots, x_n]$, their *sum* is the set $I + J := \big\{ f + g \,\big|\, f \in I \text{ and } g \in J \big\}$.

6.2.1 Lemma. *For any ideals I and J in the ring S, the sum $I + J$ is the smallest ideal containing I and J. Moreover, when $I = \langle f_1, f_2, \dots, f_r \rangle$ and $J = \langle g_1, g_2, \dots, g_s \rangle$, we have $I + J = \langle f_1, f_2, \dots, f_r, g_1, g_2, \dots, g_s \rangle$.*

For any $f_1, f_2, \dots, f_r \in S$, we have $\langle f_1, f_2, \dots, f_r \rangle = \langle f_1 \rangle + \langle f_2 \rangle + \cdots + \langle f_r \rangle$.

Proof. Given two elements h_1 and h_2 in the sum $I + J$, there exists $f_1, f_2 \in I$ and $g_1, g_2 \in J$ such that $h_i = f_i + g_i$ for all $1 \leqslant i \leqslant 2$. For any $p_1, p_2 \in S$, we have $p_1\, f_1 + p_2\, f_2 \in I$ and $p_1\, g_1 + p_2\, g_2 \in J$, so

$$p_1\, h_1 + p_2\, h_2 = (p_1\, f_1 + p_2\, f_2) + (p_1\, g_1 + p_2\, g_2) \in I + J\,.$$

We deduce that $I + J$ is an ideal.

Suppose that K is an ideal that contains I and J. It follows that K must contain all elements $f \in I$ and $g \in J$. Since K is an ideal, K must contain all $f + g$ where $f \in I$ and $g \in J$. In particular, $I + J \subseteq K$ and we see that $I + J$ is the smallest ideal containing I and J.

Finally, the ideal $\langle f_1, f_2, \dots, f_r, g_1, g_2, \dots, g_s \rangle$ contains both I and J, so $I + J \subseteq \langle f_1, f_2, \dots, f_r, g_1, g_2, \dots, g_s \rangle$. Since the reverse inclusion is tautological, we have $I + J = \langle f_1, f_2, \dots, f_r, g_1, g_2, \dots, g_s \rangle$. □

6.2.2 Proposition. *For any two ideals I and J in the ring S, we have*

$$\mathrm{V}(I + J) = \mathrm{V}(I) \cap \mathrm{V}(J)\,.$$

Proof. To demonstrate the equality of affine subvarieties, we prove containment in both directions.

$\subseteq$: For any point a in the subvariety $\mathrm{V}(I + J)$, we have $a \in \mathrm{V}(I)$ because $I \subseteq I + J$. By symmetry, we also have $a \in \mathrm{V}(J)$. It follows that $a \in \mathrm{V}(I) \cap \mathrm{V}(J)$, so $\mathrm{V}(I + J) \subseteq \mathrm{V}(I) \cap \mathrm{V}(J)$.

$\supseteq$: Suppose that the point a belongs to the intersection $\mathrm{V}(I) \cap \mathrm{V}(J)$. Given a polynomial h in the sum $I + J$, there exists $f \in I$ and $g \in J$ such that $h = f + g$. We deduce that $f(a) = 0$ and $g(a) = 0$ because $a \in \mathrm{V}(I)$ and $a \in \mathrm{V}(J)$. It follows that $h(a) = 0$. We conclude that $a \in \mathrm{V}(I + J)$ and $\mathrm{V}(I + J) \supseteq \mathrm{V}(I) \cap \mathrm{V}(J)$. □

6.2.3 Definition. For any two ideals I and J in S, their *product* is the set $I\,J := \big\{ f_1\, g_1 + f_2\, g_2 + \cdots + f_r g_r \;\big|\; f_1, f_2, \dots, f_r \in I,\, g_1, g_2, \dots, g_r \in J \big\}$.

6.2.4 Proposition. *Given $I = \langle f_1, f_2, \dots, f_r \rangle$ and $J = \langle g_1, g_2, \dots, g_s \rangle$, we have $I\,J = \big\langle f_i\, g_j \;\big|\; 1 \leqslant i \leqslant r,\, 1 \leqslant j \leqslant s \big\rangle$.*

Proof. We again establish both containments.

$\subseteq$: Any polynomial in the product $I\,J$ is a sum of polynomials of the form $f\,g$ wheres $f \in I$ and $g \in J$. Writing f and g in terms of the ideal generators, we have $f = a_1\, f_1 + a_2\, f_2 + \cdots + a_r\, f_r$ and $g = b_1\, g_1 + b_2\, g_2 + \cdots + b_s\, g_s$ for some $a_1, a_2, \dots, a_r, b_1, b_2, \dots, b_s$ in S. Hence, the product $f\,g$ can be written as a sum $\sum_{i,j} c_{i,j}\, f_i\, g_j$ where $c_{i,j} \in S$.

$\supseteq$: Obvious. □

6.2.5 Proposition. *For any ideals I and J, we have* $\mathrm{V}(I\,J) = \mathrm{V}(I) \cup \mathrm{V}(J)$.

Proof. To demonstrate the equality of affine subvarieties, we prove containment in both directions.

$\subseteq$: Given a point a in the affine subvariety $\mathrm{V}(I\,J)$, we see that $f(a)\, g(a) = 0$ for all $f \in I$ and all $g \in J$. When $f(a) = 0$ for all $f \in I$, it follows that $a \in \mathrm{V}(I)$. If $f(a) \neq 0$ for some $f \in I$, then we must have $g(a) = 0$ for all $g \in J$. In either case, we see that $a \in \mathrm{V}(I) \cup \mathrm{V}(J)$ and $\mathrm{V}(I\,J) \subseteq \mathrm{V}(I) \cup \mathrm{V}(J)$.

$\supseteq$: Suppose that the point a lies in the union $\mathrm{V}(I) \cup \mathrm{V}(J)$. Either $f(a) = 0$ for all $f \in I$ or $g(a) = 0$ for all $g \in J$. It follows that $f(a)\, g(a) = 0$ for all $f \in I$ and all $g \in J$. Thus, we have $h(a) = 0$ for all $h \in I\,J$, so $\mathrm{V}(I\,J) \supseteq \mathrm{V}(I) \cup \mathrm{V}(J)$. □

6.2.6 Definition. The *intersection* $I \cap J$ of two ideal I and J in the ring S is the set of polynomials which belong to both I and J.

6.2.7 Proposition. *For any two ideals I and J in the ring S, the intersection $I \cap J$ is an ideal. Moreover, we have* $\mathrm{V}(I \cap J) = \mathrm{V}(I) \cup \mathrm{V}(J)$.

Proof. For any two polynomials f and g in the intersection $I \cap J$ and any two polynomials r and s in the ring S, we have $r\,f + s\,g \in I$ because I is an ideal and $f, g \in I$. We also have $r\,f + s\,g \in J$ because J is an ideal and $f, g \in J$. We deduce that $r\,f + s\,g \in I \cap J$ which shows that the intersection is an ideal.

For the second part, we prove containment in both directions.

$\subseteq$: Since $I\,J \subseteq I \cap J$, we have $\mathrm{V}(I \cap J) \subseteq \mathrm{V}(I\,J) = \mathrm{V}(I) \cup \mathrm{V}(J)$.

$\supseteq$: For any point a in the union $\mathrm{V}(I) \cup \mathrm{V}(J)$, we have $a \in \mathrm{V}(I)$ or $a \in \mathrm{V}(J)$, so either $f(a) = 0$ for all $f \in I$ or $f(a) = 0$ for all $f \in J$. Since $f(a) = 0$ for all $f \in I \cap J$, we see that $a \in \mathrm{V}(I \cap J)$ and $\mathrm{V}(I \cap J) \supseteq \mathrm{V}(I) \cup \mathrm{V}(J)$. □

6.2.8 Corollary. *For any ideals I and J, we have* $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

The intersection of two ideals corresponds to the same subvariety as the product. Although the intersection is harder to compute than the product, it behaves better with respect to taking radicals.

Proof. We prove containment in both directions.

$\subseteq$: Suppose that $f \in \sqrt{I \cap J}$. By definition, there exists a positive integer m such that $f^m \in I \cap J$. Since $f^m \in I$, we have $f \in \sqrt{I}$. By symmetry, we also have $f \in \sqrt{J}$. It follows that $\sqrt{I \cap J} \subseteq \sqrt{I} \cup \sqrt{J}$.

$\supseteq$: Suppose that $f \in \sqrt{I} \cap \sqrt{J}$. There exists positive integers ℓ and m such that $f^\ell \in \sqrt{I}$ and $f^m \in J$. It follows that $f^{\ell+m} = f^\ell\, f^m \in I \cap J$, so $f \in \sqrt{I \cap J}$ and $\sqrt{I \cap J} \supseteq \sqrt{I} \cup \sqrt{J}$. □

Algebra	Geometry
radical ideals	affine subvarieties
I	$\mathrm{V}(I)$
$\mathrm{I}(X)$	X
prime ideals	irreducible subvarieties
maximal ideals	points
$I + J$	$\mathrm{V}(I) \cap \mathrm{V}(J)$
$\sqrt{\mathrm{I}(X) + \mathrm{I}(Y)}$	$X \cap Y$
$I\,J$	$\mathrm{V}(I) \cup \mathrm{V}(J)$
$\sqrt{\mathrm{I}(X)\ \mathrm{I}(Y)}$	$X \cup Y$
$I \cap J$	$\mathrm{V}(I) \cup \mathrm{V}(J)$
$\mathrm{I}(X) \cap \mathrm{I}(Y)$	$X \cup Y$

Table 6.1: Algebro-geometric dictionary when the coefficient field $\mathbb{K}$ is algebraically closed