Problems 7

Due: Friday, 26 November 2021 before 17:00 EDT

Students registered in MATH 402 should submit solutions to 4 of the following problems. Students in MATH 802 should submit solutions to all 5.

1. The ***exponential power series*** is $\exp(x) := \sum_{n \in \mathbb{N}} \dfrac{x^n}{n!} \in \mathbb{Q}[[x]]$.

(i) Let $f \in \mathbb{Q}[[x]]$. When $\frac{df}{dx} = f$, show that there exists $c \in \mathbb{Q}$ such that $f = c\,\exp(x)$.

(ii) By extracting coefficients in $\mathbb{Q}[[t, x, y]]$, demonstrate that the binomial theorem is equivalent to the identity $\exp\big(t(x+y)\big) = \exp(tx)\exp(ty)$.

(iii) For all nonnegative integers k and n, prove the multinomial theorem

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{j_1 + j_2 + \cdots + j_k = n} \frac{n!}{j_1!\, j_2! \cdots j_k!} x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k}$$

by using a similar approach.

2. Find the unique sequence $(a_0, a_1, a_2, \dots)$ of real numbers such that, for all nonnegative integers j, we have $\sum_{k=0}^{j} a_k\, a_{j-k} = 1$.

3. For all nonnegative integers n, the ***Bernoulli numbers*** B_n are defined the recurrence

$$(n+1)B_n = -\sum_{k=0}^{n-1} \binom{n+1}{k} B_k$$

and the initial condition $B_0 = 1$.

(i) Prove that

$$\frac{x}{\exp(x) - 1} = \sum_{j \in \mathbb{N}} B_j\, \frac{x^j}{j!}\,.$$

(ii) Use the part (i) to demonstrate that $B_{2j+1} = 0$ for all positive integers j.

4. For any nonnegative integers m and n, use generating series to prove that

$$\sum_{k \in \mathbb{N}} \binom{m}{k}\binom{n+k}{m} = \sum_{k \in \mathbb{N}} \binom{m}{k}\binom{n}{k} 2^k\,.$$

5. For all nonnegative integers n, the ***Laguerre polynomials*** are defined by the recurrence

$$(n+2)\,L_{n+2}(x) = \big(2(n+1) + (1-x)\big)\,L_{n+1}(x) - (n+1)\,L_n(x)\,,$$

and the initial conditions $L_0(x) = 1$ and $L_1(x) = 1 - x$.

(i) Show that the generating series for the Laguerre polynomials is

$$\Phi(t) := \sum_{n \in \mathbb{N}} L_n(x)\, t^n = \frac{1}{1-t} \exp\left(-\frac{xt}{1-t}\right).$$

(ii) Find a closed formula for $L_n(x)$.

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