Problems 5

Due: Friday, 29 October 2021 before 17:00 EDT

Students registered in MATH 402 should submit solutions to 4 of the following problems. Students in MATH 802 should submit solutions to all 5.

1. Let $p(n)$ denote the number of partitions of the nonnegative integer n. Express the number of partitions of n with no part equal to 1 as a linear combination of values $p(k)$ for some nonnegative integer k.

2. A ***complete binary tree*** is a binary tree in which every vertex has either zero or two children. For any nonnegative integer n, provide a bijective proof that the Catalan number C_n equals the number of complete binary trees with $2n+1$ vertices.

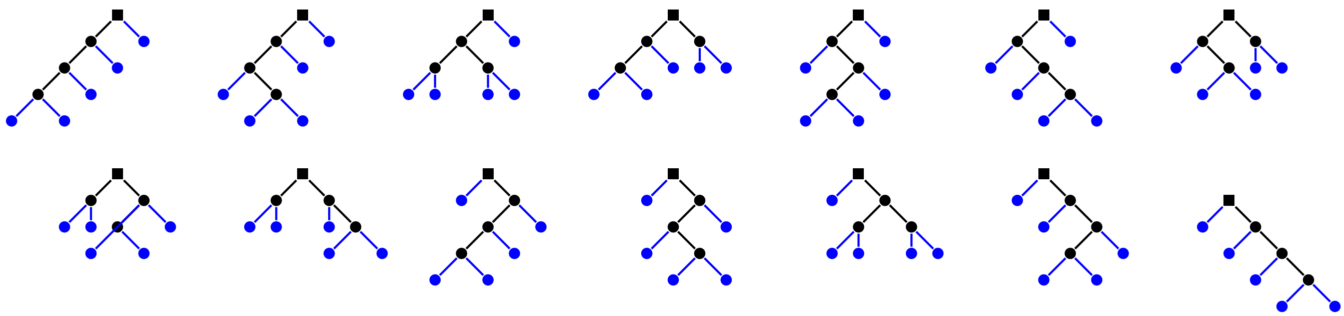

Figure 1. The 14 complete binary trees with 9 vertices

3. For any nonnegative integer n, provide a bijective proof that the Catalan number C_n counts the expressions containing n pairs of parentheses that are correctly matched.

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Figure 2. The 14 expressions containing 4 pairs of matched parentheses

4. For any nonnegative n, use a sign-reversing involution to prove that

$$\sum_{k \in \mathbb{Z}} (-1)^k \begin{bmatrix} n+2 \\ k \end{bmatrix} = 0 \,.$$

5. For all nonnegative integers m and n, use a sign-reversing involution to prove that

$$\sum_{k \in \mathbb{Z}} (-1)^k \binom{m+n}{m-k} \left(\!\!\binom{n}{k}\!\!\right) = 1 \,.$$