# Problems 3

Due: Friday, 1 October 2021 before 17:00 EDT

Students registered in MATH 402 should submit solutions to 4 of the following problems. Students in MATH 802 should submit solutions to all 5.

**1.** For any nonnegative integer $n$, let $F_n$ denote the $n$-th Fibonacci number. Prove each of the following identities via a double-counting argument.

(i) For any nonnegative integer $n$, verify that $F_{2n+2} = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \binom{n-j}{k}\binom{n-k}{j}$.

(ii) For all nonnegative integers $n$ and $m$ satisfying $n \geqslant m$, verify that

$$F_{n+m+1} = \sum_{k \in \mathbb{Z}} \binom{m}{k} F_{n-k+1} \,.$$

**2.** Give two proofs for each of the following identities: one using a double-counting argument and the other by relying on the key binomial identities.

(i) For any nonnegative integers $n$, establish that $\sum_{k \in \mathbb{Z}} \binom{n}{k}^2 = \binom{2n}{n}$.

(ii) For any nonnegative integer $n$, establish that

$$\sum_{k \in \mathbb{Z}} k(k-1)(k-2)\binom{n+3}{k} = (n+3)(n+2)(n+1)2^n \,.$$

**3.** Use the key binomial identities to solve the following problems.

(i) For any integer $k$, prove that

$$\binom{x}{k}\binom{x-\frac{1}{2}}{k} = 4^{-k}\binom{2x}{2k}\binom{2k}{k}.$$

(ii) For any nonnegative integers $m$ and $n$, express

$$\sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \binom{m-x+y}{k}\binom{n+x-y}{n-k}\binom{x}{m+n-j}\binom{k}{j}$$

in terms of $x$, $y$, $m$, and $n$.

**4.** For all positive integers $n$ and $k$, a ***composition*** of $n$ into $k$ parts is a $k$-tuple $(a_1, a_2, \dots, a_k)$ of positive integers such that $a_1 + a_2 + \cdots + a_k = n$.

(i) Provide a bijective proof that the number of compositions of $n$ into $k$ parts is $\binom{n-1}{k-1}$.

(ii) Show that the total number of compositions of $n$ is $2^{n-1}$.

(iii) Show that $\left(\!\binom{k}{n-k}\!\right) = \binom{n-1}{k-1}$ via a double-counting argument.

**5.** Prove the following identities via a double-counting argument.

(i) For all nonnegative integer $m$ and $n$, show that

$$\left(\!\binom{n}{2m+1}\!\right) = \sum_{k \in \mathbb{Z}} \left(\!\binom{k}{m}\!\right)\left(\!\binom{n-k+1}{m}\!\right).$$

(ii) For all nonnegative integer $m$, $n$, and $k$, show that

$$\left(\!\binom{m+n}{k}\!\right) = \sum_{j \in \mathbb{Z}} \left(\!\binom{m}{j}\!\right)\left(\!\binom{n}{k-j}\!\right).$$