Problems 2

Due: Friday, 24 September 2021 before 17:00 EDT

Students registered in MATH 402 should submit solutions to 4 of the following problems. Students in MATH 802 should submit solutions to all 5.

1. The ***Jacobsthal numbers*** are defined by $J_0 := 0$, $J_1 := 1$, and $J_n := J_{n-1} + 2\,J_{n-2}$ for all integers n greater than 1. This sequence begins $0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, \dots$. For all nonnegative integers n, show that J_{n+1} is the number of tilings of an $(2 \times n)$-rectangle with dominos and (2×2)-square tiles.

2. For any nonnegative integer n, let F_n denote the n-th Fibonacci number.

(i) For any nonnegative integer n, use a double-counting argument to verify that

$$F_2 + F_4 + F_6 + \cdots + F_{2n+2} = F_{2n+3} - 1\,.$$

(ii) For all nonnegative integers m and n, provide a bijective argument establishing that

$$F_{m+n+2} = F_{m+2}\,F_{n+2} - F_m\,F_n\,.$$

3. For any nonnegative integer n, the ***double factorial*** is defined by

$$n!! := \prod_{k=0}^{\lceil n/2 \rceil - 1} (n - 2k)\,.$$

This sequence begins $1, 1, 2, 3, 8, 15, 48, 105, 384, 945, 3840, 10395, \dots$.

(i) For any nonnegative integer n, show that $(2n)!!$ is the number of permutations of the set $[2n]$ such that $2i - 1$ is adjacent to $2i$ for all $1 \leqslant i \leqslant n$.

(ii) For any nonnegative integer n, use a double-counting argument to prove

$$\sum_{j=0}^{n} (2j + 1)(2j)!! = (2n + 2)!! - 1\,.$$

4. For any positive real number z, consider the integral $\Gamma(z) := \displaystyle\int_0^\infty x^{z-1} e^{-x}\, dx$.

(i) Prove that $\Gamma(z + 1) = z\,\Gamma(z)$.

(ii) For all nonnegative integers n, demonstrate that $\Gamma(n + 1) = n!$.

5. For any nonnegative integer n, consider the integral $C_n := \dfrac{1}{2\pi} \displaystyle\int_0^4 x^n \sqrt{\frac{4 - x}{x}}\, dx$.

(i) For all $n \in \mathbb{N}$, show that $C_n := \dfrac{2^{2(n+1)}}{\pi} \displaystyle\int_0^1 y^{2n} \sqrt{1 - y^2}\, dy$.

(ii) Compute C_0.

(iii) Demonstrate that $2(2n + 1)C_n = (n + 2)C_{n+1}$.