7 Generating Functions

By encoding a sequence of numbers as the coefficients of a formal power series, we gain access to a range of algebraic techniques. As a consequence, these generating series are often useful for finding a closed-form expressions for the elements in a sequence, analyzing the asymptotic properties, and proving identities.

7.0 Generating Series

In his well-known book *Generatingfunctionology*, Herbert Wilf describes a generating series as "a clothesline on which we hang up a sequence of numbers for display."

Definition 7.0.1. For any countable sequence $(a_0, a_1, a_2, \dots)$ of elements in a domain R, the corresponding ***ordinary generating series*** is the formal power series $f := \sum_{j \in \mathbb{N}} a_j \, x^j$ in $R[[x]]$.

Generating series were introduced by Abraham de Moivre in 1730 to solve the general linear recurrences.

Problem 7.0.2. For any nonnegative integer j, consider the integer sequence a_j defined by the recurrence $a_{j+1} = 2\,a_j + 1$ and the initial condition $a_0 = 0$. Find a closed-form expression for a_j.

Series solution. Consider the generating series $f := \sum_{j \in \mathbb{N}} a_j \, x^j$ in $\mathbb{Z}[x]$. The recurrence implies that, for all nonnegative integers j, we have $a_{j+1}\, x^j = (2a_j + 1)\, x^j$. It follow that

$$\frac{f - a_0}{x} = \sum_{j \in \mathbb{N}} a_{j+1}\, x^j = \sum_{j \in \mathbb{N}} (2a_j + 1)\, x^j = 2\left(\sum_{j \in \mathbb{N}} a_j \, x^j \right) + \left(\sum_{j \in \mathbb{N}} x^j \right) = 2\,f + \frac{1}{1-x} \,.$$

Hence, we obtain the functional equation

$$\left(\frac{1-2x}{x} \right) f = \left(\frac{1}{x} - 2 \right) f = \frac{1}{1-x} \quad \Rightarrow \quad f = \frac{x}{(1-x)(1-2x)} \,.$$

To find a formula for a_j, we expand f as a power series:

$$f = \frac{x}{(1-x)(1-2x)} = x\left(\frac{2}{1-2x} - \frac{1}{1-x} \right) = \sum_{j>0} (2x)^j - \sum_{j>0} x^j = \sum_{j \in \mathbb{N}} (2^j - 1)\, x^j \,,$$

so $a_j = 2^j - 1$ for all nonnegative integers j. □

Problem 7.0.3. For any nonnegative integer j, consider the integer sequence a_j defined by the recurrence $a_{j+1} = 2\,a_j + j$ and the initial condition $a_0 = 1$. Find a closed-form expression for a_j.

Series solution. Setting $f := \sum_{j \in \mathbb{N}} a_j \, x^j$ in $\mathbb{Z}[x]$, the recurrence gives

$$\frac{f - a_0}{x} = \sum_{j \in \mathbb{N}} a_{j+1} \, x^j = \sum_{j \in \mathbb{N}} (2a_j + j) \, x^j = 2 \sum_{j \in \mathbb{N}} a_j \, x^j + \sum_{j \in \mathbb{N}} j \, x^j = 2 \, f + \frac{x}{(1-x)^2} \, .$$

Hence, we obtain

$$\begin{aligned} \Big(\frac{1-2x}{x}\Big) f = \Big(\frac{1}{x} - 2\Big) f &= \frac{x}{(1-x)^2} + \frac{1}{x} \\ \Rightarrow \quad f &= \frac{(x^2 + (1-x)^2)x}{x(1-x)^2(1-2x)} = \frac{1 - 2x + 2x^2}{(1-x)^2(1-2x)} \, . \end{aligned}$$

To find an explicit formula for a_j, we first find a partial fraction decomposition

$$f = \frac{1 - 2x + 2x^2}{(1-x)^2(1-2x)} = \frac{\alpha}{(1-x)^2} + \frac{\beta}{1-x} + \frac{\gamma}{1-2x} \, .$$

Since $\alpha(1-2x) + \beta(1-x)(1-2x) + \gamma(1-x)^2 = 1 - 2x + 2x^2$, we have

$$\begin{aligned} x = 1: &\quad \alpha(-1) = 1 &&\Rightarrow \quad \alpha = -1 \\ x = \tfrac{1}{2}: &\quad \gamma\big(\tfrac{1}{2}\big)^2 = 1 - 1 + \tfrac{1}{2} &&\Rightarrow \quad \gamma = 2 \\ x = 0: &\quad -1 + \beta + 2 = 1 &&\Rightarrow \quad \beta = 0 \, . \end{aligned}$$

Hence, expanding f as a power series produces

$$\begin{aligned} f &= \frac{1 - 2x + 2x^2}{(1-x)^2(1-2x)} = \frac{(-1)}{(1-x)^2} + \frac{2}{1-2x} \\ &= -\sum_{j \in \mathbb{N}} (j+1) \, x^j + 2 \sum_{j \in \mathbb{N}} (2x)^j = \sum_{j \in \mathbb{N}} (2^{j+1} - j - 1) \, x^j \, , \end{aligned}$$

so $a_j = 2^{j+1} - j - 1$ for all nonnegative integers j. □

Proposition 7.0.4 (Fibonacci series). *For any nonnegative integer n, let F_n denote the n-th Fibonacci number. Setting $F(x) := \sum_{n \in \mathbb{N}} F_n \, x^n$, we have*

$$F(x) = \frac{x}{1 - x - x^2} \, .$$

Proof. The Fibonacci recurrence gives

$$\frac{F(x) - F_0 - F_1 \, x}{x^2} = \sum_{n \in \mathbb{N}} F_{n+2} \, x^n = \sum_{n \in \mathbb{N}} (F_{n+1} + F_n) \, x^n = \frac{F(x) - F_0}{x} + F(x) \, .$$

Using the initial conditions $F_0 = 0$ and $F_1 = 1$, we obtain

$$\begin{aligned} \Big(\frac{1 - x - x^2}{x^2}\Big) F(x) = \Big(\frac{1}{x^2} - \frac{1}{x} - 1\Big) F(x) &= \frac{1}{x} \\ \Rightarrow \quad F(x) &= \frac{x}{1 - x - x^2} \, . \end{aligned}$$ □

Problem 7.0.5. Use Fibonacci series to determine a closed-form for the Fibonacci numbers.

Series solution. We first find a partial fraction decomposition for the Fibonacci series. By setting $\varphi_{\pm} := \frac{1}{2}(1 \pm \sqrt{5})$, it follows that $1 - x - x^2 = (1 - \varphi_+ \, x)(1 - \varphi_- \, x)$, so we obtain

$$F = \frac{x}{1 - x - x^2} = \frac{\alpha}{1 - \varphi_+ \, x} + \frac{\beta}{1 - \varphi_- \, x} \, .$$

Since $\alpha(1-\varphi_-\,x)+\beta(1-\varphi_+\,x)=x$, we have

$$
\begin{aligned}
x = \tfrac{1}{\varphi_+} : \ \alpha\,(1-\tfrac{\varphi_-}{\varphi_+}) = \tfrac{1}{\varphi_+} \quad &\Rightarrow \quad \alpha = \tfrac{1}{\varphi_+}\,(\tfrac{\varphi_+}{\varphi_+-\varphi_-}) = \tfrac{1}{\sqrt{5}}\,,\\
x = \tfrac{1}{\varphi_-} : \ \beta\,(1-\tfrac{\varphi_+}{\varphi_-}) = \tfrac{1}{\varphi_-} \quad &\Rightarrow \quad \beta = \tfrac{1}{\varphi_-}\,(\tfrac{\varphi_-}{\varphi_--\varphi_+}) = -\tfrac{1}{\sqrt{5}}\,.
\end{aligned}
$$

Hence, expanding f as a power series produces

$$
\begin{aligned}
F(x) &= \tfrac{1}{\sqrt{5}}\left(\frac{1}{1-\varphi_+\,x} - \frac{1}{1-\varphi_-\,x}\right)\\
&= \tfrac{1}{\sqrt{5}}\left[\sum_{n\in\mathbb{N}}(\varphi_+\,x)^n - \sum_{n\in\mathbb{N}}(\varphi_-\,x)^n\right] = \sum_{n\in\mathbb{N}}\tfrac{1}{\sqrt{5}}(\varphi_+^n - \varphi_-^n)\,x^n
\end{aligned}
$$

so $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$ for all nonnegative integers j. □

Problem 7.0.6. For all nonnegative integers n, let F_n denote the n-th Fibonacci number. Show that $F_0+F_1+F_2+\cdots+F_n = F_{n+2}-1$.

Series solution. Combining Problem 6.3.3 and Proposition 7.0.4 shows that

$$
\sum_{n\in\mathbb{N}}(F_0+F_1+F_2+\cdots+F_n)\,x^n = \frac{F(x)}{(1-x)} = \frac{x}{(1-x)(1-x-x^2)}\,.
$$

On the other hand, we also have

$$
\begin{aligned}
\sum_{n\in\mathbb{N}}(F_{n+2}-1)\,x^n &= \frac{F-F_0-F_1\,x}{x^2} - \frac{1}{1-x}\\
&= \frac{1}{x(1-x-x^2)} - \frac{1}{x} - \frac{1}{1-x}\\
&= \frac{(1-x)-(1-x-x^2)(1-x)-x(1-x-x^2)}{x(1-x)(1-x-x^2)}\\
&= \frac{x}{(1-x)(1-x-x^2)}\,.
\end{aligned}
$$

Comparing coefficients of x^n in these two formal power series yields the identity. □

7.1 Identities from Generating Series

We illustrate the power of generating series by proving and reproving combinatorial identities.

Problem 7.1.1. For any nonnegative integer n, let F_n be the n-th Fibonacci number. Prove that $F_1+F_3+F_5+\cdots+F_{2n+1} = F_{2n+2}$.

Series solution. Proposition 7.0.4 gives $F(x) = \frac{x}{1-x-x^2} = \sum_{n\in\mathbb{N}} F_n\,x^n$. Since

$$
\frac{1}{2x}\big(F(x)-F(-x)\big) = \frac{1}{x}\sum_{n\in\mathbb{N}}\frac{1}{2}\big(F_n-(-1)^n\,F_n\big)\,x^n = \frac{1}{x}\sum_{n\in\mathbb{N}}F_{2n+1}\,x^{2n+1} = \sum_{n\in\mathbb{N}}F_{2n+1}\,x^{2n}\,,
$$

Problem 6.3.3 shows that the generating series for the left side of the desired identity is

$$\sum_{n\in\mathbb{N}}(F_1 + F_3 + \cdots + F_{2n+1})\,(x^{2n} + x^{2n+1}) = \frac{F(x) - F(-x)}{2x(1-x)}$$
$$= \frac{1}{2x(1-x)}\left(\frac{x}{1-x-x^2} + \frac{x}{1+x-x^2}\right) = \frac{1+x-x^2+1-x-x^2}{2(1-x)(1-x-x^2)(1+x-x^2)}$$
$$= \frac{1+x}{(1-x-x^2)(1+x-x^2)}\,.$$

Similarly, we have

$$\frac{1}{2}\big(F(x) + F(-x)\big) = \sum_{n\in\mathbb{N}}\frac{1}{2}\big(F_n + (-1)^n F_n\big)\,x^n = \sum_{n\in\mathbb{N}} F_{2n}\,x^{2n}\,,$$

so the generating series for the right side is

$$\sum_{n\in\mathbb{N}} F_{2n+2}\,(x^{2n} + x^{2n+1}) = \left(\frac{1+x}{2x^2}\right)\big(F(x) + F(-x)\big)$$
$$= \frac{1+x}{2x^2}\left(\frac{x}{1-x-x^2} - \frac{x}{1+x-x^2}\right) = \frac{1+x}{2x}\left(\frac{1+x-x^2-1+x+x^2}{(1-x-x^2)(1+x-x^2)}\right)$$
$$= \frac{1+x}{(1-x-x^2)(1+x-x^2)}\,.$$

By extracting the coefficients of x^{2n} or x^{2n+1}, we establish the desired identity on Fibonacci numbers. □

Alternative solution. Because we have

$$\sum_{n\in\mathbb{N}} F_n\,x^n = \frac{x}{1-x-x^2}\left(\frac{1+x-x^2}{1+x-x^2}\right) = \frac{x+x^2-x^3}{1-3x^2+x^4} = x\frac{1-x^2}{1-3x^2+x^4} + \frac{x^2}{1-3x^2+x^4}\,,$$

extracting even and odd powers shows

$$\sum_{n\in\mathbb{N}} F_{2n}\,x^n = \frac{x}{1-3x+x^2} \quad\text{and}\quad \sum_{n\in\mathbb{N}} F_{2n+1}\,x^n = \frac{1-x}{1-3x+x^2}\,.$$

Hence, the generating series for the left side is

$$\sum_{n\in\mathbb{N}}(F_1 + F_3 + \cdots + F_{2n+1})\,x^n = \frac{1-x}{(1-x)(1-3x+x^2)} = \frac{1}{1-3x+x^2}\,,$$

and the generating series for the right side is

$$\sum_{n\in\mathbb{N}} F_{2n+2}\,x^n = \frac{x}{x(1-3x+x^2)} = \frac{1}{1-3x+x^2}\,.$$

The two generating series equal the same rational function, so the desired identity holds. □

Problem 7.1.2. For any nonnegative integer n, let F_n denote the n-th Fibonacci number. Prove that

$$F_2 + F_4 + F_5 + \cdots + F_{2n} = F_{2n+1} - 1\,.$$

Series solution. The alternative solution above shows that the generating series for the right side is

$$\sum_{n \in \mathbb{N}} (F_{2n+1} - 1)\, x^n = \frac{1-x}{1-3x+x^2} - \frac{1}{1-x} = \frac{(1-x)^2 - 1 + 3x - x^2}{(1-x)(1-3x+x^2)} = \frac{x}{(1-x)(1-3x+x^2)}$$

and the generating series for the left side is

$$\sum_{n \in \mathbb{N}} (F_0 + F_2 + \cdots + F_{2n})\, x^n = \frac{x}{(1-x)(1-3x+x^2)} \,.$$

The two generating series equal the same rational function, so the desired identity holds. □

Proposition 7.1.3 (Catalan series)**.** *For any nonnegative integer n, let C_n be the n-th Catalan number. Setting $C(x) := \sum_{n \in \mathbb{N}} C_n\, x^n$, we have*

$$C(x) = \frac{1 - \sqrt{1-4x}}{2x} \,.$$

Proof. Since $C_0 = 1$ and the Catalan recurrence [1.2.5] asserts that $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$, we obtain

$$\frac{C(x) - C_0}{x} = \sum_{n \in \mathbb{N}} C_{n+1}\, x^n = \sum_{n \in \mathbb{N}} \left(\sum_{k=0}^{n} C_k C_{n-k} \right) x^n = \big(C(x)\big)^2$$

It follows that $x\big(C(x)\big)^2 - C(x) + 1 = 0$ and $C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$. Since the binomial theorem [6.2.5] for complex exponents gives $\sqrt{1-4x} = 1 - 2x + \cdots$, we have

$$\frac{1+\sqrt{1-4x}}{2x} = \frac{1 + (1 - 2x + \cdots)}{2x} = \frac{1}{x} - 1 + \cdots \,,$$
$$\frac{1-\sqrt{1-4x}}{2x} = \frac{1 - (1 - 2x + \cdots)}{2x} = 1 + \cdots \,,$$

and we conclude that $C(x) = \frac{1-\sqrt{1-4x}}{2x}$. □

From the Catan series, we can recover the closed-formula for the individual Catalan numbers.

Corollary 7.1.4. *For any nonnegative integer n, we have*

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!(n+1)!} = \frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n}\binom{2n}{n-1}.$$

Proof. The binomial theorem [6.2.5] for complex exponents and the absorption identity [2.1.3] for binomial coefficients give

$$\begin{aligned}
\sqrt{1-4x} &= \sum_{n \in \mathbb{N}} \binom{1/2}{n} (-4x)^n = 1 + \sum_{n \geqslant 1} \frac{1}{2n} \binom{-1/2}{n-1} (-4)^n\, x^n \\
&= 1 + \sum_{n \geqslant 1} \frac{1}{2n} \left(\frac{(-\frac{1}{2})(-\frac{1}{2} - 1) \cdots (-\frac{1}{2} - n + 2)}{(n-1)!} \right) (-4)^n x^n \\
&= 1 - \sum_{n \geqslant 1} \frac{2^n}{n} \left(\frac{(1)(1+2)\cdots(1+2n-4)}{(n-1)!} \right) x^n \\
&= 1 - 2 \sum_{n \geqslant 1} \frac{1}{n} \left(\frac{(1)(2)(1+2)(4)\cdots(2n-3)(2n-2)}{(n-1)!(n-1)!} \right) x^n \\
&= 1 - 2 \sum_{n \geqslant 1} \frac{1}{n} \binom{2n-2}{n-1} x^n = 1 - 2 \sum_{n \in \mathbb{N}} \frac{1}{n+1} \binom{2n}{n} x^{n+1}
\end{aligned}$$

we obtain $C(x) = \frac{1}{2x}\big(1 - \sqrt{1 - 4\,x}\big) = \sum_{n \in \mathbb{N}} \frac{1}{n+1} \binom{2n}{n} x^n$. □

Exercises

Problem 7.1.5. For any nonnegative integer n, let C_n denote the n-th Catalan number and set $C(x) := \sum_{n \in \mathbb{N}} C_n\, x^n$. Use generating series to prove the identity

$$\sum_{k=0}^{n} C_{2k}\, C_{2(n-k)} = 4^n\, C_n \, .$$

Problem 7.1.6. For any nonnegative integer n, let C_n be the n-th Catalan number and set $C(x) := \sum_{n \in \mathbb{N}} C_n\, x^n$. Prove that

$$\sum_{n \in \mathbb{N}} x^n\, t^n \big(C(t)\big)^n = \frac{1 - x + xt\, C(t)}{1 - x + x^2 t} \, .$$

7.2 Binomial Identities from Series

Generating series provide independent verification of the key identities involving binomial and multichoose coefficients.

Theorem 7.2.1 (Binomial)**.** *For any nonnegative integer n, we have*

$$(1 + x)^n = \sum_{k \in \mathbb{N}} \binom{n}{k} x^k$$

Inductive proof. For any nonnegative integer n, set

$$P_n(x) := \sum_{k \in \mathbb{N}} \binom{n}{k} x^k \, .$$

When $n = 0$, we see that $P_0(x) = 1$ because $\binom{0}{0} = 1$ and $\binom{0}{k+1} = 0$ for all nonnegative integers k. The addition identity [2.0.6] for binomial coefficients gives

$$\begin{aligned} P_{n+1}(x) - \binom{n+1}{0} &= \sum_{k \geqslant 1} \binom{n}{k} x^k = \sum_{k \geqslant 1} \left[\binom{n}{k} + \binom{n}{k-1} \right] x^k \\ &= \sum_{k \geqslant 1} \binom{n}{k} x^k + \sum_{k \in \mathbb{N}} \binom{n}{k} x^{k+1} = \left[P_n(x) - \binom{n}{0} \right] + x\, P_n(x) \, . \end{aligned}$$

Since $\binom{n+1}{0} = 1 = \binom{n}{0}$, we deduce that $P_{n+1}(x) = (1 + x)\, P_n(x)$. The induction hypothesis is $P_n(x) = (1 + x)^n$, so we conclude that $P_{n+1}(x) = (1 + x)^{n+1}$. □

Problem 7.2.2. For all nonnegative integers k, n, and m, reprove the trinomial revision identity [2.1.4]:

$$\binom{n}{k} \binom{n-k}{n-m} = \binom{n}{m} \binom{m}{k} .$$

Series solution. Using the binomial theorem four times gives

$$\sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}}\binom{n}{k}\binom{n-k}{j}x^j\,y^k = \sum_{k\in\mathbb{N}}\binom{n}{k}\left[\sum_{j\in\mathbb{N}}\binom{n-k}{j}x^j\right]y^k = \sum_{k\in\mathbb{N}}\binom{n}{k}(1+x)^{n-k}\,y^k$$
$$= \big((1+x)+y\big)^n = (1+x+y)^n = \big(x+(1+y)\big)^n$$
$$= \sum_{m\in\mathbb{N}}\binom{n}{m}x^{n-m}\,(1+y)^m = \sum_{m\in\mathbb{N}}\binom{n}{m}x^{n-m}\left[\sum_{k\in\mathbb{N}}\binom{m}{k}y^k\right] = \sum_{m\in\mathbb{N}}\sum_{k\in\mathbb{N}}\binom{n}{m}\binom{m}{k}x^{n-m}y^k\,.$$

Extracting the coefficient of $x^{n-m}y^k$ proves the trinomial revision identity. This argument also explains the name. □

Problem 7.2.3. For all nonnegative integers m and n, reprove the parallel sum identity [2.3.4] for binomial coefficients:

$$\sum_{j=0}^{n}\binom{m+j}{j} = \binom{m+n+1}{n}.$$

Series solution. The generalized binomial theorem [6.2.3] states that $(1-x)^{-m-1} = \sum_{n\in\mathbb{N}}\binom{m+n}{n}x^n$. Hence, extracting the coefficient of x^n from both sides of the equation

$$\sum_{n\in\mathbb{N}}\left[\sum_{j=0}^{n}\binom{m+j}{j}\right]x^n = \left(\frac{1}{(1-x)^{m+1}}\right)\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^{m+2}}$$
$$= \sum_{n\in\mathbb{N}}\binom{m+n+1}{m+1}x^n = \sum_{n\in\mathbb{N}}\binom{m+n+1}{n}x^n\,,$$

establishes the parallel sum identity. □

Problem 7.2.4. For all nonnegative integers m and n, re-confirm the upper sum identity [2.0.8] for binomial coefficients:

$$\sum_{j=0}^{n}\binom{j}{m} = \binom{n+1}{m+1}.$$

Series Solution. The generalized binomial theorem [6.2.3], together with the symmetry [2.0.5] of binomial coefficients, implies that

$$\frac{1}{(1-x)^{m+1}} = \sum_{n\in\mathbb{N}}\binom{m+n}{n}x^n = \sum_{n\in\mathbb{N}}\binom{m+n}{m}x^n\,.$$

Multiplying both sides by x^m and reindexing the sum yields

$$\frac{x^m}{(1-x)^{m+1}} = \sum_{j\in\mathbb{N}}\binom{j}{m}x^j\,.$$

Hence, the generating function for the left side the identity is

$$\sum_{n\in\mathbb{N}}\left[\sum_{j=0}^{n}\binom{j}{m}\right]x^n = \left(\frac{x^m}{(1-x)^{m+1}}\right)\left(\frac{1}{1-x}\right) = \frac{x^m}{(1-x)^{m+2}}$$

and the generating function for the right side is

$$\sum_{n\in\mathbb{N}}\binom{n+1}{m+1}x^n = \frac{1}{x}\left[\frac{x^{m+2}}{(1-x)^{m+2}} - \binom{0}{m+1}\right] = \frac{x^m}{(1-x)^{m+2}}\,.$$

The two generating series equal the same rational function, so the desired identity holds. □

Problem 7.2.5. For all nonnegative integers k and n, reprove the parallel sum identity [2.3.3] for multichoose coefficients:

$$\left(\!\!\binom{n+1}{k}\!\!\right) = \sum_{j=0}^{k} \left(\!\!\binom{n}{j}\!\!\right).$$

Series solution. The generalized binomial theorem [6.2.3] and Problem 6.3.3 give

$$\sum_{k \in \mathbb{N}} \left(\!\!\binom{n+1}{k}\!\!\right) x^k = \frac{1}{(1-x)^{n+1}} = \left(\frac{1}{(1-x)^n}\right)\left(\frac{1}{1-x}\right) = \sum_{k \in \mathbb{N}} \left[\sum_{j=0}^{k} \left(\!\!\binom{n}{j}\!\!\right)\right] x^k \,,$$

Extracting the coefficients of x^k proves the identity. □

Theorem 7.2.6 (Generalized binomial)**.** *For any nonnegative integer n, we have*

$$\frac{1}{(1-x)^n} = \sum_{k \in \mathbb{N}} \left(\!\!\binom{n}{k}\!\!\right) x^k \,.$$

Inductive proof. For any nonnegative integer n, set

$$M_n(x) := \sum_{k \in \mathbb{N}} \left(\!\!\binom{n}{k}\!\!\right) x^k \,.$$

When $n = 0$, we have $M_0(x) = 1$ because $\left(\!\binom{0}{0}\!\right) = 1$ and $\left(\!\binom{0}{k+1}\!\right) = 0$ for all nonnegative integers k. The addition formula [2.2.5] for multichoose coefficients gives

$$\begin{aligned} M_{n+1}(x) - \left(\!\!\binom{n+1}{0}\!\!\right) &= \sum_{k \geqslant 1} \left(\!\!\binom{n+1}{k}\!\!\right) x^k = \sum_{k \geqslant 1} \left[\left(\!\!\binom{n+1}{k-1}\!\!\right) + \left(\!\!\binom{n}{k}\!\!\right)\right] x^k \\ &= \sum_{k \in \mathbb{N}} \left(\!\!\binom{n+1}{k}\!\!\right) x^{k+1} + \sum_{k \geqslant 1} \left(\!\!\binom{n}{k}\!\!\right) x^k = x\, M_{n+1}(x) + M_n(x) - \left(\!\!\binom{n}{0}\!\!\right). \end{aligned}$$

Since $\left(\!\binom{n+1}{0}\!\right) = 1 = \left(\!\binom{n}{0}\!\right)$, we deduce that $(1-x)\, M_{n+1}(x) = M_n(x)$. The induction hypothesis states that $M_n(x) = (1-x)^{-n}$, so we conclude that $M_{n+1}(x) = (1-x)^{-n-1}$. □

Exercises

Problem 7.2.7. For all nonnegative integers m and n, the ***Delannoy numbers*** $D_{m,n}$ satisfy $D_{m+1,n+1} = D_{m,n+1} + D_{m+1,n} + D_{m,n}$ and the initial conditions $D_{m,0} = D_{0,n} = 1$.

(i) Express the generating series $D(x, y) := \sum_{m \in \mathbb{N}} \sum_{n \in \mathbb{N}} D_{m,n}\, x^m\, y^n$ as a rational function.

(ii) Show that $D_{m,n} = \sum_{k \in \mathbb{N}} \binom{m}{k}\binom{m+n-k}{m}$.

Problem 7.2.8. For all nonnegative integers m and n, use generating series to demonstrate that

$$\sum_{k \in \mathbb{N}} \binom{m}{k}\binom{n+k}{m} = \sum_{k \in \mathbb{N}} \binom{m}{k}\binom{n}{k} 2^k \,.$$